Research Article

Modified Hybrid Steepest-Descent Methods for General Systems of Variational Inequalities with Solutions to Zeros of m-Accretive Operators in Banach Spaces

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The purpose of this paper is to introduce and analyze modified hybrid steepest-descent methods for a general system of variational inequalities (GSVI), with solutions being also zeros of an m-accretive operator A in the setting of real uniformly convex and 2-uniformly smooth Banach space X. Here the modified hybrid steepest-descent methods are based on Korpelevich's extragradient method, hybrid steepest-descent method, and viscosity approximation method. We propose and consider modified implicit and explicit hybrid steepest-descent algorithms for finding a common element of the solution set of the GSVI and the set $A^{-1}(0)$ of zeros of A in X. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literature.

1. Introduction

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The set of fixed points of T is denoted by $\text{Fix}(T)$. We use the notation $\rightharpoonup$ to indicate the weak convergence and $\to$ to indicate the strong convergence. A mapping $A : C \to X$ is said to be

(i) accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge 0; \tag{2}$$

(ii) α-strongly accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge \alpha \|x - y\|^2, \tag{3}$$

for some $\alpha \in (0, 1)$;

(iii) β-inverse strongly accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \ge \beta \|Ax - Ay\|^2, \tag{4}$$

for some $\beta > 0$;

(iv) λ-strictly pseudocontractive [1] (see also [2]) if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \le \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2 \tag{5}$$

for some $\lambda \in (0, 1)$.

It is worth emphasizing that the definition of the inverse strongly accretive mapping is based on that of the inverse

strongly monotone mapping, which was studied by so many authors; see, for example, [3–5]. Let $U = \{x \in X : \|x\| = 1\}$ denote the unite sphere of X. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that for all $x, y \in U$

$$\|x - y\| \geq \epsilon \implies \frac{\|x + y\|}{2} \leq 1 - \delta. \tag{6}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{7}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gateaux differentiable norm. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$; in this case, X is also said to have a uniformly Frechet differentiable norm. The norm of X is said to be the Frechet differential if for each $x \in U$ this limit is attained uniformly for $y \in U$. In the meantime, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho(\tau) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : \right.$$
$$\left. x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \tag{8}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [6], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X.

Let A be an operator with domain $D(A)$ and range $R(A)$ in X is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$ there exists $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1) \rangle \geq 0. \tag{9}$$

An accretive operator A is said to be m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$. Denote by J_r the resolvent of A; that is, for each $r > 0$,

$$J_r = (I + rA)^{-1}. \tag{10}$$

It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex. In 2008, Chen and Zhu [7] derived the following strong convergence theorems for viscosity approximation methods for accretive operators in a uniformly smooth Banach space X.

Theorem CZ1. *Let X be a uniformly smooth Banach space. Suppose that A is an m-accretive operator in X such that*

$C = \overline{D(A)}$ *is convex and $f : C \to C$ is a fixed contractive map. For each $t \in (0, 1)$, $\{x_{t,n}\}$ is defined by*

$$x_{t,n} = tf(x_{t,n}) + (1 - t) J_{r_n} x_{t,n}. \tag{11}$$

Then as $t \to 0$, $\{x_{t,n}\}$ converges strongly to a zero of A.

Theorem CZ2. *Let X be a uniformly smooth Banach space. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex and $f : C \to C$ is a fixed contractive map. The sequence $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n} x_n, \quad \forall n \geq 0, \tag{12}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $r_n \geq \varepsilon$ for all n and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then as $n \to \infty$, $\{x_n\}$ converges strongly to a zero of A.

In the meantime, Ceng et al. [8] derived some strong convergence theorems of composite iterative schemes for zeros of m-accretive operators in uniformly smooth Banach spaces. Furthermore, motivated by strong convergence results for hybrid steepest-descent methods in [9, 10], Ceng et al. [11] established some strong convergence theorems for hybrid steepest-descent methods for nonexpansive and m-accretive operators in a uniformly smooth Banach space X. Subsequently, Ceng et al. [12] introduced hybrid viscosity approximation method for finding zeros of m-accretive operators, which combine viscosity approximation method with hybrid steepest-descent method, and obtained the following strong convergence theorems.

Theorem CASY1. *Let X be a uniformly smooth Banach space, let A be an m-accretive operator in X with $A^{-1}(0) \neq \emptyset$, and let $f : X \to C (= \overline{D(A)})$ be a contractive map. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. For each $t \in (0, 1)$ and each integer $n \geq 0$, let $\{x_{t,n}\}$ be defined by*

$$x_{t,n} = tf(x_{t,n}) + (1 - t) \left[J_{r_n} x_{t,n} - \theta_t F(x_{t,n}) \right], \tag{13}$$

where $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$ and $\{\theta_t : t \in (0, 1)\} \subset [0, 1)$ with $\lim_{t \to 0}(\theta_t/t) = 0$. Then as $t \to 0$, $\{x_{t,n}\}$ converges strongly to a zero p of A, which is a unique solution of the variational inequality problem (VIP)

$$\langle (I - f) p, J(p - u) \rangle \leq 0, \quad \forall u \in A^{-1}(0). \tag{14}$$

Theorem CASY2. *Let X be a uniformly smooth Banach space, let A be an m-accretive operator in X with $A^{-1}(0) \neq \emptyset$, and let $f : X \to C (= \overline{D(A)})$ be a contractive map. Assume that $F : X \to X$ is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Given sequences $\{\lambda_n\}_{n=0}^{\infty}, \{\mu_n\}_{n=0}^{\infty}$ in $[0, 1]$, $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ in $(0, 1]$, and $\{r_n\}_{n=0}^{\infty}$*

in $[\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:

(i) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\lim_{n \to \infty} (\lambda_n \mu_n) / \beta_n = 0$,

(iii) $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then for any given point $x_0 \in X$, the sequence $\{x_n\}$ generated by

$$y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n,$$
$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) \left[J_{r_n} y_n - \lambda_n \mu_n F(J_{r_n} y_n) \right], \quad (15)$$
$$\forall n \geq 0,$$

converges strongly to a zero p of A, which is a unique solution of the VIP as above.

On the other hand, Cai and Bu [13] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in C, \quad (16)$$

where C is a nonempty, closed and convex subset of X, $B_1, B_2 : C \to X$ are two nonlinear mappings, and μ_1 and μ_2 are two positive constants. Here the set of solutions of GSVI (16) is denoted by GSVI (C, B_1, B_2). In particular, if $X = H$, a real Hilbert space, then GSVI (16) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \quad (17)$$

where μ_1 and μ_2 are two positive constants. The set of solutions of problem (17) is still denoted by GSVI (C, B_1, B_2). In particular, if $B_1 = B_2 = A$, then problem (17) reduces to the new system of variational inequalities (NSVI), introduced and studied by Verma [14]. Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (18)$$

The solution set of the VIP (18) is denoted by VI (C, A). Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [15]. This alternative formulation has been used to suggest and analyze projection iterative method for solving variational

inequalities under the condition that the involved operator must be strongly monotone and Lipschitz continuous.

Recently, Ceng et al. [16] transformed problem (17) into a fixed point problem in the following way.

Lemma 1 (see [16]). *For given $\overline{x}, \overline{y} \in C, (\overline{x}, \overline{y})$ is a solution of problem* (17) *if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C \left[P_C \left(x - \mu_2 B_2 x \right) - \mu_1 B_1 P_C \left(x - \mu_2 B_2 x \right) \right], \quad (19)$$
$$\forall x \in C,$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$ and P_C is the the projection of H onto C.

In particular, if the mapping $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

In 1976, Korpelevič [17] proposed an iterative algorithm for solving the VIP (18) in Euclidean space $\mathbf{R}^n$:

$$y_n = P_C(x_n - \tau A x_n), \quad x_{n+1} = P_C(x_n - \tau A y_n), \quad (20)$$
$$\forall n \geq 0,$$

with $\tau > 0$ being a given number, which is known as the extragradient method (see also [18]). The literature on the VIP is vast, and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, for example, [3, 13, 19–32] and references therein, to name but a few.

In particular, whenever X is still a real smooth Banach space, $B_1 = B_2 = A$, and $x^* = y^*$, then GSVI (16) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \quad (21)$$

which was considered by Aoyama et al. [33]. Note that VIP (21) is connected with the fixed point problem for nonlinear mapping (see, e.g., [34]), the problem of finding a zero point of a nonlinear operator (see, e.g., [35]), and so on. It is clear that VIP (21) extends VIP (18) from Hilbert spaces to Banach spaces.

In order to find a solution of VIP (21), Aoyama et al. [33] introduced the following iterative scheme for an accretive operator A:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \Pi_C (x_n - \lambda_n A x_n), \quad \forall n \geq 1, \quad (22)$$

where Π_C is a sunny nonexpansive retraction from X onto C. Then they proved a weak convergence theorem.

Beyond doubt, it is an interesting and valuable problem of constructing some algorithms with strong convergence for solving GSVI (16) which contains VIP (21) as a special case. Very recently, Cai and Bu [13] constructed an iterative algorithm for solving GSVI (16) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the proposed algorithm by virtue of the following inequality in a 2-uniformly smooth Banach space X.

Lemma 2 (see [36]). *Let X be a 2-uniformly smooth Banach space. Then*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x) \rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X, \quad (23)$$

where κ is the 2-uniformly smooth constant of X and J is the normalized duality mapping from X into X^*.

Define the mapping $G : C \to C$ as follows:

$$G(x) := \Pi_C (I - \mu_1 B_1) \Pi_C (I - \mu_2 B_2) x, \quad \forall x \in C. \quad (24)$$

The fixed point set of G is denoted by Ω. Then their strong convergence theorem on the proposed method is stated as follows.

Theorem CB (see [13, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse strongly accretive with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Let f be a contraction of C into itself with coefficient $\rho \in (0, 1)$. Let $\{S_n\}_{n=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself such that $\Delta = \bigcap_{i=1}^\infty \text{Fix}(S_i) \cap \Omega \ne \emptyset$, where Ω is the fixed point set of the mapping G defined by (24). For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n y_n,$$

$$y_n = \alpha_n f(x_n) + (1 - \alpha_n) z_n,$$

$$z_n = \Pi_C (u_n - \mu_1 B_1 u_n), \quad (25)$$

$$u_n = \Pi_C (x_n - \mu_2 B_2 x_n),$$

$$\forall n \ge 1.$$

Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$.

Assume that $\sum_{n=1}^\infty \sup_{x \in D} \|S_{n+1} x - S_n x\| < \infty$ for any bounded subset D of C, let S be a mapping of C into X defined by $Sx = \lim_{n \to \infty} S_n x$ for all $x \in C$, and suppose that $\text{Fix}(S) = \bigcap_{n=1}^\infty \text{Fix}(S_n)$. Then $\{x_n\}$ converges strongly to $p \in \Delta$, which solves the VIP:

$$\langle (I - f) p, J(p - u) \rangle \le 0, \quad \forall u \in \Delta. \quad (26)$$

It is easy to see that the iterative scheme in Theorem CB is essentially equivalent to the following two-step iterative scheme:

$$y_n = \alpha_n f(x_n) + (1 - \alpha_n) Gx_n,$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S_n y_n, \quad (27)$$

$$\forall n \ge 1.$$

For the convenience of implementing the argument techniques in [16], the authors [13] have used the following inequality in a real smooth and uniform convex Banach space X.

Proposition 3 (see [37]). *Let X be a real smooth and uniform convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$, such that*

$$g(\|x - y\|) \le \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in B_r, \quad (28)$$

where $B_r = \{x \in X : \|x\| \le r\}$.

Let C be a nonempty closed convex subset of a real uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Motivated and inspired by the research going on in this area, we introduce and analyze modified hybrid steepest-descent methods for the GSVI (16), with solutions being also zeros of an m-accretive operator A in X. Here the modified hybrid steepest-descent methods are based on Korpelevich's extragradient method, hybrid steepest-descent method, and viscosity approximation method. We propose and consider modified implicit and explicit hybrid steepest-descent algorithms for finding a common element of the solution set of the GSVI (16) and the set $A^{-1}(0)$ of zeros of A in X. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literature [12, 13, 16, 32].

2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 4 can be found in [38]. Lemma 5 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 4. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n) a_n + b_n c_n, \quad \forall n \ge 0, \quad (29)$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset [0, 1]$ and $\sum_{n=0}^\infty b_n = \infty$,

(ii) *either* $\limsup_{n \to \infty} c_n \le 0$ *or* $\sum_{n=0}^\infty |b_n c_n| < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$.

Lemma 5. *In a real smooth Banach space X, there holds the inequality*

$$\|x\|^2 + 2\langle y, J(x) \rangle \le \|x + y\|^2$$

$$\le \|x\|^2 + 2\langle y, J(x + y) \rangle, \quad \forall x, y \in X, \quad (30)$$

where $J : X \to X^*$ is the normalized duality mapping.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi\left[\Pi(x) + t(x - \Pi(x))\right] = \Pi(x), \qquad (31)$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 6 (see [39]). *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then the following are equivalent:*

(i) *Π is sunny and nonexpansive,*

(ii) *$\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y)) \rangle, \ \forall x, y \in C$,*

(iii) *$\langle x - \Pi(x), J(y - \Pi(x)) \rangle \leq 0, \ \forall x \in C, \ y \in D$.*

It is well known that if $X = H$, a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C. The following lemma follows easily from Lemma 6.

Lemma 7. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let $B_1, B_2 : C \to X$ be nonlinear mappings. For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of GSVI (16) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

In terms of Lemma 7, we observe that

$$x^* = \Pi_C\left[\Pi_C\left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 \Pi_C\left(x^* - \mu_2 B_2 x^*\right)\right], \qquad (32)$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

Lemma 8 is the resolvent identity which can be found in [40], and Lemma 9 can be found in [41].

Lemma 8. *For $\lambda, \mu > 0$, there holds the identity*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right), \quad \forall x \in X. \qquad (33)$$

Lemma 9. *Assume that $c_2 \geq c_1 > 0$. Then $\|J_{c_1} x - x\| \leq 2\|J_{c_2} x - x\|$ for all $x \in X$.*

Lemma 10 (see [42]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings on C. Suppose that*

$\bigcap_{n=0}^\infty \mathrm{Fix}(T_n)$ *is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^\infty \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^\infty \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive, and $\mathrm{Fix}(S) = \bigcap_{n=0}^\infty \mathrm{Fix}(T_n)$ holds.*

Let μ be a mean if μ is a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbf{N}$ if and only if

$$\inf\{a_n : n \in \mathbf{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbf{N}\} \qquad (34)$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbf{N}$ is called a Banach limit if and only if

$$\mu_n(a_n) = \mu_n(a_{n+1}) \qquad (35)$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. We know that if μ is a Banach limit, then

$$\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n \qquad (36)$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. So if $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots) \in l^\infty$, and $a_n \to c$ (resp., $a_n - b_n \to 0$), as $n \to \infty$, we have

$$\mu_n(a_n) = \mu(a) = c \quad (\text{resp.}, \mu_n(a_n) = \mu_n(b_n)). \qquad (37)$$

Further, it is well known that there holds the following result.

Lemma 11 (see [43]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $\{x_n\}$ be a bounded sequence of X, let μ be a mean on $\mathbf{N}$, and let $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2 \qquad (38)$$

if and only if

$$\mu_n \langle y - z, J(x_n - z) \rangle \leq 0, \quad \forall y \in C, \qquad (39)$$

where J is the normalized duality mapping of X.

Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let Ξ_C be the set of all contractive self-mappings on C. For $t \in (0, 1)$ and $f \in \Xi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1 - t)Tx$ on C; that is,

$$x_t = tf(x_t) + (1 - t)Tx_t. \qquad (40)$$

Lemma 12 (see [34]). *Let X be a uniformly smooth Banach space. Let C be a nonempty closed convex subset of X, let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and let $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If one defines a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t \to 0} x_t$, for all $f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\langle (I - f)Q(f), J(Q(f) - u) \rangle \leq 0,$$
$$\forall f \in \Xi_C, \ u \in \mathrm{Fix}(T). \qquad (41)$$

The following proposition will be used frequently throughout this paper.

Proposition 13 (see [11]). *Let X be a real smooth Banach space and let $F : X \rightarrow X$ be a mapping.*

(i) *If F is λ-strictly pseudocontractive, then F is Lipschitzian with constant $1 + 1/\lambda$.*

(ii) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.*

(iii) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, then, for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda})$.*

3. Main Results

In this section, we introduce our modified hybrid steepest-descent schemes and show the strong convergence theorems. We will need the following useful lemmas in the sequel.

Lemma 14 (see [13, Lemma 2.8]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \rightarrow X$ be α_i-inverse strongly accretive. Then, one has*

$$\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\|^2$$
$$\leq \|x - y\|^2 + 2\mu_i \left(\mu_i \kappa^2 - \alpha_i\right) \|B_i x - B_i y\|^2, \quad \forall x, y \in C, \tag{42}$$

for $i = 1, 2$, where $\mu_i > 0$. In particular, if $0 < \mu_i \leq \alpha_i/\kappa^2$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 15 (see [13, Lemma 2.9]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \rightarrow X$ be α_i-inverse strongly accretive for $i = 1, 2$. Let $G : C \rightarrow C$ be the mapping defined by*

$$Gx = \Pi_C \left[\Pi_C \left(x - \mu_2 B_2 x\right) - \mu_1 B_1 \Pi_C \left(x - \mu_2 B_2 x\right)\right], \tag{43}$$
$$\forall x \in C.$$

If $0 < \mu_i \leq \alpha_i/\kappa^2$ for $i = 1, 2$, then $G : C \rightarrow C$ is nonexpansive.

Let X be a real smooth Banach space and let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $F : X \rightarrow X$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Let the mapping $B_i : C \rightarrow X$ be α_i-inverse strongly accretive for $i = 1, 2$ and let $f : X \rightarrow X$ be a contractive map with coefficient $\rho \in (0, 1)$. In this section, we will consider the problem of finding a point $p \in \Delta = A^{-1}(0) \cap \Omega(\neq \emptyset)$, which is a unique solution of the VIP:

$$\langle (I - f)p, J(p - u) \rangle \leq 0, \quad \forall u \in \Delta, \tag{44}$$

where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i \leq \alpha_i/\kappa^2$ for $i = 1, 2$. For each $t \in (0, 1)$ and each integer $n \geq 0$, we choose a number $\theta_t \in [0, 1)$ arbitrarily and then consider the following mapping $\Gamma_{t,n} : X \rightarrow X$ defined as

$$\Gamma_{t,n}x = tf(x) + (1 - t)\left[G\left(J_{r_n}x\right) - \theta_t FG\left(J_{r_n}x\right)\right], \quad \forall x \in X. \tag{45}$$

Then, $\Gamma_{t,n} : X \rightarrow X$ is a contractive map. Indeed, utilizing Proposition 13(iii) and Lemma 15, we have for all $x, y \in X$

$$\left\|\Gamma_{t,n}x - \Gamma_{t,n}y\right\|$$
$$= \left\|tf(x) + (1 - t)\left[G\left(J_{r_n}x\right) - \theta_t FG\left(J_{r_n}x\right)\right]\right.$$
$$\left. - tf(y) - (1 - t)\left[G\left(J_{r_n}y\right) - \theta_t FG\left(J_{r_n}y\right)\right]\right\|$$
$$\leq t\left\|f(x) - f(y)\right\|$$
$$\quad + (1 - t)\left\|(I - \theta_t F)G\left(J_{r_n}x\right) - (I - \theta_t F)G\left(J_{r_n}y\right)\right\|$$
$$\leq t\rho\|x - y\| + (1 - t)\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \delta}{\lambda}}\right)\right)$$
$$\quad \times \left\|G\left(J_{r_n}x\right) - G\left(J_{r_n}y\right)\right\|$$
$$\leq t\rho\|x - y\| + (1 - t)\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \delta}{\lambda}}\right)\right)$$
$$\quad \times \left\|J_{r_n}x - J_{r_n}y\right\|$$
$$\leq t\rho\|x - y\| + (1 - t)\left(1 - \theta_t\left(1 - \sqrt{\frac{1 - \delta}{\lambda}}\right)\right)\|x - y\|$$
$$\leq t\rho\|x - y\| + (1 - t)\|x - y\|$$
$$= (1 - (1 - \rho)t)\|x - y\|, \tag{46}$$

and hence $\Gamma_{t,n} : X \rightarrow X$ is contractive due to $(1 - \rho)t \in (0, 1)$. By Banach's Contraction Mapping Principle, there exists a unique fixed point $x_{t,n}$ of $\Gamma_{t,n}$ in X; that is,

$$x_{t,n} = tf(x_{t,n}) + (1 - t)\left[G\left(J_{r_n}x_{t,n}\right) - \theta_t FG\left(J_{r_n}x_{t,n}\right)\right]. \tag{47}$$

Theorem 16. *Let X be a uniformly convex and 2-uniformly smooth Banach space and let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \rightarrow X$ be α_i-inverse strongly accretive for $i = 1, 2$, let $f : X \rightarrow C$ be a contractive map with coefficient $\rho \in (0, 1)$, and let $F : X \rightarrow X$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Assume that $\Delta = A^{-1}(0) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. For each $t \in (0, 1)$*

and each integer $n \geq 0$, let $\{x_{t,n}\}$ be defined by (47), where $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$ and $\{\theta_t : t \in (0,1)\} \subset [0,1)$ with $\lim_{t \to 0}(\theta_t/t) = 0$. Then as $t \to 0$, $\{x_{t,n}\}$ converges strongly to $p \in \Delta$, which is a unique solution of the VIP (44).

Proof. First let us show that for some $a \in (0,1)$, $\{x_{t,n} : t \in (0,a], n \geq 0\}$ is bounded. Indeed, since $\{\theta_t : t \in (0,1)\} \subset [0,1)$ with $\lim_{t \to 0}(\theta_t/t) = 0$, there exists some $a \in (0,1)$ such that $0 \leq \theta_t/t < 1$ for all $t \in (0,a]$.

Take $p \in \Delta$. Then utilizing Proposition 13, we have

$$
\begin{aligned}
&\|x_{t,n} - p\| \\
&\leq t\|f(x_{t,n}) - p\| + (1-t) \\
&\quad \times \left\|[G(J_{r_n}x_{t,n}) - \theta_t FG(J_{r_n}x_{t,n})] - p\right\| \\
&\leq t\rho\|x_{t,n} - p\| + t\|f(p) - p\| + (1-t) \\
&\quad \times \left[\left\|(I - \theta_t F)G(J_{r_n}x_{t,n}) - (I - \theta_t F)G(J_{r_n}p)\right\| \right. \\
&\quad\quad \left. + \left\|(I - \theta_t F)G(J_{r_n}p) - p\right\|\right] \\
&\leq t\rho\|x_{t,n} - p\| + t\|f(p) - p\| + (1-t) \\
&\quad \times \left(1 - \theta_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\left\|G(J_{r_n}x_{t,n}) - G(J_{r_n}p)\right\| \\
&\quad + (1-t)\theta_t\|F(p)\| \\
&\leq t\rho\|x_{t,n} - p\| + t\|f(p) - p\| + (1-t) \\
&\quad \times \left(1 - \theta_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\left\|J_{r_n}x_{t,n} - J_{r_n}p\right\| \\
&\quad + \theta_t\|F(p)\| \\
&\leq t\rho\|x_{t,n} - p\| + t\|f(p) - p\| + (1-t) \\
&\quad \times \left(1 - \theta_t\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_{t,n} - p\| + \theta_t\|F(p)\| \\
&\leq t\rho\|x_{t,n} - p\| + t\|f(p) - p\| + (1-t)\|x_{t,n} - p\| \\
&\quad + \theta_t\|F(p)\|,
\end{aligned}
\tag{48}
$$

and, hence, for all $t \in (0,a]$

$$
\begin{aligned}
\|x_{t,n} - p\| &\leq \frac{1}{1-\rho}\left(\|f(p) - p\| + \frac{\theta_t}{t}\|F(p)\|\right) \\
&\leq \frac{1}{1-\rho}(\|f(p) - p\| + \|F(p)\|).
\end{aligned}
\tag{49}
$$

Thus, this implies that $\{x_{t,n} : t \in (0,a], n \geq 0\}$ is bounded and so are $\{f(x_{t,n}) : t \in (0,a], n \geq 0\}$, $\{J_{r_n}x_{t,n} : t \in (0,a], n \geq 0\}$, and $\{G(J_{r_n}x_{t,n}) : t \in (0,a], n \geq 0\}$.

Let us show that $\|J_{r_n}x_{t,n} - G(J_{r_n}x_{t,n})\| \to 0$ as $t \to 0$.

Indeed, for simplicity we put $q = \Pi_C(I - \mu_2 B_2)p$, $\widehat{x}_{t,n} = J_{r_n}x_{t,n}$, $u_{t,n} = \Pi_C(I - \mu_2 B_2)\widehat{x}_{t,n}$, and $v_{t,n} = \Pi_C(I - \mu_1 B_1)u_{t,n}$.

Then it is clear that $p = \Pi_C(I - \mu_1 B_1)q$ and $v_{t,n} = G(\widehat{x}_{t,n}) = G(J_{r_n}x_{t,n})$. Taking into account $x_{t,n} - p = t(f(x_{t,n}) - p) + (1-t)[G(J_{r_n}x_{t,n}) - \theta_t FG(J_{r_n}x_{t,n}) - p]$, we get

$$
\begin{aligned}
&\|x_{t,n} - p\|^2 \\
&= t\left\langle f(x_{t,n}) - p, J(x_{t,n} - p)\right\rangle \\
&\quad + (1-t)\left\langle G(J_{r_n}x_{t,n}) - p, J(x_{t,n} - p)\right\rangle \\
&\quad - (1-t)\theta_t\left\langle FG(J_{r_n}x_{t,n}), J(x_{t,n} - p)\right\rangle \\
&\leq t\left\langle f(x_{t,n}) - f(p), J(x_{t,n} - p)\right\rangle \\
&\quad + t\left\langle f(p) - p, J(x_{t,n} - p)\right\rangle \\
&\quad + (1-t)\left\|G(J_{r_n}x_{t,n}) - p\right\|\left\|J(x_{t,n} - p)\right\| \\
&\quad + (1-t)\theta_t\left\|FG(J_{r_n}x_{t,n})\right\|\left\|J(x_{t,n} - p)\right\| \\
&\leq \rho t\|x_{t,n} - p\|^2 + t\|f(p) - p\|\|x_{t,n} - p\| \\
&\quad + (1-t)\|v_{t,n} - p\|\|x_{t,n} - p\| \\
&\quad + \theta_t\|Fv_{t,n}\|\|x_{t,n} - p\|.
\end{aligned}
\tag{50}
$$

From Lemma 14 we have

$$
\begin{aligned}
\|u_{t,n} - q\|^2 &= \left\|\Pi_C(\widehat{x}_{t,n} - \mu_2 B_2\widehat{x}_{t,n}) - \Pi_C(p - \mu_2 B_2 p)\right\|^2 \\
&\leq \left\|\widehat{x}_{t,n} - p - \mu_2(B_2\widehat{x}_{t,n} - B_2 p)\right\|^2 \\
&\leq \left\|\widehat{x}_{t,n} - p\right\|^2 - 2\mu_2\left(\alpha_2 - \kappa^2\mu_2\right)\left\|B_2\widehat{x}_{t,n} - B_2 p\right\|^2, \\
\|v_{t,n} - p\|^2 &= \left\|\Pi_C(u_{t,n} - \mu_1 B_1 u_{t,n}) - \Pi_C(q - \mu_1 B_1 q)\right\|^2 \\
&\leq \left\|u_{t,n} - q - \mu_1(B_1 u_{t,n} - B_1 q)\right\|^2 \\
&\leq \left\|u_{t,n} - q\right\|^2 - 2\mu_1\left(\alpha_1 - \kappa^2\mu_1\right)\left\|B_1 u_{t,n} - B_1 q\right\|^2.
\end{aligned}
\tag{51}
$$

From the last two inequalities, we obtain

$$
\begin{aligned}
\|v_{t,n} - p\|^2 &\leq \left\|\widehat{x}_{t,n} - p\right\|^2 - 2\mu_2\left(\alpha_2 - \kappa^2\mu_2\right)\left\|B_2\widehat{x}_{t,n} - B_2 p\right\|^2 \\
&\quad - 2\mu_1\left(\alpha_1 - \kappa^2\mu_1\right)\left\|B_1 u_{t,n} - B_1 q\right\|^2 \\
&\leq \|x_{t,n} - p\|^2 - 2\mu_2\left(\alpha_2 - \kappa^2\mu_2\right)\left\|B_2\widehat{x}_{t,n} - B_2 p\right\|^2 \\
&\quad - 2\mu_1\left(\alpha_1 - \kappa^2\mu_1\right)\left\|B_1 u_{t,n} - B_1 q\right\|^2,
\end{aligned}
\tag{52}
$$

which, together with (50), implies that

$$
\begin{aligned}
&\|x_{t,n} - p\|^2 \\
&\leq \rho t\|x_{t,n} - p\|^2 + t\|f(p) - p\|\|x_{t,n} - p\| \\
&\quad + (1-t)\|v_{t,n} - p\|\|x_{t,n} - p\| + \theta_t\|Fv_{t,n}\|\|x_{t,n} - p\|
\end{aligned}
$$

$$\leq \rho t \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\|$$
$$+ (1-t)\frac{1}{2}\left(\|x_{t,n} - p\|^2 + \|v_{t,n} - p\|^2\right)$$
$$+ \theta_t \|F v_{t,n}\| \|x_{t,n} - p\|$$
$$\leq \rho t \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\|$$
$$+ (1-t)\frac{1}{2}\left\{\|x_{t,n} - p\|^2 + \|x_{t,n} - p\|^2\right.$$
$$- 2\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_{t,n} - B_2 p\|^2$$
$$\left. - 2\mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right\}$$
$$+ \theta_t \|F v_{t,n}\| \|x_{t,n} - p\|$$
$$= (1 - t(1-\rho)) \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\|$$
$$- (1-t)\left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_{t,n} - B_2 p\|^2\right.$$
$$\left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right]$$
$$+ \theta_t \|F v_{t,n}\| \|x_{t,n} - p\|$$
$$\leq \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\|$$
$$- (1-t)\left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_{t,n} - B_2 p\|^2\right.$$
$$\left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right]$$
$$+ \theta_t \|F v_{t,n}\| \|x_{t,n} - p\|.$$
$$\tag{53}$$

So it immediately follows that

$$(1-t)\left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 \widehat{x}_{t,n} - B_2 p\|^2\right.$$
$$\left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right] \tag{54}$$
$$\leq t \|f(p) - p\| \|x_{t,n} - p\| + \theta_t \|F v_{t,n}\| \|x_{t,n} - p\|.$$

Since $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$, we have

$$\lim_{t \to 0} \|B_2 \widehat{x}_{t,n} - B_2 p\| = 0,$$
$$\lim_{t \to 0} \|B_1 u_{t,n} - B_1 q\| = 0. \tag{55}$$

Utilizing Proposition 3 and Lemma 6, we have

$$\|u_{t,n} - q\|^2 = \|\Pi_C(\widehat{x}_{t,n} - \mu_2 B_2 \widehat{x}_{t,n}) - \Pi_C(p - \mu_2 B_2 p)\|^2$$
$$\leq \langle \widehat{x}_{t,n} - \mu_2 B_2 \widehat{x}_{t,n} - (p - \mu_2 B_2 p), J(u_{t,n} - q)\rangle$$
$$= \langle \widehat{x}_{t,n} - p, J(u_{t,n} - q)\rangle$$
$$+ \mu_2 \langle B_2 p - B_2 \widehat{x}_{t,n}, J(u_{t,n} - q)\rangle$$

$$\leq \frac{1}{2}\left[\|\widehat{x}_{t,n} - p\|^2 + \|u_{t,n} - q\|^2\right.$$
$$\left. - g_1 \left(\|\widehat{x}_{t,n} - u_{t,n} - (p - q)\|\right)\right]$$
$$+ \mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\|,$$
$$\tag{56}$$

which implies that

$$\|u_{t,n} - q\|^2 \leq \|\widehat{x}_{t,n} - p\|^2 - g_1 \left(\|\widehat{x}_{t,n} - u_{t,n} - (p - q)\|\right)$$
$$+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\|. \tag{57}$$

In the same way, we derive

$$\|v_{t,n} - p\|^2$$
$$= \|\Pi_C(u_{t,n} - \mu_1 B_1 u_{t,n}) - \Pi_C(q - \mu_1 B_1 q)\|^2$$
$$\leq \langle u_{t,n} - \mu_1 B_1 u_{t,n} - (q - \mu_1 B_1 q), J(v_{t,n} - p)\rangle$$
$$= \langle u_{t,n} - q, J(v_{t,n} - p)\rangle + \mu_1 \langle B_1 q - B_1 u_{t,n}, J(v_{t,n} - p)\rangle$$
$$\leq \frac{1}{2}\left[\|u_{t,n} - q\|^2 + \|v_{t,n} - p\|^2\right.$$
$$\left. - g_2 \left(\|u_{t,n} - v_{t,n} + (p - q)\|\right)\right]$$
$$+ \mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|,$$
$$\tag{58}$$

which implies that

$$\|v_{t,n} - p\|^2 \leq \|u_{t,n} - q\|^2 - g_2 \left(\|u_{t,n} - v_{t,n} + (p - q)\|\right)$$
$$+ 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|. \tag{59}$$

Substituting (57) for (59), we get

$$\|v_{t,n} - p\|^2 \leq \|\widehat{x}_{t,n} - p\|^2 - g_1 \left(\|\widehat{x}_{t,n} - u_{t,n} - (p - q)\|\right)$$
$$- g_2 \left(\|u_{t,n} - v_{t,n} + (p - q)\|\right)$$
$$+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\|$$
$$+ 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|$$
$$\leq \|x_{t,n} - p\|^2 - g_1 \left(\|\widehat{x}_{t,n} - u_{t,n} - (p - q)\|\right)$$
$$- g_2 \left(\|u_{t,n} - v_{t,n} + (p - q)\|\right)$$
$$+ 2\mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\|$$
$$+ 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|,$$
$$\tag{60}$$

which, together with (50), implies that

$$
\begin{aligned}
\|x_{t,n} &- p\|^2 \\
&\leq \rho t \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\| \\
&\quad + (1-t)\frac{1}{2}\left(\|x_{t,n} - p\|^2 + \|v_{t,n} - p\|^2\right) \\
&\quad + \theta_t \|Fv_{t,n}\| \|x_{t,n} - p\| \\
&\leq \rho t \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\| \\
&\quad + (1-t)\frac{1}{2}\Big\{\|x_{t,n} - p\|^2 + \|x_{t,n} - p\|^2 \\
&\qquad\qquad - g_1\left(\|\widehat{x}_{t,n} - u_{t,n} - (p-q)\|\right) \\
&\qquad\qquad - g_2\left(\|u_{t,n} - v_{t,n} + (p-q)\|\right) \\
&\qquad\qquad + 2\mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\| \\
&\qquad\qquad + 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|\Big\} \\
&\quad + \theta_t \|Fv_{t,n}\| \|x_{t,n} - p\| \\
&= (1 - t(1-\rho)) \|x_{t,n} - p\|^2 \\
&\quad + t \|f(p) - p\| \|x_{t,n} - p\| \\
&\quad - (1-t)\frac{1}{2}\big[g_1\left(\|\widehat{x}_{t,n} - u_{t,n} - (p-q)\|\right) \\
&\qquad\qquad + g_2\left(\|u_{t,n} - v_{t,n} + (p-q)\|\right)\big] \\
&\quad + (1-t)\big[\mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\| \\
&\qquad\qquad + \mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|\big] \\
&\quad + \theta_t \|Fv_{t,n}\| \|x_{t,n} - p\| \\
&\leq \|x_{t,n} - p\|^2 + t \|f(p) - p\| \|x_{t,n} - p\| \\
&\quad - (1-t)\frac{1}{2}\big[g_1\left(\|\widehat{x}_{t,n} - u_{t,n} - (p-q)\|\right) \\
&\qquad\qquad + g_2\left(\|u_{t,n} - v_{t,n} + (p-q)\|\right)\big] \\
&\quad + \mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\| \\
&\quad + \mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\| \\
&\quad + \theta_t \|Fv_{t,n}\| \|x_{t,n} - p\|.
\end{aligned}
\tag{61}
$$

So it immediately follows that

$$
\begin{aligned}
(1-t)\frac{1}{2}&\big[g_1\left(\|\widehat{x}_{t,n} - u_{t,n} - (p-q)\|\right) \\
&\quad + g_2\left(\|u_{t,n} - v_{t,n} + (p-q)\|\right)\big] \\
&\leq t \|f(p) - p\| \|x_{t,n} - p\| \\
&\quad + \mu_2 \|B_2 p - B_2 \widehat{x}_{t,n}\| \|u_{t,n} - q\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\| \\
&\quad + \theta_t \|Fv_{t,n}\| \|x_{t,n} - p\|.
\end{aligned}
\tag{62}
$$

Hence, from (55) we conclude that

$$
\begin{aligned}
\lim_{t\to 0} g_1\left(\|\widehat{x}_{t,n} - u_{t,n} - (p-q)\|\right) &= 0, \\
\lim_{t\to 0} g_2\left(\|u_{t,n} - v_{t,n} + (p-q)\|\right) &= 0.
\end{aligned}
\tag{63}
$$

Utilizing the properties of g_1 and g_2, we get

$$
\begin{aligned}
\lim_{t\to 0} \|\widehat{x}_{t,n} - u_{t,n} - (p-q)\| &= 0, \\
\lim_{t\to 0} \|u_{t,n} - v_{t,n} + (p-q)\| &= 0,
\end{aligned}
\tag{64}
$$

which leads to

$$
\begin{aligned}
\|\widehat{x}_{t,n} &- v_{t,n}\| \\
&\leq \|\widehat{x}_{t,n} - u_{t,n} - (p-q)\| \\
&\quad + \|u_{t,n} - v_{t,n} + (p-q)\| \longrightarrow 0 \quad \text{as } t \longrightarrow 0.
\end{aligned}
\tag{65}
$$

That is,

$$
\lim_{t\to 0} \left\|J_{r_n} x_{t,n} - G\left(J_{r_n} x_{t,n}\right)\right\| = \lim_{t\to 0} \|\widehat{x}_{t,n} - v_{t,n}\| = 0.
\tag{66}
$$

Note that $\text{Fix}(J_{r_n}) = A^{-1}(0)$ for all $n \geq 0$ and that $\{x_{t,n} : t \in (0,a],\ n \geq 0\}$ is bounded and so are $\{f(x_{t,n}) : t \in (0,a],\ n \geq 0\}$, $\{J_{r_n} x_{t,n} : t \in (0,a],\ n \geq 0\}$, and $\{G(J_{r_n} x_{t,n}) : t \in (0,a],\ n \geq 0\}$. Hence, we have

$$
\begin{aligned}
\Big\|x_{t,n} &- G\left(J_{r_n} x_{t,n}\right)\Big\| \\
&= t\left\|f(x_{t,n}) - G\left(J_{r_n} x_{t,n}\right) - (1-t)\frac{\theta_t}{t} FG\left(J_{r_n} x_{t,n}\right)\right\| \longrightarrow 0,
\end{aligned}
\tag{67}
$$

as $t \to 0$. Also, observe that

$$
\left\|x_{t,n} - J_{r_n} x_{t,n}\right\| \leq \left\|x_{t,n} - G\left(J_{r_n} x_{t,n}\right)\right\| + \left\|G\left(J_{r_n} x_{t,n}\right) - J_{r_n} x_{t,n}\right\|.
\tag{68}
$$

This, together with (66) and (67), implies that

$$
\lim_{t\to 0} \left\|x_{t,n} - J_{r_n} x_{t,n}\right\| = 0.
\tag{69}
$$

Utilizing the nonexpansivity of G, we obtain from (67) and (69) that

$$
\begin{aligned}
\left\|x_{t,n} - Gx_{t,n}\right\| &\leq \left\|x_{t,n} - G\left(J_{r_n} x_{t,n}\right)\right\| \\
&\quad + \left\|G\left(J_{r_n} x_{t,n}\right) - Gx_{t,n}\right\| \\
&\leq \left\|x_{t,n} - G\left(J_{r_n} x_{t,n}\right)\right\| \\
&\quad + \left\|J_{r_n} x_{t,n} - x_{t,n}\right\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0.
\end{aligned}
\tag{70}
$$

Since $r_n \geq \varepsilon$ for all n, utilizing Lemma 9 we have

$$\|x_{t,n} - J_\varepsilon x_{t,n}\| \leq 2 \|x_{t,n} - J_{r_n} x_{t,n}\| \longrightarrow 0, \quad \text{as } t \longrightarrow 0. \quad (71)$$

For any integer $n \geq 0$, for simplicity put $w_t = x_{t,n}$ for all $t \in (0, a]$. Now let $\{t_k\}$ be a sequence in $(0, a]$ that converges to 0 as $k \to \infty$ and define a function g on C by

$$g(w) = \mu_k \frac{1}{2} \|w_{t_k} - w\|^2, \quad \forall w \in C, \quad (72)$$

where μ is a Banach limit. Define the set

$$K := \{w \in C : g(w) = \min\{g(y) : y \in C\}\} \quad (73)$$

and the mapping

$$Wx = (1 - \theta) J_\varepsilon x + \theta G x, \quad \forall x \in C, \quad (74)$$

where θ is a constant in $(0, 1)$. Then by Lemma 10, we know that $\text{Fix}(W) = \text{Fix}(J_\varepsilon) \cap \text{Fix}(G) = \Delta$. We observe that

$$\|w_t - Ww_t\| = \|(1 - \theta)(w_t - J_\varepsilon w_t) + \theta(w_t - Gw_t)\|$$
$$\leq (1 - \theta) \|w_t - J_\varepsilon w_t\| + \theta \|w_t - Gw_t\|. \quad (75)$$

So from (70) and (71) we obtain

$$\lim_{n \to \infty} \|w_t - Ww_t\| = 0. \quad (76)$$

Since X is a uniformly smooth Banach space, K is a nonempty bounded closed convex subset of C; for more details, see [43]. We claim that K is also invariant under the nonexpansive mapping W. Indeed, noticing (76), we have for $w \in K$

$$g(Ww) = \mu_k \frac{1}{2} \|w_{t_k} - Ww\|^2 = \mu_k \frac{1}{2} \|Ww_{t_k} - Ww\|^2$$
$$\leq \mu_k \frac{1}{2} \|w_{t_k} - w\|^2 = g(w). \quad (77)$$

Since every nonempty closed bounded convex subset of uniformly smooth Banach space X has the fixed point property for nonexpansive mappings and W is a nonexpansive mapping of K, W has a fixed point in K, say p. Utilizing Lemma 11, we get

$$\mu_k \langle x - p, J(w_{t_k} - p) \rangle \leq 0, \quad \forall x \in C. \quad (78)$$

Putting $x = f(p)$ we have

$$\mu_k \langle f(p) - p, J(w_{t_k} - p) \rangle \leq 0, \quad \forall x \in C. \quad (79)$$

Since $w_{t_k} - p = t_k(f(w_{t_k}) - p) + (1 - t_k)[G(J_{r_n} w_{t_k}) - \theta_{t_k} FG(J_{r_n} w_{t_k}) - p]$, we get

$$\|w_{t_k} - p\|^2$$
$$= t_k \langle f(w_{t_k}) - p, J(w_{t_k} - p) \rangle$$
$$\quad + (1 - t_k) \langle G(J_{r_n} w_{t_k}) - p, J(w_{t_k} - p) \rangle$$
$$\quad - (1 - t_k)\theta_{t_k} \langle FG(J_{r_n} w_{t_k}), J(w_{t_k} - p) \rangle$$
$$\leq t_k \langle f(w_{t_k}) - f(p), J(w_{t_k} - p) \rangle$$
$$\quad + t_k \langle f(p) - p, J(w_{t_k} - p) \rangle \quad (80)$$
$$\quad + (1 - t_k) \|G(J_{r_n} w_{t_k}) - p\| \|J(w_{t_k} - p)\|$$
$$\quad + (1 - t_k)\theta_{t_k} \|FG(J_{r_n} w_{t_k})\| \|J(w_{t_k} - p)\|$$
$$\leq \rho t_k \|w_{t_k} - p\|^2 + t_k \langle f(p) - p, J(w_{t_k} - p) \rangle$$
$$\quad + (1 - t_k) \|w_{t_k} - p\|^2$$
$$\quad + \theta_{t_k} \|FG(J_{r_n} w_{t_k})\| \|w_{t_k} - p\|.$$

It follows that

$$\|w_{t_k} - p\|^2 \leq \frac{1}{1 - \rho} \left[\langle f(p) - p, J(w_{t_k} - p) \rangle \right.$$
$$\left. + \frac{\theta_{t_k}}{t_k} \|FG(J_{r_n} w_{t_k})\| \|w_{t_k} - p\| \right]. \quad (81)$$

Since $\lim_{k \to \infty}(\theta_{t_k}/t_k) = 0$, from (79) and the boundedness of sequences $\{FG(J_{r_n} w_{t_k})\}$, $\{w_{t_k}\}$, it follows that

$$\mu_k \|w_{t_k} - p\|^2$$
$$\leq \frac{1}{1 - \rho} \mu_k \left[\langle f(p) - p, J(w_{t_k} - p) \rangle \right.$$
$$\left. + \frac{\theta_{t_k}}{t_k} \|FG(J_{r_n} w_{t_k})\| \|w_{t_k} - p\| \right]$$
$$= \frac{1}{1 - \rho} \left[\mu_k \langle f(p) - p, J(w_{t_k} - p) \rangle \right.$$
$$\left. + \mu_k \left(\frac{\theta_{t_k}}{t_k} \|FG(J_{r_n} w_{t_k})\| \|w_{t_k} - p\| \right) \right] \leq 0. \quad (82)$$

Therefore, it is known that for any sequence $\{w_{t_k}\}$ in $\{w_t : t \in (0, a]\}$ there exists a subsequence which is still denoted by $\{w_{t_k}\}$ that converges strongly to some fixed point p of W. To prove that the net $\{w_t : t \in (0, a]\}$ converges strongly to p as $t \to 0$, suppose that there exists another subsequence $\{w_{s_k}\} \subset \{w_t\}$ such that $w_{s_k} \to q$ as $s_k \to 0$, and then we also have $q \in \text{Fix}(W) = A^{-1}(0) \cap \Omega =: \Delta$ due to (76).

Since the sets $\{w_t - u : t \in (0, a]\}$ and $\{w_t - f(w_t) : t \in (0, a]\}$ are bounded and the duality map J is single-valued and norm-to-norm uniformly continuous on bounded sets of uniformly smooth Banach space X, for any $u \in \Delta = A^{-1}(0) \cap \Omega$, from $w_{s_k} \to q$ $(s_k \to 0)$ we obtain

$$\left\| (I - f) w_{s_k} - (I - f) q \right\| \longrightarrow 0 \quad (s_k \to 0),$$

$$\left| \left\langle w_{s_k} - f(w_{s_k}), J(w_{s_k} - u) \right\rangle - \left\langle (I - f) q, J(q - u) \right\rangle \right|$$

$$= \left| \left\langle (I - f) w_{s_k} - (I - f) q, J(w_{s_k} - u) \right\rangle \right.$$

$$\left. + \left\langle (I - f) q, J(w_{s_k} - u) - J(q - u) \right\rangle \right|$$

$$\leq \left\| (I - f) w_{s_k} - (I - f) q \right\| \left\| w_{s_k} - u \right\|$$

$$+ \left| \left\langle (I - f) q, J(w_{s_k} - u) - J(q - u) \right\rangle \right| \longrightarrow 0,$$

$$(83)$$

as $s_k \to 0$. Therefore,

$$\left\langle (I - f) q, J(q - u) \right\rangle = \lim_{s_k \to 0} \left\langle w_{s_k} - f(w_{s_k}), J(w_{s_k} - u) \right\rangle. \tag{84}$$

Since $u \in \Delta = A^{-1}(0) \cap \Omega$,

$$\left\| x_{t,n} - [(1-t)u + tf(x_{t,n})] \right\|^2$$

$$= \left\| (1-t) G(J_{r_n} x_{t,n}) + tf(x_{t,n}) - (1-t) u - tf(x_{t,n}) \right.$$

$$\left. - (1-t)\theta_t FG(J_{r_n} x_{t,n}) \right\|^2$$

$$\leq \left[(1-t) \left\| G(J_{r_n} x_{t,n}) - u \right\| + (1-t) \theta_t \left\| FG(J_{r_n} x_{t,n}) \right\| \right]^2$$

$$\leq \left[(1-t) \left\| x_{t,n} - u \right\| + (1-t) \theta_t \left\| FG(J_{r_n} x_{t,n}) \right\| \right]^2$$

$$= (1-t)^2 \left\| x_{t,n} - u \right\|^2 + 2(1-t)^2 \theta_t \left\| x_{t,n} - u \right\| \left\| FG(J_{r_n} x_{t,n}) \right\|$$

$$+ (1-t)^2 \theta_t^2 \left\| FG(J_{r_n} x_{t,n}) \right\|^2. \tag{85}$$

Utilizing Lemma 5, we have

$$\left\| x_{t,n} - [(1-t)u + tf(x_{t,n})] \right\|^2$$

$$= \left\| (1-t)(x_{t,n} - u) + t(x_{t,n} - f(x_{t,n})) \right\|^2$$

$$\geq (1-t)^2 \left\| x_{t,n} - u \right\|^2$$

$$+ 2t(1-t) \left\langle x_{t,n} - f(x_{t,n}), J(x_{t,n} - u) \right\rangle. \tag{86}$$

Consequently, from the last two inequalities we deduce that

$$2t(1-t) \left\langle x_{t,n} - f(x_{t,n}), J(x_{t,n} - u) \right\rangle$$

$$\leq 2(1-t)^2 \theta_t \left\| x_{t,n} - u \right\| \left\| FG(J_{r_n} x_{t,n}) \right\| \tag{87}$$

$$+ (1-t)^2 \theta_t^2 \left\| FG(J_{r_n} x_{t,n}) \right\|^2,$$

and hence

$$\left\langle x_{t,n} - f(x_{t,n}), J(x_{t,n} - u) \right\rangle$$

$$\leq (1-t) \frac{\theta_t}{t} \left\| x_{t,n} - u \right\| \left\| FG(J_{r_n} x_{t,n}) \right\| \tag{88}$$

$$+ (1-t) \frac{\theta_t^2}{2t} \left\| FG(J_{r_n} x_{t,n}) \right\|^2.$$

Noticing (84), from $\lim_{t \to 0}(\theta_t/t) = 0$ and the boundedness of sequences $\{FG(J_{r_n} x_{t,n})\}$, $\{x_{t,n}\}$, we conclude that

$$\left\langle (I - f) q, J(q - u) \right\rangle \leq 0. \tag{89}$$

Interchanging p and u leads to

$$\left\langle (I - f) q, J(q - p) \right\rangle \leq 0. \tag{90}$$

Interchanging q and p leads to

$$\left\langle (I - f) p, J(p - q) \right\rangle \leq 0. \tag{91}$$

This implies that

$$\left\langle (q - p) - (f(q) - f(p)), J(p - q) \right\rangle \leq 0, \tag{92}$$

and hence

$$\left\| q - p \right\|^2 \leq \rho \left\| q - p \right\|^2. \tag{93}$$

Taking into account $\rho \in (0, 1)$, we obtain $p = q$. Furthermore, by the careful analysis of the above proof, we can readily see that p is also a unique solution of the VIP:

$$\left\langle (I - f) p, J(p - u) \right\rangle \leq 0, \quad \forall u \in \Delta. \tag{94}$$

This completes the proof. $\qquad \square$

Remark 17. In the assertion of Theorem 16, "as $t \to 0$, $\{x_{t,n}\}$ converges strongly to $p \in \Delta$;" this p does not depend on n. Indeed, it is known that there holds the condition that $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$. Moreover, in the proof of Theorem 16 it can be readily seen that p is first found out as a fixed point of the nonexpansive self-mapping W of K. This shows that p depends on neither n nor t.

Theorem 18. *Let X be a uniformly convex and 2-uniformly smooth Banach space and let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse strongly accretive for $i = 1, 2$, $f : X \to C$ be a contractive map with coefficient $\rho \in (0, 1)$, and let $F : X \to X$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Assume that $\Delta = A^{-1}(0) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Given sequences $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ in $[0, 1]$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0, 1]$, and $\{r_n\}_{n=0}^{\infty}$ in $[\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:*

(i) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\lim_{n \to \infty} (\lambda_n \mu_n)/\beta_n = 0$,

(iii) $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$ and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then for any given point $x_0 \in X$, the sequence $\{x_n\}$ generated by

$$y_n = \alpha_n x_n + (1 - \alpha_n) G\left(J_{r_n} x_n\right),$$

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)\left[G\left(J_{r_n} y_n\right) - \lambda_n \mu_n F G\left(J_{r_n} y_n\right)\right],$$

$$\forall n \geq 0, \tag{95}$$

converges strongly to $p \in \Delta$, which is a unique solution of the VIP (44).

Proof. First, let us show that $\{x_n\}$ is bounded. Indeed, taking a fixed $u \in \Delta$ arbitrarily, we have

$$\begin{aligned}
\|y_n - u\| &= \left\|\alpha_n x_n + (1 - \alpha_n) G\left(J_{r_n} x_n\right) - u\right\| \\
&\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \left\|G\left(J_{r_n} x_n\right) - u\right\| \\
&\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \left\|J_{r_n} x_n - u\right\| \\
&\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|x_n - u\| \\
&= \|x_n - u\|.
\end{aligned} \tag{96}$$

So $\|y_n - u\| \leq \|x_n - u\|$ for all $n \geq 0$. Thus, by Proposition 13 (iii), we have

$$\begin{aligned}
&\|x_{n+1} - u\| \\
&= \left\|\beta_n f(x_n) + (1 - \beta_n)\left[G\left(J_{r_n} y_n\right) - \lambda_n \mu_n F G\left(J_{r_n} y_n\right)\right] - u\right\| \\
&\leq \beta_n \|f(x_n) - u\| + (1 - \beta_n) \\
&\quad \times \left\|\lambda_n (I - \mu_n F) G\left(J_{r_n} y_n\right) + (1 - \lambda_n) G\left(J_{r_n} y_n\right) - u\right\| \\
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n)\left[\lambda_n \left\|(I - \mu_n F) G\left(J_{r_n} y_n\right) - u\right\| \right. \\
&\qquad\qquad\qquad\quad \left. + (1 - \lambda_n)\left\|G\left(J_{r_n} y_n\right) - u\right\|\right] \\
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n)\left[\lambda_n \left\|(I - \mu_n F) G\left(J_{r_n} y_n\right) - (I - \mu_n F) u\right\| \right. \\
&\qquad\qquad\qquad + \lambda_n \left\|(I - \mu_n F) u - u\right\| \\
&\qquad\qquad\qquad\quad \left. + (1 - \lambda_n)\left\|G\left(J_{r_n} y_n\right) - u\right\|\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n)\left[\lambda_n \left(1 - \mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\left\|G\left(J_{r_n} y_n\right) - u\right\| \right. \\
&\qquad\qquad\qquad \left. + \lambda_n \mu_n \|Fu\| + (1 - \lambda_n)\left\|G\left(J_{r_n} y_n\right) - u\right\|\right] \\[2mm]
&= \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n)\left[\left(1 - \lambda_n \mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\left\|G\left(J_{r_n} y_n\right) - u\right\| \right. \\
&\qquad\qquad\qquad \left. + \lambda_n \mu_n \|Fu\|\right] \\[2mm]
&= \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n)\left[\left(1 - \lambda_n \mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\left\|G\left(J_{r_n} y_n\right) - u\right\| \right. \\
&\qquad\quad \left. + \lambda_n \mu_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right] \\[2mm]
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n) \max\left\{\left\|G\left(J_{r_n} y_n\right) - u\right\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\} \\[2mm]
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n) \max\left\{\|y_n - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\} \\[2mm]
&\leq \beta_n \rho \|x_n - u\| + \beta_n \|f(u) - u\| \\
&\quad + (1 - \beta_n) \max\left\{\|x_n - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\} \\[2mm]
&\leq \beta_n \|f(u) - u\| + (1 - (1 - \rho)\beta_n) \\
&\quad \times \max\left\{\|x_n - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\} \\[2mm]
&= (1 - \rho)\beta_n \frac{\|f(u) - u\|}{1 - \rho} + (1 - (1 - \rho)\beta_n) \\
&\quad \times \max\left\{\|x_n - u\|, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\} \\[2mm]
&\leq \max\left\{\|x_n - u\|, \frac{\|f(u) - u\|}{1 - \rho}, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1}\|Fu\|\right\}.
\end{aligned} \tag{97}$$

By induction,

$$\|x_n - u\|$$

$$\leq \max \left\{ \|x_0 - u\|, \frac{\|f(u) - u\|}{1 - \rho}, \left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^{-1} \|Fu\| \right\},$$

$$\forall n \geq 0. \tag{98}$$

Thus, $\{x_n\}$ is bounded and so is $\{y_n\}$. Because G and J_{r_n} are nonexpansive for all $n \geq 0$, f is contractive, and F is Lipschitzian, $\{J_{r_n} x_n\}$, $\{J_{r_n} y_n\}$, $\{G(J_{r_n} x_n)\}$, $\{G(J_{r_n} y_n)\}$, $\{f(x_n)\}$, and $\{FG(J_{r_n} y_n)\}$ are bounded. From conditions (i), (ii) we have

$$\|x_{n+1} - G(J_{r_n} y_n)\|$$

$$\leq \beta_n \|f(x_n) - G(J_{r_n} y_n)\|$$

$$+ (1 - \beta_n) \|G(J_{r_n} y_n) - \lambda_n \mu_n FG(J_{r_n} y_n) - G(J_{r_n} y_n)\|$$

$$= \beta_n \|f(x_n) - G(J_{r_n} y_n)\|$$

$$+ (1 - \beta_n) \lambda_n \mu_n \|FG(J_{r_n} y_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{99}$$

Now, we claim that

$$\|x_{n+1} - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{100}$$

In order to prove (100), we estimate $\|x_{n+1} - x_n\|$ first. From (95) we have

$$y_n = \alpha_n x_n + (1 - \alpha_n) G(J_{r_n} x_n),$$

$$y_{n-1} = \alpha_{n-1} x_{n-1} + (1 - \alpha_{n-1}) G(J_{r_{n-1}} x_{n-1}). \tag{101}$$

Simple calculations show that

$$y_n - y_{n-1} = (1 - \alpha_n) \left(G(J_{r_n} x_n) - G(J_{r_{n-1}} x_{n-1})\right)$$

$$+ \alpha_n (x_n - x_{n-1}) \tag{102}$$

$$+ \left(x_{n-1} - G(J_{r_{n-1}} x_{n-1})\right) (\alpha_n - \alpha_{n-1}).$$

It follows that

$$\|y_n - y_{n-1}\| \leq (1 - \alpha_n) \|G(J_{r_n} x_n) - G(J_{r_{n-1}} x_{n-1})\|$$

$$+ \alpha_n \|x_n - x_{n-1}\|$$

$$+ \|x_{n-1} - G(J_{r_{n-1}} x_{n-1})\| |\alpha_n - \alpha_{n-1}|$$

$$\leq (1 - \alpha_n) \|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\|$$

$$+ \alpha_n \|x_n - x_{n-1}\|$$

$$+ \|x_{n-1} - G(J_{r_{n-1}} x_{n-1})\| |\alpha_n - \alpha_{n-1}|. \tag{103}$$

On the other hand, if $r_{n-1} \leq r_n$, using the resolvent identity in Lemma 8

$$J_{r_n} x_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} x_n\right), \tag{104}$$

we get

$$\|J_{r_n} x_n - J_{r_{n-1}} G_{n-1}\|$$

$$= \left\|J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} x_n\right) - J_{r_{n-1}} x_{n-1}\right\|$$

$$\leq \frac{r_{n-1}}{r_n} \|x_n - x_{n-1}\| + \left(1 - \frac{r_{n-1}}{r_n}\right) \|J_{r_n} x_n - x_{n-1}\|$$

$$\leq \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{r_n} \|J_{r_n} x_n - x_{n-1}\|$$

$$\leq \|x_n - x_{n-1}\| + \frac{1}{\varepsilon} |r_n - r_{n-1}| \|J_{r_n} x_n - x_{n-1}\|. \tag{105}$$

If $r_n \leq r_{n-1}$, it is easy to see that

$$\|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\|$$

$$\leq \|x_{n-1} - x_n\| + \frac{1}{\varepsilon} |r_{n-1} - r_n| \|J_{r_{n-1}} x_{n-1} - x_n\|. \tag{106}$$

So combining the above cases we obtain

$$\|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\|$$

$$\leq \|x_{n-1} - x_n\| + \frac{|r_{n-1} - r_n|}{\varepsilon}$$

$$\times \sup_{n \geq 1} \left\{\|J_{r_n} x_n - x_{n-1}\| + \|J_{r_{n-1}} x_{n-1} - x_n\|\right\}, \tag{107}$$

$$\forall n \geq 1.$$

In the similar way we can derive

$$\|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\|$$

$$\leq \|y_{n-1} - y_n\| + \frac{|r_{n-1} - r_n|}{\varepsilon}$$

$$\times \sup_{n \geq 1} \left\{\|J_{r_n} y_n - y_{n-1}\| + \|J_{r_{n-1}} y_{n-1} - y_n\|\right\}, \tag{108}$$

$$\forall n \geq 1.$$

Therefore, we have

$$\|J_{r_n} x_n - J_{r_{n-1}} x_{n-1}\| \leq \|x_{n-1} - x_n\| + |r_{n-1} - r_n| M_0,$$

$$\|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \leq \|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_0, \tag{109}$$

for all $n \geq 1$, where $\sup_{n \geq 1}\{(1/\varepsilon)(\|J_{r_n} x_n - x_{n-1}\| + \|J_{r_{n-1}} x_{n-1} - x_n\|)\} \leq M_0$ and $\sup_{n \geq 1}\{(1/\varepsilon)(\|J_{r_n} y_n - y_{n-1}\| + \|J_{r_{n-1}} y_{n-1} - y_n\|)\} \leq M_0$ for some $M_0 > 0$.

Substituting (109) for (103), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\leq (1 - \alpha_n) \left\| J_{r_n} x_n - J_{r_{n-1}} x_{n-1} \right\| \\
&\quad + \alpha_n \|x_n - x_{n-1}\| \\
&\quad + \left\| x_{n-1} - G\left(J_{r_{n-1}} x_{n-1}\right) \right\| |\alpha_n - \alpha_{n-1}| \\
&\leq (1 - \alpha_n) \left(\|x_n - x_{n-1}\| + |r_n - r_{n-1}| M_0 \right) \\
&\quad + \alpha_n \|x_n - x_{n-1}\| \\
&\quad + \left\| x_{n-1} - G\left(J_{r_{n-1}} x_{n-1}\right) \right\| |\alpha_n - \alpha_{n-1}| \\
&\leq \|x_n - x_{n-1}\| \\
&\quad + M_1 \left(|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}| \right),
\end{aligned}
\tag{110}
$$

where $\sup_{n \geq 0} \{ M_0, \|x_n - G(J_{r_n} x_n)\| \} \leq M_1$ for some constant $M_1 > 0$.

In the meantime, it follows from (95) that

$$
\begin{aligned}
x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n) \left[G\left(J_{r_n} y_n\right) - \lambda_n \mu_n F G\left(J_{r_n} y_n\right) \right], \\
x_n &= \beta_{n-1} f(x_{n-1}) + (1 - \beta_{n-1}) \\
&\quad \times \left[G\left(J_{r_{n-1}} y_{n-1}\right) - \lambda_{n-1} \mu_{n-1} F G\left(J_{r_{n-1}} y_{n-1}\right) \right].
\end{aligned}
\tag{111}
$$

Simple calculations show that

$$
\begin{aligned}
x_{n+1} &- x_n \\
&= (\beta_n - \beta_{n-1}) f(x_{n-1}) + \beta_n (f(x_n) - f(x_{n-1})) \\
&\quad + (\beta_{n-1} - \beta_n) (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \\
&\quad + (1 - \beta_n) \left[(I - \lambda_n \mu_n F) G\left(J_{r_n} y_n\right) \right. \\
&\qquad\qquad \left. - (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right] \\
&= (\beta_n - \beta_{n-1}) f(x_{n-1}) + \beta_n (f(x_n) - f(x_{n-1})) \\
&\quad + (\beta_{n-1} - \beta_n) (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \\
&\quad + (1 - \beta_n) \left[(I - \lambda_n \mu_n F) G\left(J_{r_n} y_n\right) \right. \\
&\qquad\qquad - (I - \lambda_n \mu_n F) G\left(J_{r_{n-1}} y_{n-1}\right) \\
&\qquad\qquad \left. + (\lambda_{n-1} \mu_{n-1} - \lambda_n \mu_n) F G\left(J_{r_{n-1}} y_{n-1}\right) \right].
\end{aligned}
\tag{112}
$$

It follows from Proposition 13(iii) and (109) that

$$
\begin{aligned}
\|x_{n+1} &- x_n\| \\
&\leq |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \|f(x_n) - f(x_{n-1})\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| + (1 - \beta_n) \\
&\quad \times \left[\left\| (I - \lambda_n \mu_n F) G\left(J_{r_n} y_n\right) - (I - \lambda_n \mu_n F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right. \\
&\qquad \left. + |\lambda_{n-1} \mu_{n-1} - \lambda_n \mu_n| \left\| F G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right] \\
&\leq |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \rho \|x_n - x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \\
&\quad + (1 - \beta_n) \left[\left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1 - \delta}{\lambda}} \right) \right) \right. \\
&\qquad\qquad \times \left\| G\left(J_{r_n} y_n\right) - G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \\
&\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| \left\| F G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right] \\
&\leq |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \rho \|x_n - x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \\
&\quad + (1 - \beta_n) \left[\left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1 - \delta}{\lambda}} \right) \right) \right. \\
&\qquad\qquad \times \left\| J_{r_n} y_n - J_{r_{n-1}} y_{n-1} \right\| \\
&\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| \left\| F G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right] \\
&\leq |\beta_n - \beta_{n-1}| \|f(x_{n-1})\| + \beta_n \rho \|x_n - x_{n-1}\| \\
&\quad + |\beta_{n-1} - \beta_n| \left\| (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \\
&\quad + (1 - \beta_n) \left[\left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1 - \delta}{\lambda}} \right) \right) \right. \\
&\qquad\qquad \times \left(\|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_0 \right) \\
&\qquad\qquad \left. + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| \left\| F G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right] \\
&\leq \beta_n \rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \\
&\quad \times \left(\|f(x_{n-1})\| + \left\| (I - \lambda_{n-1} \mu_{n-1} F) G\left(J_{r_{n-1}} y_{n-1}\right) \right\| \right) \\
&\quad + (1 - \beta_n) \|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_0 \\
&\quad + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| \left\| F G\left(J_{r_{n-1}} y_{n-1}\right) \right\|
\end{aligned}
$$

$$\leq \beta_n \rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M_2$$
$$+ (1 - \beta_n) \|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_2$$
$$+ |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| M_2$$
$$= \beta_n \rho \|x_n - x_{n-1}\| + (1 - \beta_n) \|y_n - y_{n-1}\|$$
$$+ M_2 (|\beta_n - \beta_{n-1}| + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}|$$
$$+ |r_n - r_{n-1}|), \tag{113}$$

where $\sup_{n\geq 0} \{\|f(x_n)\| + \|(I - \lambda_n \mu_n F)G(J_{r_n} y_n)\|, \|FG(J_{r_n} y_n)\|,$
$M_0, M_1\} \leq M_2$ for some $M_2 > 0$.

Substituting (110) for (113), we get

$$\|x_{n+1} - x_n\|$$
$$\leq \beta_n \rho \|x_n - x_{n-1}\| + (1 - \beta_n) \|y_n - y_{n-1}\|$$
$$+ M_2 (|\beta_n - \beta_{n-1}| + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}|$$
$$+ |r_n - r_{n-1}|)$$
$$\leq \beta_n \rho \|x_n - x_{n-1}\| + (1 - \beta_n)$$
$$\times [\|x_n - x_{n-1}\| + M_1 (|r_n - r_{n-1}| + |\alpha_n - \alpha_{n-1}|)]$$
$$+ M_2 (|\beta_n - \beta_{n-1}| + |\lambda_n \mu_n - \lambda_{n-1} \mu_{n-1}| + |r_n - r_{n-1}|)$$
$$\leq (1 - (1 - \rho) \beta_n) \|x_n - x_{n-1}\|$$
$$+ 2 M_2 [|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|$$
$$+ |\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}| + |r_n - r_{n-1}|]. \tag{114}$$

Since it follows from conditions (i) and (iv) that $\sum_{n=0}^{\infty}(1 - \rho)\beta_n = \infty$ and

$$\sum_{n=0}^{\infty} 2 M_2 [|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\lambda_n - \lambda_{n-1}|$$
$$+ |\mu_n - \mu_{n-1}| + |r_n - r_{n-1}|] < \infty, \tag{115}$$

Lemma 4 is applicable to (114) and we obtain

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{116}$$

By condition (iii) and (95), we have

$$\|y_n - x_n\|$$
$$= (1 - \alpha_n) \|G(J_{r_n} x_n) - x_n\|$$
$$\leq (1 - a) (\|G(J_{r_n} x_n) - G(J_{r_n} y_n)\|$$
$$+ \|G(J_{r_n} y_n) - x_{n+1}\| + \|x_{n+1} - x_n\|)$$
$$\leq (1 - a) (\|x_n - y_n)\| + \|G(J_{r_n} y_n) - x_{n+1}\|$$
$$+ \|x_{n+1} - x_n\|), \tag{117}$$

which implies that

$$\|y_n - x_n\| \leq \frac{1 - a}{a} (\|G(J_{r_n} y_n) - x_{n+1}\| + \|x_{n+1} - x_n\|). \tag{118}$$

This together with (99)-(100) implies that

$$\lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{119}$$

So we obtain

$$\|x_n - G(J_{r_n} x_n)\| \leq \|x_n - y_n\| + \|y_n - G(J_{r_n} x_n)\|$$
$$\leq \|x_n - y_n\| + \alpha_n \|x_n - G(J_{r_n} x_n)\| \tag{120}$$
$$\leq \|x_n - y_n\| + b \|x_n - G(J_{r_n} x_n)\|,$$

which implies that

$$\|x_n - G(J_{r_n} x_n)\| \leq \frac{1}{1 - b} \|x_n - y_n\|, \tag{121}$$

and hence

$$\lim_{n \to \infty} \|x_n - G(J_{r_n} x_n)\| = 0. \tag{122}$$

Next let us show that $\lim_{n \to \infty} \|x_n - J_{r_n} x_n\| = 0$ and $\lim_{n \to \infty} \|x_n - G x_n\| = 0$.

Indeed, for simplicity, put $v = \Pi_C(u - \mu_2 B_2 u)$, $\hat{x}_n = J_{r_n} x_n$, $u_n = \Pi_C(\hat{x}_n - \mu_2 B_2 \hat{x}_n)$ and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then $u = \Pi_C(v - \mu_1 B_1 v)$, and $v_n = G\hat{x}_n = G(J_{r_n} x_n)$ for all $n \geq 0$. It is clear from (95) that

$$\|y_n - u\|^2 \leq \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|G(J_{r_n} x_n) - u\|^2$$
$$= \alpha_n \|x_n - u\|^2 + (1 - \alpha_n) \|v_n - u\|^2. \tag{123}$$

Utilizing Lemma 14, we have

$$\|u_n - v\|^2 = \|\Pi_C(\hat{x}_n - \mu_2 B_2 \hat{x}_n) - \Pi_C(u - \mu_2 B_2 u)\|^2$$
$$\leq \|\hat{x}_n - u - \mu_2(B_2 \hat{x}_n - B_2 u)\|^2$$
$$\leq \|\hat{x}_n - u\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 \hat{x}_n - B_2 u\|^2, \tag{124}$$

$$\|v_n - u\|^2 = \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(v - \mu_1 B_1 v)\|^2$$
$$\leq \|u_n - v - \mu_1(B_1 u_n - B_1 v)\|^2$$
$$\leq \|u_n - v\|^2 - 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 v\|^2. \tag{125}$$

Substituting (124) for (125), we obtain

$$\|v_n - u\|^2 \leq \|\hat{x}_n - u\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 \hat{x}_n - B_2 u\|^2$$
$$- 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 v\|^2$$
$$\leq \|x_n - u\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 \hat{x}_n - B_2 u\|^2$$
$$- 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 v\|^2, \tag{126}$$

which, together with (123), implies that

$$
\begin{aligned}
& \left\| y_n - u \right\|^2 \\
& \leq \alpha_n \left\| x_n - u \right\|^2 + (1 - \alpha_n) \left\| v_n - u \right\|^2 \\
& \leq \alpha_n \left\| x_n - u \right\|^2 + (1 - \alpha_n) \\
& \quad \times \left[\left\| x_n - u \right\|^2 - 2\mu_2 \left(\alpha_2 - \kappa^2 \mu_2 \right) \left\| B_2 \widehat{x}_n - B_2 u \right\|^2 \right. \\
& \qquad \left. - 2\mu_1 \left(\alpha_1 - \kappa^2 \mu_1 \right) \left\| B_1 u_n - B_1 v \right\|^2 \right] \\
& = \left\| x_n - u \right\|^2 - 2(1 - \alpha_n) \\
& \quad \times \left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2 \right) \left\| B_2 \widehat{x}_n - B_2 u \right\|^2 \right. \\
& \qquad \left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1 \right) \left\| B_1 u_n - B_1 v \right\|^2 \right].
\end{aligned}
\tag{127}
$$

It immediately follows that

$$
\begin{aligned}
& 2(1 - \alpha_n) \left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2 \right) \left\| B_2 \widehat{x}_n - B_2 u \right\|^2 \right. \\
& \qquad \left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1 \right) \left\| B_1 u_n - B_1 v \right\|^2 \right] \\
& \leq \left\| x_n - u \right\|^2 - \left\| y_n - u \right\|^2 \\
& \leq \left(\left\| x_n - u \right\| + \left\| y_n - u \right\| \right) \left\| x_n - y_n \right\|.
\end{aligned}
\tag{128}
$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$, we deduce from (119) and condition (iii) that

$$
\begin{aligned}
& \lim_{n \to \infty} \left\| B_2 \widehat{x}_n - B_2 u \right\| = 0, \\
& \lim_{n \to \infty} \left\| B_1 u_n - B_1 v \right\| = 0.
\end{aligned}
\tag{129}
$$

Utilizing Proposition 3 and Lemma 6, we have

$$
\begin{aligned}
& \left\| u_n - v \right\|^2 \\
& = \left\| \Pi_C(\widehat{x}_n - \mu_2 B_2 \widehat{x}_n) - \Pi_C(u - \mu_2 B_2 u) \right\|^2 \\
& \leq \left\langle \widehat{x}_n - \mu_2 B_2 \widehat{x}_n - (u - \mu_2 B_2 u), J(u_n - v) \right\rangle \\
& = \left\langle \widehat{x}_n - u, J(u_n - v) \right\rangle + \mu_2 \left\langle B_2 p - B_2 \widehat{x}_n, J(u_n - v) \right\rangle \\
& \leq \frac{1}{2} \left[\left\| \widehat{x}_n - u \right\|^2 + \left\| u_n - v \right\|^2 - g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \right] \\
& \quad + \mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\|,
\end{aligned}
\tag{130}
$$

which implies that

$$
\begin{aligned}
\left\| u_n - v \right\|^2 & \leq \left\| \widehat{x}_n - u \right\|^2 - g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \\
& \quad + 2\mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\|.
\end{aligned}
\tag{131}
$$

In the same way, we derive

$$
\begin{aligned}
& \left\| v_n - u \right\|^2 \\
& = \left\| \Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(v - \mu_1 B_1 v) \right\|^2 \\
& \leq \left\langle u_n - \mu_1 B_1 u_n - (v - \mu_1 B_1 v), J(v_n - u) \right\rangle \\
& = \left\langle u_n - v, J(v_n - u) \right\rangle + \mu_1 \left\langle B_1 v - B_1 u_n, J(v_n - u) \right\rangle \\
& \leq \frac{1}{2} \left[\left\| u_n - v \right\|^2 + \left\| v_n - u \right\|^2 - g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \right] \\
& \quad + \mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\|,
\end{aligned}
\tag{132}
$$

which implies that

$$
\begin{aligned}
\left\| v_n - u \right\|^2 & \leq \left\| u_n - v \right\|^2 - g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \\
& \quad + 2\mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\|.
\end{aligned}
\tag{133}
$$

Substituting (131) for (133), we get

$$
\begin{aligned}
\left\| v_n - u \right\|^2 & \leq \left\| \widehat{x}_n - u \right\|^2 - g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \\
& \quad - g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \\
& \quad + 2\mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\| \\
& \quad + 2\mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\| \\
& \leq \left\| x_n - u \right\|^2 - g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \\
& \quad - g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \\
& \quad + 2\mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\| \\
& \quad + 2\mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\|,
\end{aligned}
\tag{134}
$$

which, together with (123), implies that

$$
\begin{aligned}
& \left\| y_n - u \right\|^2 \\
& \leq \alpha_n \left\| x_n - u \right\|^2 + (1 - \alpha_n) \left\| v_n - u \right\|^2 \\
& \leq \alpha_n \left\| x_n - u \right\|^2 + (1 - \alpha_n) \\
& \quad \times \left[\left\| x_n - u \right\|^2 - g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \right. \\
& \qquad - g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \\
& \qquad + 2\mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\| \\
& \qquad \left. + 2\mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\| \right] \\
& = \left\| x_n - u \right\|^2 - (1 - \alpha_n) \\
& \quad \times \left[g_1 \left(\left\| \widehat{x}_n - u_n - (u - v) \right\| \right) \right. \\
& \qquad \left. + g_2 \left(\left\| u_n - v_n + (u - v) \right\| \right) \right] \\
& \quad + 2(1 - \alpha_n) \left(\mu_2 \left\| B_2 u - B_2 \widehat{x}_n \right\| \left\| u_n - v \right\| \right. \\
& \qquad \left. + \mu_1 \left\| B_1 v - B_1 u_n \right\| \left\| v_n - u \right\| \right).
\end{aligned}
\tag{135}
$$

It immediately follows that

$$(1 - \alpha_n) \left[g_1 \left(\|\widehat{x}_n - u_n - (u - v)\| \right) \right.$$
$$\left. + g_2 \left(\|u_n - v_n + (u - v)\| \right) \right]$$
$$\leq \|x_n - u\|^2 - \|y_n - u\|^2$$
$$+ 2 (1 - \alpha_n) \left(\mu_2 \|B_2 u - B_2 \widehat{x}_n\| \|u_n - v\| \right.$$
$$\left. + \mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\| \right) \quad (136)$$
$$\leq \left(\|x_n - u\| + \|y_n - u\| \right) \|x_n - y_n\|$$
$$+ 2\mu_2 \|B_2 u - B_2 \widehat{x}_n\| \|u_n - v\|$$
$$+ 2\mu_1 \|B_1 v - B_1 u_n\| \|v_n - u\|.$$

Since $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ are bounded, we deduce from (119), (129), and condition (iii) that

$$\lim_{n \to \infty} g_1 \left(\|\widehat{x}_n - u_n - (u - v)\| \right) = 0,$$
$$\lim_{n \to \infty} g_2 \left(\|u_n - v_n + (u - v)\| \right) = 0. \quad (137)$$

Utilizing the properties of g_1 and g_2, we get

$$\lim_{n \to \infty} \|\widehat{x}_n - u_n - (u - v)\| = 0,$$
$$\lim_{n \to \infty} \|u_n - v_n + (u - v)\| = 0, \quad (138)$$

which hence yields

$$\|\widehat{x}_n - v_n\| \leq \|\widehat{x}_n - u_n - (u - v)\|$$
$$+ \|u_n - v_n + (u - v)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (139)$$

That is,

$$\lim_{n \to \infty} \left\| J_{r_n} x_n - G \left(J_{r_n} x_n \right) \right\| = \lim_{n \to \infty} \|\widehat{x}_n - v_n\| = 0. \quad (140)$$

Note that

$$\left\| x_n - J_{r_n} x_n \right\| \leq \left\| x_n - G \left(J_{r_n} x_n \right) \right\| + \left\| G \left(J_{r_n} x_n \right) - J_{r_n} x_n \right\|. \quad (141)$$

So from (122) and (140) we have

$$\lim_{n \to \infty} \left\| x_n - J_{r_n} x_n \right\| = 0, \quad (142)$$

which, together with (122), leads to

$$\left\| x_n - G x_n \right\| \leq \left\| x_n - G \left(J_{r_n} x_n \right) \right\| + \left\| G \left(J_{r_n} x_n \right) - G x_n \right\|$$
$$\leq \left\| x_n - G \left(J_{r_n} x_n \right) \right\| \quad (143)$$
$$+ \left\| J_{r_n} x_n - x_n \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

That is,

$$\lim_{n \to \infty} \|x_n - G x_n\| = 0. \quad (144)$$

In addition, utilizing Lemma 9 we obtain from $\{r_n\} \subset [\varepsilon, \infty)$ that

$$\|x_n - J_\varepsilon x_n\| \leq 2 \|x_n - J_{r_n} x_n\|, \quad (145)$$

which, together with (142), implies that

$$\lim_{n \to \infty} \|x_n - J_\varepsilon x_n\| = 0. \quad (146)$$

Define a mapping

$$W x = (1 - \theta) J_\varepsilon x + \theta G x, \quad \forall x \in C, \quad (147)$$

where θ is a constant in $(0, 1)$. Then by Lemma 10, we know that $\text{Fix}(W) = \text{Fix}(J_\varepsilon) \cap \text{Fix}(G) = \Delta$. We observe that

$$\|x_n - W x_n\| = \|(1 - \theta) (x_n - J_\varepsilon x_n) + \theta (x_n - G x_n)\|$$
$$\leq (1 - \theta) \|x_n - J_\varepsilon x_n\| + \theta \|x_n - G x_n\|. \quad (148)$$

So from (144) and (146) we obtain

$$\lim_{n \to \infty} \|x_n - W x_n\| = 0. \quad (149)$$

Now, we claim that

$$\limsup_{n \to \infty} \left\langle f(p) - p, J(x_n - p) \right\rangle \leq 0, \quad (150)$$

where $p = s - \lim_{t \to 0} x_t$, with x_t being the fixed point of the contraction $x \mapsto t f(x) + (1 - t) W x$ of C into itself (due to Lemma 12). Then x_t solves the fixed point equation $x_t = t f(x_t) + (1 - t) W x_t$. Thus we have

$$\|x_t - x_n\| = \|(1 - t)(W x_t - x_n) + t(f(x_t) - x_n)\|. \quad (151)$$

By Lemma 5 we conclude that

$$\|x_t - x_n\|^2 = \|(1 - t)(W x_t - x_n) + t(f(x_t) - x_n)\|^2$$
$$\leq (1 - t)^2 \|W x_t - x_n\|^2$$
$$+ 2t \left\langle f(x_t) - x_n, J(x_t - x_n) \right\rangle$$
$$\leq (1 - t)^2 \left(\|W x_t - W x_n\| + \|W x_n - x_n\| \right)^2$$
$$+ 2t \left\langle f(x_t) - x_n, J(x_t - x_n) \right\rangle$$
$$\leq (1 - t)^2 \left(\|x_t - x_n\| + \|W x_n - x_n\| \right)^2$$
$$+ 2t \left\langle f(x_t) - x_n, J(x_t - x_n) \right\rangle$$
$$= (1 - t)^2 \left[\|x_t - x_n\|^2 + 2 \|x_t - x_n\| \|W x_n - x_n\| \right.$$
$$\left. + \|W x_n - x_n\|^2 \right]$$
$$+ 2t \left\langle f(x_t) - x_t, J(x_t - x_n) \right\rangle$$
$$+ 2t \left\langle x_t - x_n, J(x_t - x_n) \right\rangle$$
$$= \left(1 - 2t + t^2 \right) \|x_t - x_n\|^2 + f_n(t)$$
$$+ 2t \left\langle f(x_t) - x_t, J(x_t - x_n) \right\rangle + 2t \|x_t - x_n\|^2, \quad (152)$$

where

$$f_n(t) = (1-t)^2 \left(2\|x_t - x_n\| + \|x_n - Wx_n\|\right)$$
$$\times \|x_n - Wx_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{153}$$

It follows from (152) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{154}$$

Letting $n \to \infty$ in (154) and noticing (153), we derive

$$\limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_3, \tag{155}$$

where $M_3 > 0$ is a constant such that $\|x_t - x_n\|^2 \le M_3$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (155), we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{156}$$

On the other hand, we have

$$\langle f(p) - p, J(x_n - p)\rangle$$
$$= \langle f(p) - p, J(x_n - p)\rangle - \langle f(p) - p, J(x_n - x_t)\rangle$$
$$+ \langle f(p) - p, J(x_n - x_t)\rangle - \langle f(p) - x_t, J(x_n - x_t)\rangle$$
$$+ \langle f(p) - x_t, J(x_n - x_t)\rangle - \langle f(x_t) - x_t, J(x_n - x_t)\rangle$$
$$+ \langle f(x_t) - x_t, J(x_n - x_t)\rangle$$
$$= \langle f(p) - p, J(x_n - p) - J(x_n - x_t)\rangle$$
$$+ \langle x_t - p, J(x_n - x_t)\rangle + \langle f(p) - f(x_t), J(x_n - x_t)\rangle$$
$$+ \langle f(x_t) - x_t, J(x_n - x_t)\rangle. \tag{157}$$

It follows that

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle$$
$$\le \limsup_{n\to\infty} \langle f(p) - p, J(x_n - p) - J(x_n - x_t)\rangle$$
$$+ \|x_t - p\| \limsup_{n\to\infty} \|x_n - x_t\| \tag{158}$$
$$+ \rho\|p - x_t\| \limsup_{n\to\infty} \|x_n - x_t\|$$
$$+ \limsup_{n\to\infty} \langle f(x_t) - x_t, J(x_n - x_t)\rangle.$$

Taking into account that $x_t \to p$ as $t \to 0$, we have from (156)

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle$$
$$= \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(p) - p, J(x_n - p)\rangle$$
$$\le \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(p) - p, J(x_n - p) - J(x_n - x_t)\rangle. \tag{159}$$

Since X has a uniformly Frechet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable and hence (150) holds. It is clear from (150) that

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_{n+1} - p)\rangle \le 0. \tag{160}$$

Finally, let us show that $x_n \to p$ as $n \to \infty$. We observe that

$$\|y_n - p\|^2 \le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|G(J_{r_n}x_n) - p\|^2$$
$$\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \tag{161}$$
$$= \|x_n - p\|^2,$$

and hence

$$\|x_{n+1} - p\|^2$$
$$= \|\beta_n(f(x_n) - f(p))$$
$$\quad + (1-\beta_n)\left[G(J_{r_n}y_n) - \lambda_n\mu_n FG(J_{r_n}y_n) - p\right]$$
$$\quad + \beta_n(f(p) - p)\|^2$$
$$\le \|\beta_n(f(x_n) - f(p))$$
$$\quad + (1-\beta_n)\left[G(J_{r_n}y_n) - \lambda_n\mu_n FG(J_{r_n}y_n) - p\right]\|^2$$
$$\quad + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle$$
$$\le \beta_n\|f(x_n) - f(p)\|^2$$
$$\quad + (1-\beta_n)\|G(J_{r_n}y_n) - \lambda_n\mu_n FG(J_{r_n}y_n) - p\|^2$$
$$\quad + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle$$
$$= \beta_n\|f(x_n) - f(p)\|^2$$
$$\quad + (1-\beta_n)\|(I - \lambda_n\mu_n F)G(J_{r_n}y_n)$$
$$\qquad - (I - \lambda_n\mu_n F)p - \lambda_n\mu_n Fp\|^2$$
$$\quad + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle$$
$$\le \beta_n\rho\|x_n - p\|^2$$
$$\quad + (1-\beta_n)\left[\|(I - \lambda_n\mu_n F)G(J_{r_n}y_n) - (I - \lambda_n\mu_n F)p\|\right.$$
$$\qquad \left. + \lambda_n\mu_n\|Fp\|\right]^2$$
$$\quad + 2\beta_n\langle f(p) - p, J(x_{n+1} - p)\rangle$$

$$\leq \beta_n \rho \|x_n - p\|^2 + (1 - \beta_n)$$

$$\times \left[\left(1 - \lambda_n \mu_n \left(1 - \sqrt{\frac{1 - \delta}{\lambda}} \right) \right) \|G(J_{r_n} y_n) - p\| \right.$$

$$\left. + \lambda_n \mu_n \|Fp\| \right]^2$$

$$+ 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle$$

$$\leq \beta_n \rho \|x_n - p\|^2$$

$$+ (1 - \beta_n) (\|y_n - p\| + \lambda_n \mu_n \|Fp\|)^2$$

$$+ 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle$$

$$\leq \beta_n \rho \|x_n - p\|^2$$

$$+ (1 - \beta_n) (\|x_n - p\| + \lambda_n \mu_n \|Fp\|)^2$$

$$+ 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle$$

$$= \beta_n \rho \|x_n - p\|^2$$

$$+ (1 - \beta_n) \left[\|x_n - p\|^2 + \lambda_n \mu_n \|Fp\| \right.$$

$$\left. \times (2 \|x_n - p\| + \lambda_n \mu_n \|Fp\|) \right]$$

$$+ 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle$$

$$\leq (1 - (1 - \rho)\beta_n) \|x_n - p\|^2$$

$$+ \lambda_n \mu_n \|Fp\| (2 \|x_n - p\| + \lambda_n \mu_n \|Fp\|)$$

$$+ 2\beta_n \langle f(p) - p, J(x_{n+1} - p) \rangle$$

$$= (1 - (1 - \rho)\beta_n) \|x_n - p\|^2$$

$$+ (1 - \rho) \beta_n \left\{ \frac{\lambda_n \mu_n}{\beta_n} \frac{\|Fp\| (2 \|x_n - p\| + \lambda_n \mu_n \|Fp\|)}{1 - \rho} \right.$$

$$\left. + \frac{2 \langle f(p) - p, J(x_{n+1} - p) \rangle}{1 - \rho} \right\}. \tag{162}$$

Taking into account (160) and conditions (i), (ii), we obtain that $\sum_{n=0}^{\infty}(1 - \rho)\beta_n = \infty$ and

$$\limsup_{n \to \infty} \left\{ \frac{\lambda_n \mu_n}{\beta_n} \frac{\|Fp\| (2 \|x_n - p\| + \lambda_n \mu_n \|Fp\|)}{1 - \rho} \right.$$

$$\left. + \frac{2 \langle f(p) - p, J(x_{n+1} - p) \rangle}{1 - \rho} \right\} \leq 0. \tag{163}$$

Therefore, applying Lemma 4 to (162), we infer that

$$\lim_{n \to \infty} \|x_n - p\| = 0. \tag{164}$$

This completes the proof. $\qquad\square$

Remark 19. As pointed out in [12, Remark 3.2], the sequences $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\beta_n\}$ can be taken, which satisfy the conditions in Theorem 18. As a matter of fact, put $\lambda_n = (1 + n)^{-5/6}$, $\mu_n = 1$, and $\beta_n = (1 + n)^{-2/3}$ for all $n \geq 0$. Then there hold the following statements:

(i) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\lim_{n \to \infty} (\lambda_n \mu_n)/\beta_n = 0$,

(iii) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\sum_{n=0}^{\infty} |\mu_{n+1} - \mu_n| < \infty$.

By the careful analysis of the proof of Theorem 18, we can obtain the following result. Because its proof is much simpler than that of Theorem 18, we omit its proof.

Theorem 20. *Let X be a uniformly convex and 2-uniformly smooth Banach space and let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse strongly accretive for $i = 1, 2$, let $f : X \to C$ be a contractive map with coefficient $\rho \in (0, 1)$, and let $F : X \to X$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$. Assume that $\Delta = A^{-1}(0) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Given sequences $\{\lambda_n\}_{n=0}^{\infty}$ in $[0, 1]$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in $(0, 1]$, and $\{r_n\}_{n=0}^{\infty}$ in $[\varepsilon, \infty)$ for some $\varepsilon > 0$, suppose that there hold the following conditions:*

(i) $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\lim_{n \to \infty} \lambda_n/\beta_n = 0$ and $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

(iii) $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$,

(iv) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, and $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then for any given point $x_0 \in X$, the sequence $\{x_n\}$ generated by

$$y_n = \alpha_n x_n + (1 - \alpha_n) G(J_{r_n} x_n),$$

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) [y_n - \lambda_n F(y_n)], \tag{165}$$

$$\forall n \geq 0,$$

converges strongly to $p \in \Delta$, which is a unique solution of the VIP (44).

Remark 21. Our Theorems 16–20 improve, extend, supplement, and develop Cai and Bu [13, Theorem 3.1] and Ceng et al. [12, Theorems 3.1–3.3] in the following aspects.

(i) The problem of finding a point $p \in A^{-1}(0) \cap \Omega$ in our Theorems 16–20 is very different from everyone of both the problem of finding a point $p \in \bigcap_n \text{Fix}(S_n) \cap \Omega$ in Cai and Bu [13, Theorem 3.1] and the problem of finding a point $p \in A^{-1}(0)$ in Ceng et al. [12, Theorems 3.1–3.3]. There is no doubt that our problem of finding a point $p \in A^{-1}(0) \cap \Omega$ is more general than the problem of finding a point $p \in A^{-1}(0)$ in [12, Theorems 3.1–3.3].

(ii) Compared with the choice of iterative parameters in [12, Theorems 3.1–3.3], the choice of iterative parameters in our Theorems 16–20 is the same as that in [12, Theorems 3.1–3.3].

(iii) The iterative schemes in [12, Theorems 3.1–3.3] are extended to develop the iterative schemes in our Theorems 16–20 by virtue of the iterative scheme of [13, Theorems 3.1]. The iterative schemes in our Theorems 16–20 are more advantageous and more flexible than the iterative schemes of [12, Theorems 3.1–3.3] because they involve solving two problems: the GSVI (16) and the problem of finding zeros of an m-accretive operator.

(iv) The iterative schemes in our Theorems 16–20 are very different from everyone in both [13, Theorem 3.1] and [12, Theorems 3.1–3.3] because the iterative scheme in our Theorem 16 is implicit and because the mapping G in [13, Theorem 3.1] and the mapping J_{r_n} in [12, Theorems 3.1–3.3] are replaced by the same composite mapping $G \circ J_{r_n}$ in the iterative schemes of our Theorems 16–20.

(v) Cai and Bu's proof in [13, Theorem 3.1] depends on the argument techniques in [16], the inequality in 2-uniformly smooth Banach spaces (see Lemma 2), and the inequality in smooth and uniform convex Banach spaces (see Proposition 3). Because the composite mapping $G \circ J_{r_n}$ appears in the iterative schemes in our Theorems 16–20, the proof of our Theorems 16–20 depends on the argument techniques in [16], the inequality in 2-uniformly smooth Banach spaces (see Lemma 2), the inequality in smooth and uniform convex Banach spaces (see Proposition 3), and the properties of the resolvent of an m-accretive operator (see Lemmas 8 and 9), the Banach limit (see Lemma 11) and the strongly accretive and strictly pseudocontractive mapping (see Proposition 13).

Acknowledgments

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