Research Article

Lower and Upper Solutions Method for Positive Solutions of Fractional Boundary Value Problems

R. Darzi,1 B. Mohammadzadeh,2 A. Neamaty,3 and D. Băleanu4,5,6

1 Department of Mathematics, Neka Branch, Islamic Azad University, P.O. Box 48411-86114, Neka, Iran
2 Department of Mathematics, Sari Branch, Islamic Azad University, P.O. Box 48161-19318, Sari, Iran
3 Department of Mathematics, University of Mazandaran, P.O. Box 47446-95447, Babolsar, Iran
4 Department of Mathematics and Computer Science, Faculty of Art and Sciences, Cankaya University, Yenimahalle, 06810 Ankara, Turkey
5 Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box 80204, Jeddah 21589, Saudi Arabia
6 Institute of Space Sciences, P.O. Box, MG-23, R 76900 Magurele, Bucharest, Romania

Correspondence should be addressed to R. Darzi; r.drazi@yahoo.com

Received 10 May 2013; Revised 11 July 2013; Accepted 1 August 2013

Academic Editor: Juan J. Trujillo

Copyright © 2013 R. Darzi et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We apply the lower and upper solutions method and fixed-point theorems to prove the existence of positive solution to fractional boundary value problem

\[ D_0^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \]

with the boundary conditions

\[ u(0) = u_0, \quad u'(0) = u'(1) = 0. \]

As an application, one example is given to illustrate the main result.

1. Introduction

In the recent years, fractional calculus has been one of the most interesting issues that have attracted many scientists, especially in the fields of mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, and viscoelasticity try to present a model of these phenomena by boundary value problems of fractional differential equations [1–4]. In order to achieve extra information in fractional calculus, interested readers can refer to more valuable books that are written by other authors [5–20].

The existence and multiplicity of solutions or positive solutions of nonlinear fractional differential equation (FDE) by the use of fixed point theorems, Leray-Shauder theory, and so forth are mentioned in some papers [6, 8, 12, 20, 21]. Few papers have considered the boundary value problems of fractional differential equations [12, 14]. By the use of some fixed point theorems on cones, Zhang [15] obtained the existence of positive solution for the equation

\[ D_0^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha < 2, \]

with the boundary conditions

\[ u(0) = u(1) = 0. \]

In [22], Liang and Zhang applied lower and upper solutions method and fixed point theorems to obtain some results on the existence of positive solutions for the following BVPs:

\[ D_0^\alpha u(t) = f(t, u(t)), \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \]

\[ u(0) = u(1) = u'(0) = u'(1) = 0, \]

where \( D_0^\alpha \) denotes Riemann-Liouville fractional derivative.
In this paper, we investigate the existence of positive solution for a nonlocal BVP of FDE,
\[ D_0^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \]
\[ u(0) = u'(0) = 0, \quad D_0^{\alpha-1}u(1) = \beta u(\xi), \] (4) (5)
using lower and upper solutions method and fixed point theorem, where \( D_0^\alpha \) denotes standard Riemann-Liouville fractional derivative, \( \beta \xi^{\alpha-1} \geq 2 \Gamma(\alpha), \) and \( f \in C([0,1] \times [0,\infty), [0,\infty)). \)

The main result of this paper can be seen in Theorem 10. In this section, we present the necessary definitions and lemmas that will be used to prove our new results.

2. Basic Definitions and Preliminaries

In this section, we present the necessary definitions and lemmas that will be used to prove our new results.

Definition 1 (see [5, 6]). The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f : (0,\infty) \rightarrow \mathbb{R} \) is defined by
\[ I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad n-1 < \alpha \leq n, \]
provided that the right-hand side is pointwise defined on \((0,\infty)).\]

Definition 2 (see [5, 6]). The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) of a function \( f : (0,\infty) \rightarrow \mathbb{R} \) is defined by
\[ D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds, \]
\[ n-1 < \alpha \leq n, \]
where \( n = [\alpha] + 1 \), provided that the right-hand side is pointwise defined on \((0,\infty)).\]

Definition 3 (see [5, 6]). A function \( \mu(t) \in C^2[0,1] \) is called a lower solution of problem (4)-(5) if \( \mu(t) \) satisfies
\[ -D_0^\alpha \mu(t) \leq f(t, \mu(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \]
\[ \mu(0) \leq 0, \quad \mu'(0) \leq 0, \quad D_0^{\alpha-1}\mu(t) \leq \beta \mu(\xi). \] (8)

Definition 4 (see [7, 8]). A function \( \mu(t) \in C^2[0,1] \) is called an upper solution of problem (4)-(5) if \( \mu(t) \) satisfies
\[ -D_0^\alpha \mu(t) \geq f(t, \mu(t)), \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \]
\[ \mu(0) \geq 0, \quad \mu'(0) \geq 0, \quad D_0^{\alpha-1}\mu(t) \geq \beta \mu(\xi). \] (9)

Lemma 5 (see [7, 8]). Let \( u \in C(0,1) \cap L^1(0,1) \). Then the fractional differential equation
\[ D_0^n u(t) = 0 \] (10)
has
\[ u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \]
for some \( c_i \in \mathbb{R}, \quad i = 1, \ldots, n, \)
as a unique solution.

Lemma 6 (see [7, 8]). Let \( u \in C(0,1) \cap L^1(0,1) \) with a fractional derivative of order \( \alpha > 0 \), \( n-1 < \alpha \leq n \) \((n \in \mathbb{N})\), that belongs to \( C(0,1) \cap L(0,1) \). Then
\[ I_0^n D_0^n u(t) = u(t) + \sum_{i=1}^n c_i t^{\alpha-i}, \]
for some \( c_i \in \mathbb{R}, \quad i = 1, \ldots, n, \)
as a unique solution.

Lemma 7. If \( \beta \xi^{\alpha-1} \geq 2 \Gamma(\alpha) \), then for \( 0 \leq y(t) \in C[0,1] \), the problem,
\[ D_0^n u(t) + y(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \]
\[ u(0) = u'(0) = 0, \quad D_0^{\alpha-1} u(1) = \beta u(\xi), \] (13) (14)
has a unique positive solution
\[ u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \]
\[ + \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \int_0^1 t^{\alpha-1} y(s) \, ds \]
\[ + \frac{\beta}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \Gamma(\alpha) \int_0^\xi (\xi-s)^{\alpha-1} y(s) \, ds. \] (15)

Proof. We can apply Lemma 6 to reduce (13) to an equivalent integral equation
\[ u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \] (16)
for some \( c_1, c_2, c_3 \in \mathbb{R}. \) From \( u(0) = 0 \) and \( u'(0) = 0 \) in (14), we have \( c_2 = c_3 = 0. \) On the other hand, \( D_0^{\alpha-1} u(1) = \beta u(\xi) \) yields
\[ c_1 = \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \int_0^\xi y(s) \, ds \]
\[ + \frac{\beta}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \Gamma(\alpha) \int_0^\xi (\xi-s)^{\alpha-1} y(s) \, ds. \] (17)
Then, the unique solution of problem is given by \( u(t). \) Obviously, \( u(t) \geq 0 \) if \( y(t) \geq 0 \) on \( t \in [0,1]. \) The proof is complete.
3. Main Result

In this section, we present and prove our main result.

Lemma 8. Suppose that \( \beta \xi^{\alpha-1} \geq 2 \Gamma(\alpha) \). Given that \( y \in C[0, 1] \), the Green function for the problem (13)-(14) is given by

\[
G(t, s) = \begin{cases} 
\left[\left(-t-s\right)^{\alpha-1} \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] + \Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1} \right] \\
\times \left[ \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha) \right]^{-1}, & 0 \leq s \leq t \leq 1, \ s \leq \xi, \\
\left(-t-s\right)^{\alpha-1} \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] + \Gamma(\alpha) t^{\alpha-1} \\
\times \left[ \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha) \right]^{-1}, & 0 \leq \xi \leq s \leq t \leq 1, \\
\frac{\Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1}}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)}, & 0 \leq t \leq s \leq \xi \leq 1, \\
\beta \xi^{\alpha-1} - \Gamma(\alpha), & 0 \leq t \leq s \geq \xi.
\end{cases}
\]

(18)

Proof. By Lemma 7, for \( t \leq \xi \), we have

\[
u(t) = -\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \left( -s \right)^{\alpha-1} y(s) \, ds + \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \times \left[ \left( \int_{0}^{t} + \int_{t}^{1} + \int_{0}^{t} \right) t^{\alpha-1} y(s) \, ds \right]
\]

\[
+ \frac{\beta}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} \times \left[ \left( \int_{0}^{t} + \int_{t}^{1} \right) \left( -s \right)^{\alpha-1} y(s) \, ds \right]
\]

\[
= \int_{0}^{t} \left( -s \right)^{\alpha-1} \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] + \Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1} \\
\times \left[ \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha) \right]^{-1} y(s) \, ds
\]

\[
+ \int_{t}^{1} \frac{\Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1}}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} y(s) \, ds
\]

\[
+ \int_{\xi}^{1} \frac{\Gamma(\alpha) t^{\alpha-1}}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} y(s) \, ds
\]

\[
= \int_{0}^{t} G(t, s) y(s) \, ds.
\]

(19)

For \( t \geq \xi \), we have

\[
u(t) = -\frac{1}{\Gamma(\alpha)} \left[ \left( \int_{0}^{t} + \int_{t}^{1} + \int_{0}^{t} \right) t^{\alpha-1} y(s) \, ds \right]
\]

\[
+ \frac{\beta}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} \times \left[ \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha) \right]^{-1} y(s) \, ds
\]

\[
= \int_{0}^{t} \left( -s \right)^{\alpha-1} \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] + \Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1} \\
\times \left[ \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha) \right]^{-1} y(s) \, ds
\]

\[
+ \int_{t}^{1} \frac{\Gamma(\alpha) t^{\alpha-1} + \beta t^{\alpha-1} (\xi - s)^{\alpha-1}}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} y(s) \, ds
\]

\[
+ \int_{\xi}^{1} \frac{\Gamma(\alpha) t^{\alpha-1}}{\left[ \beta \xi^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha)} y(s) \, ds
\]

\[
= \int_{0}^{1} G(t, s) y(s) \, ds.
\]

(20)

The proof is complete. \( \square \)

Lemma 9. Suppose that \( u(t) \in C^2[0, 1] \) and is a positive solution of (4)-(5). Then

\[
m q(t) \leq u(t) \leq M q(t),
\]

(21)

where

\[
q(t) = \frac{1}{c \Gamma(\alpha + 1)} \left[ \beta \xi^{\alpha-1} - \Gamma(\alpha + 1) \right] t^{\alpha-1} - ct^\alpha,
\]

(22)

where \( c = \beta \xi^{\alpha-1} - \Gamma(\alpha) \), \( \beta \xi^{\alpha-1} \neq \Gamma(\alpha) \), and \( m \) and \( M \) are two constants.

Proof. Since \( u(t) \in C^2[0, 1] \), there exists \( T > 0 \) so that \( |u(t)| \leq T \) for \( t \in [0, 1] \). We define

\[
m := \min_{(t,u) \in [0,1] \times [0,T]} f(t, u(t)),
\]

\[
M := \max_{(t,u) \in [0,1] \times [0,T]} f(t, u(t)).
\]

(23)

Therefore, we have

\[
m \int_{0}^{1} G(t, s) \, ds \leq \int_{0}^{1} G(t, s) f(s, u(s)) \, ds \leq M \int_{0}^{1} G(t, s) \, ds.
\]

(24)
On the other hand, by direct computation, we get

\[
\int_0^1 G(t, s) \, ds = \frac{1}{\left[ \Gamma^{\alpha-1} - \Gamma(\alpha) \right] \Gamma(\alpha + 1) + \left[ \beta \right]^{\alpha-1} - \left[ \beta \right]^{\alpha}}. \\
\times \left( \left[ \beta \right]^{\alpha-1} - \Gamma(\alpha + 1) \right)^{\alpha-1} - \left[ \beta \right]^{\alpha-1} - \Gamma(\alpha) \right) t^\alpha. \\
\left(25\right)
\]

This completes the proof of the lemma.

**Theorem 10.** The fractional boundary value problem (4)-(5) has a positive solution \( u(t) \) if the conditions \( \left(H_1\right) \)–\( \left(H_2\right) \) are satisfied.

**Proof.** Suppose that

\[
\alpha_1 = \min \left\{ 1, \inf_{t \in [a, b]} f(t, \varphi(t)) \right\}, \\
\alpha_2 = \max \left\{ 1, \sup_{t \in [a, b]} f(t, \varphi(t)) \right\}, \\
0 < k_1 \leq \min \left\{ \frac{1}{\alpha_2}, (\alpha_1)^{\lambda/(1-\lambda)} \right\}, \\
k_2 \geq \max \left\{ \frac{1}{\alpha_2}, (\alpha_1)^{\lambda/(1-\lambda)} \right\},
\]

and \( h(t) = \int_0^1 G(t, s) f(s, \varphi(s)) \, ds. \) We show that \( \mu(t) = k_1 h(t) \) and \( \nu(t) = k_2 h(t) \) satisfy the conditions \( \left(H_1\right) \)–\( \left(H_2\right) \), respectively. From Lemma 7, \( h(t) \) is a positive solution of the following problem:

\[
-D_0^\alpha u(t) \leq f(t, \varphi(t)), \quad 0 < t < 1, \, 2 < \alpha \leq 3, \\
\quad u(0) = u'(0) = 0, \quad D_0^\alpha u(1) = \beta u(\xi). \\
\left(27\right)
\]

We know that \( \alpha_1 \varphi(t) \leq h(t) \leq \alpha_2 \varphi(t). \) Now, using the assumption of the theorem, we get

\[
k_1 \alpha_1 \leq \frac{h(t)}{\varphi(t)} \leq k_2 \alpha_2, \quad \frac{1}{k_1 \alpha_2} \leq \frac{\varphi(t)}{h(t)} \leq \frac{1}{k_2 \alpha_1} \leq 1, \\
(k_1 \alpha_1)^{\lambda} \geq k_1, \quad (k_2 \alpha_2)^{\lambda} \geq k_2.
\left(28\right)
\]

Therefore, from \( \left(H_2\right) \) and since \( (k_1 \alpha_1)^{\lambda} \geq k_1 \), the following relations satisfy

\[
\begin{align*}
&\quad f(t, \mu(t)) = f(t, \frac{\mu(t)}{\varphi(t)} \varphi(t)) \geq \left( \frac{\mu(t)}{\varphi(t)} \right)^{\lambda} f(t, \varphi(t)) \\
&\quad \geq (k_1 \alpha_1)^{\lambda} f(t, \varphi(t)) > k_1 f(t, \varphi(t)), \\
&\quad k_2 f(t, \nu(t)) = f(t, \frac{\varphi(t)}{\nu(t)} \nu(t)) \geq k_2 (\frac{\varphi(t)}{\nu(t)})^{\lambda} f(t, \nu(t)) \\
&\quad \geq k_2 (k_1 \alpha_1)^{\lambda} f(t, \nu(t)) > f(t, \nu(t)).
\end{align*}
\left(29\right)
\]

Consequently

\[
-D_0^\alpha \mu(t) = k_1 f(t, \varphi(t)) \leq f(t, \mu(t)), \quad 0 < t < 1, \, 2 < \alpha \leq 3, \\
-D_0^\alpha \nu(t) = k_2 f(t, \varphi(t)) \geq f(t, \nu(t)), \quad 0 < t < 1, \, 2 < \alpha \leq 3.
\left(30\right)
\]

Since \( \mu(t) = k_1 h(t) \) and \( \nu(t) = k_2 h(t) \) satisfy the boundary conditions, \( \mu(t) \) and \( \nu(t) \) are lower and upper solutions of (4)-(5), respectively. Now, we suppose that

\[
g(t, u(t)) = \begin{cases}
\begin{align*}
f(t, \mu(t)), \quad u(t) \leq \mu(t), \\
f(t, u(t)), \quad \mu(t) \leq u(t) \leq \nu(t), \\
f(t, \nu(t)), \quad u(t) \geq \nu(t),
\end{cases}
\end{align*}
\left(31\right)
\]

and prove that FBVP,

\[
-D_0^\alpha u(t) = g(t, u(t)), \quad 0 < t < 1, \, 2 < \alpha \leq 3, \\
u(0) = u'(0) = 0, \quad D_0^\alpha u(1) = \beta u(\xi),
\left(33\right)
\]

has a solution. Consider operator \( T : C^2[0, 1] \rightarrow C^2[0, 1] \), with \( Tu(t) = \int_0^1 G(t, s) g(s, u(s)) \, ds, \) where \( G(t, s) \) is defined as in Lemma 8. It is easy to see that \( T \) is continuous in \( C^2[0, 1] \). Since \( f \) is nondecreasing in \( u \) (from \( \left(H_1\right) \)), for \( u \in C^2[0, 1] \), we have

\[
f(t, \mu(t)) \leq g(t, u(t)) \leq f(t, \nu(t)), \quad t \in [0, 1].
\left(34\right)
\]

So, there exists a positive constant \( M \), such that \( |g(t, u(t))| \leq M \). We will show that the operator \( T \) is equicontinuous.

**Case 1.** If \( s \leq \xi \),

\[
\begin{align*}
&Tu(t_1) - Tu(t_2) \\
&= \left| \int_0^1 \left[ G(t_2, s) - G(t_1, s) \right] g(s, u(s)) \, ds \right| \\
&\leq \int_0^1 \left| G(t_2, s) - G(t_1, s) \right| g(s, u(s)) \, ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds \\
&\quad + \int_0^1 \left| \Gamma(\alpha) \int_0^s \frac{\beta \xi^{\alpha-1} - \Gamma(\alpha)}{\Gamma(\alpha)} g(s, u(s)) \, ds \right| ds.
\end{align*}
\]
\[
\begin{align*}
&+ \frac{\beta t_2^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_2}^{1} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&- \int_0^{t_1} (t_2 - s)^{\alpha-1} \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \, ds \\
&- \frac{\Gamma(\alpha) t_2^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_2}^{1} g(s, u(s)) \, ds \\
&- \frac{\beta t_1^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{t_2} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&- \frac{\Gamma(\alpha) t_2^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \times \int_{t_1}^{1} g(s, u(s)) \, ds \\
&+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \, ds \\
&+ \frac{\Gamma(\alpha) t_2^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_2}^{1} g(s, u(s)) \, ds \\
&+ \frac{\beta (t_2^{\alpha-1} - t_1^{\alpha-1})}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{t_2} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&+ \frac{\beta t_2^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \times \int_{t_1}^{t_2} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&+ \frac{\Gamma(\alpha) (t_2^{\alpha-1} - t_1^{\alpha-1})}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{1} g(s, u(s)) \, ds \\
&- \frac{\Gamma(\alpha) t_1^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \times \int_{t_1}^{t_2} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&- \frac{\beta t_1^{\alpha-1}}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{t_2} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \\
&+ 2M \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} (t_2 - t_1)^{\alpha} + \frac{M \Gamma(\alpha) (t_2^{\alpha-1} - t_1^{\alpha-1}) t_1}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \\
&+ \frac{M \Gamma(\alpha) t_2^{\alpha-1} (t_2 - t_1)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \\
&+ \frac{M \beta t_2^{\alpha-1} (t_2 - t_1)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \\
&+ \frac{M \Gamma(\alpha) t_2^{\alpha-1} (1 - t_2)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} + \frac{M \Gamma(\alpha) t_2^{\alpha-1} (t_2 - t_1)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \\
&+ \frac{M \beta t_2^{\alpha-1} (1 - t_2)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} + \frac{M \beta t_2^{\alpha-1} (t_2 - t_1)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} .
\end{align*}
\]

Case 2. If \( s \geq \xi \),

\[
[T u(t_1) - T u(t_2)] 
\leq \left| \int_0^{t_1} [G(t_2, s) - G(t_1, s)] g(s, u(s)) \, ds \right|
\]

\[
\leq \left| \int_0^{t_1} \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} + \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{1} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \right] g(s, u(s)) \, ds \right|
\]

\[
\leq \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \left| \int_0^{t_1} \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} + \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{1} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \right] g(s, u(s)) \, ds \right|
\]

\[
\leq \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \left| \int_0^{t_1} \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} + \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{1} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \right] g(s, u(s)) \, ds \right|
\]

\[
\leq \frac{1}{\beta \xi^{\alpha-1} - \Gamma(\alpha)} \left| \int_0^{t_1} \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} + \frac{\Gamma(\alpha)}{[\beta \xi^{\alpha-1} - \Gamma(\alpha)] \Gamma(\alpha)} \int_{t_1}^{1} (\xi - s)^{\alpha-1} g(s, u(s)) \, ds \right] g(s, u(s)) \, ds \right|
\]
\[
\frac{M_\tau \Gamma(2 - \alpha)(t_2 - t_1)}{\Gamma(\alpha - 1)} + \frac{M_\tau \Gamma(1 - \alpha)(t_2 - t_1)}{\Gamma(\alpha - 1)} (1 - t_2) + \frac{M_\tau \Gamma(1 - \alpha)(t_2 - t_1)}{\Gamma(\alpha - 1)}
\]

Therefore, the operator \( T \) is equicontinuous, and by Arzela-Ascoli theorem, \( T \) is a compact operator. Now, the Schauder fixed-point theorem [23] shows that the operator \( T \) has a fixed-point theorem and so FBVP (32)-(33) has a solution. Finally, we will prove that FBVP (4)-(5) has a positive solution. Suppose that \( u_1(t) \) is a solution of FBVP (32)-(33). Since the function \( f \) is nondecreasing in \( u \), we have

\[
f(t, u(t)) \leq g(t, u_1(t)) \leq f(t, v(t)), \quad t \in [0, 1].
\]

Assuming \( X(t) = v(t) - u_1(t) \),

\[
D^\alpha X(t) \geq f(t, v(t)) - g(t, u_1(t)) \geq 0,
\]

\[
X(0) = X'(0) = 0, \quad D^{\alpha - 1} X(1) = \beta X(\xi).
\]

By Lemma 7, \( X(t) \geq 0 \); that is, \( u_1(t) \leq v(t) \) for \( t \in [0, 1] \). Similarly, \( \mu(t) \leq u_1(t) \) for \( t \in [0, 1] \). Therefore \( u_1(t) \) is a positive solution of FBVP (4)-(5). The proof is complete. \( \square \)

**Example 2.** Consider the following fractional boundary value problem:

\[
-D^{3/2}_0 u(t) = f(t, u(t)), \quad 0 < t < 1,
\]

\[
u(0) = u'(0) = 0, \quad D^{\alpha - 1}_0 u(1) = \beta u(\xi),
\]

where

\[
f(t, u(t)) = \sin\left(\frac{\pi t}{2}\right) + \sqrt{u}.
\]

For \( 0 \leq k \leq 1 \), we have \( \sqrt{k} \leq k \). Therefore

\[
k^{1/2} f(t, u(t)) = k^{1/2} \left(\sin\frac{\pi t}{2} + \sqrt{u}\right)
\]

\[
\leq \sin\frac{\pi t}{2} + \sqrt{k} u = f(t, ku(t)).
\]

Now, by Theorem 10, we obtain that the FBVP (39) has a positive solution.

**References**


