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Research Article

Biharmonic Maps and Laguerre Minimal Surfaces

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A Laguerre surface is known to be minimal if and only if its corresponding isotropic map is biharmonic. For every Laguerre surface Φ is its associated surface $\Psi = (1 + |u|^2)\Phi$, where u lies in the unit disk. In this paper, the projection of the surface Ψ associated to a Laguerre minimal surface is shown to be biharmonic. A complete characterization of Ψ is obtained under the assumption that the corresponding isotropic map of the Laguerre minimal surface is harmonic. A sufficient and necessary condition is also derived for Ψ to be a graph. Estimates of the Gaussian curvature to the Laguerre minimal surface are obtained, and several illustrative examples are given.

1. Introduction

Surfaces in $\mathbb{R}^3$ that minimize geometric energies are of great interest to architects because of their stability over other surfaces. These surfaces are used in the design and construction process of certain discrete meshed surfaces such as surfaces covered by special quadrilateral meshes with planar faces and conical meshes [1–3]. Of the many minimal surfaces, the Laguerre minimal surfaces are widely used.

Laguerre minimal surfaces were introduced by Weingarten in 1888 [3–6] and later studied in detail by Blaschke in a series of papers dating from 1924 [4–6].

A Laguerre minimal (L-minimal) surface S is an $\mathbb{R}^3$ surface that minimizes the geometric energy

$$W = \int_S \frac{H^2 - K}{K} dA, \tag{1}$$

where H is the mean curvature and K the Gaussian curvature in the isotropic sense. This will be given more light in Section 2. Of interest to Weingarten and Blaschke was the fact that W is invariant under the group of Laguerre transformations. These are transformations on the space of oriented spheres which preserve oriented contact of spheres and take planes into planes in $\mathbb{R}^3$ [3, 6]. Section 2 will give a brief description of the Laguerre geometry used in this paper.

Our keen interest in the geometric aspect of biharmonic maps [7–13] moved us to study L-minimal surfaces. The link between the two comes from the fact that isotropic models of L-minimal surfaces are described by biharmonic functions [3, 6].

In the sequel, the Laguerre surface (L-surface) is denoted by $\Phi := \Phi(a(x, y), b(x, y), c(x, y))$, x, y real, $\Psi = (1 + x^2 + y^2)\Phi$ is the associated L-surface, and $\Phi^i : z = f(x, y)$ is the corresponding isotropic graph. The surface Ψ is called an associated L-minimal surface when Φ is L-minimal. Note that Ψ itself need not be L-minimal.

In Section 3, we write Ψ in the form

$$\Psi = \Psi(w(u, \overline{u}), C), \tag{2}$$

where $u = x+iy$ lies in the unit disk $\mathbb{U}$. Assuming that Φ is a L-minimal surface, it is shown in Lemma 3 that the projection w and C of Ψ are biharmonic. Additionally, if the isotropic map $\Phi^i : z = f(x, y)$ is harmonic, then C is harmonic and w takes the form

$$w(u, \overline{u}) = -u\overline{G(u)} + H(u), \tag{3}$$

with G analytic and H harmonic.

In Theorem 7, the associated L-minimal surface Ψ is completely characterized when the isotropic map is harmonic. It is shown that Ψ is an associated L-minimal surface if and only if the projection map w is a biharmonic map of the form (3). The associated surface Ψ is given more emphasis than the L-minimal surface because the coordinates of Ψ are either biharmonic or harmonic, and therefore are much easier to handle. We also give in Proposition 9 an estimate for the Gaussian curvature K (isotropic sense) of an L-minimal surface when the function G in (3) is analytic univalent satisfying $G(0) = 0$ and $G'(0) = 1$.

Section 4 considers the case when Ψ is a graph; that is, it is a nonparametric surface. When the function $w(u, \overline{u})$ is univalent and biharmonic, it is shown in Theorem 10 that Ψ is an L-minimal graph. In Theorem 12, Landau's theorem for biharmonic maps [7, 9, 12] is used to find a uniform disk centered at 0 over which Ψ is locally a graph. In Theorem 14, a universal disk is obtained over which Ψ is a graph when $G(u) = F(u^2)$ and F a normalized analytic univalent function. Neither one of the uniform disks described in Theorems 12 and 14 is sharp. Theorem 14 does not hold though over the entire class of normalized analytic univalent functions. Finally, three examples of graphs and local graphs are given to illustrate the results obtained.

Recall that a function F is harmonic [11] if $\Delta F = 0$, and F is biharmonic if $\Delta(\Delta F) = 0$, where

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} \tag{4}$$

is the Laplacian operator. It is easy to show that a mapping F is biharmonic in a simply connected domain $\mathscr{D}$ if and only if F has the representation

$$F = r^2 G + K, \quad re^{i\theta} \in \mathscr{D}, \tag{5}$$

where G and K are complex-valued harmonic functions in $\mathscr{D}$, with

$$G = g_1 + \overline{g_2}, \tag{6}$$

g_1, g_2 being analytic in $\mathscr{D}$ (for details see [7, 8, 10–12]). The Jacobian of a map W is given by

$$J_W = |W_z|^2 - |W_{\overline{z}}|^2. \tag{7}$$

2. Laguerre Geometry

For the sake of completeness, the basic essentials of Laguerre geometry is presented in this section. Additional details may be obtained from the works of [1–6, 14, 15].

2.1. Isotropic Curvature for Graphs. The i-curvature of a regular surface ϕ given by the function $z = f(x, y)$ is the curvature along unit vectors in the xy-plane. It is known that the principle i-curvatures m and M at a point on the surface are the eigenvalues of the Hessian matrix $\nabla^2 f$ given by $\nabla^2 f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$. Hence the i-mean curvature H is given by

$$H = \frac{m + M}{2} = \frac{\Delta f}{2}, \tag{8}$$

where Δf is the Laplacian of f, while the i-Gaussian curvature K is

$$K = mM = \det \nabla^2 f. \tag{9}$$

These curvatures are much easier to deal with compared to the Euclidean curvatures.

2.2. Duality between Surfaces of Graphs. Let ϕ^* be the dual of $\phi : z = f(x, y)$ given by the components of the tangent plane, specifically,

$$x^* = f_x(x, y), \qquad y^* = f_y(x, y), \qquad z^* = xf_x + yf_y - f. \tag{10}$$

If σ is the corresponding map between ϕ and ϕ^*, then σ has an inverse σ^*. Hence, if H^* and K^* are the corresponding i-mean and i-Gaussian curvatures of ϕ^*, then [2, 6]

$$H^* = \frac{H}{K}, \qquad K^* = \frac{1}{K}, \tag{11}$$

where H, K are, respectively, the i-mean and i-Gaussian curvatures of ϕ.

2.3. Laguerre Geometry. In Laguerre geometry, a point on a surface in $\mathbb{R}^3$ is represented by its oriented tangent plane. An oriented plane P is given by

$$n^T \cdot x + h = 0, \tag{12}$$

where n is the unit normal vector. An oriented sphere S, with center m and signed radius R (R can be negative), is tangent to an oriented plane P if the signed distance from the center m to P equals R; that is, $n^T \cdot m + h = R$. Points are viewed as oriented spheres with zero radius. The interested reader is referred to [2, 6] for additional details.

2.4. The Isotropic Image of an Oriented Plane. Let $P : n_1 x + n_2 y + n_3 z + h = 0$ be an oriented plane with unit normal vector $n = (n_1, n_2, n_3)$, and associate P with the point $(n_1, n_2, n_3, h) \in \mathbb{R}^4$. Next replace n with its stereographic image

$$\left(\frac{n_1}{n_3 + 1}, \frac{n_2}{n_3 + 1}, 0\right) \tag{13}$$

under the projection of the unit sphere S^2 from $(0, 0, -1)$ onto the plane $z = 0$. Then the isotropic image P^i of P is defined as

$$P = (n_1, n_2, n_3, h) \longrightarrow P^i = \left(\frac{n_1}{n_3 + 1}, \frac{n_2}{n_3 + 1}, \frac{h}{n_3 + 1}\right). \tag{14}$$

If we let $u = x + iy$ and write $n_1 + in_2 = (2x + i2y)/(1 + x^2 + y^2)$, then $n_3 = (1 - x^2 - y^2)/(1 + x^2 + y^2)$, $n_3 + 1 = 2/(1 + x^2 + y^2)$, and the unit vector n in complex variables becomes

$$n = (n_1, n_2, n_3) = \frac{1}{1 + |u|^2} \left(2 \operatorname{Re} u, 2 \operatorname{Im} u, 1 - |u|^2\right). \tag{15}$$

In this case, (14) becomes

$$P : (u, \overline{u}, h) \longrightarrow P^i : \left(x, y, \frac{\left(1 + |u|^2\right) h}{2}\right). \tag{16}$$

2.5. Laguerre Surface. Let Φ be a Laguerre surface in $\mathbb{R}^3$. Any regular point P on Φ is thus represented as in (16). Denote the corresponding isotropic surface by Φ^i with P^i given by (16). By duality, their corresponding curvatures are related by

$$H^* = \frac{H}{K}, \qquad K^* = \frac{1}{K}. \tag{17}$$

Blaschke [6] defined the middle tangent sphere to be the tangent to the tangent plane P with radius

$$R = \frac{R_1 + R_2}{2}, \tag{18}$$

where $R_1 = 1/m$, $R_2 = 1/M$, and m, M are the principal curvatures of the L-surface Φ. Let Φ_M denote the middle surface consisting of centers of the middle spheres. It is shown in [6] that Φ_M is invariant under Laguerre transformations.

A surface Φ is an L-minimal surface when Φ_M minimizes the area functional

$$\begin{aligned}\Omega &= \frac{1}{4}\int_{\Phi}(R_1 - R_2)^2 K dA \\ &= \int_{\Phi}\frac{H^2 - K}{K} dA = \int_{\Phi^i}\left(H_i^2 - K_i\right) dA_i\end{aligned} \tag{19}$$

(see (8) and (9)). This is also invariant under L-transforms.

If, in (16), P on Φ is given by $h = h(u, \overline{u})$, then [5, 6]

$$\Omega = \frac{i}{2}\iint h_{uu} h_{\overline{uu}} du d\overline{u}, \tag{20}$$

($du d\overline{u} = -2i dx dy$), and when Ω is minimal, then

$$h_{uu\overline{uu}} = 0. \tag{21}$$

The latter implies that h is biharmonic. Assume now that Φ^i is given by the function $z = f(x, y)$. Since $z = h/2$, it follows from (20) that $z = f(x, y)$ is also biharmonic.

This leads to the following result.

Theorem 1 (see [2, 3]). *Let Φ be a Laguerre surface and Φ^i its corresponding isotropic surface related as in* (16). *Suppose Φ^i is given by the function $z = f(x, y)$. Then Φ is minimal if and only if f is biharmonic.*

3. Projection of L-Minimal Surface onto a Plane

In (16), the Laguerre surface Φ is expressed in terms of the Laguerre coordinates $(u, \overline{u}, h)$. In this section, the Euclidean coordinates are used instead. Simple calculations from (16) and use of Theorem 1 lead to the following known result.

Theorem 2 (see [3]). *Let Φ^i be the graph of the biharmonic function $z = f(x, y)$. Then the parametric equations of the corresponding L-minimal surface Φ are given by*

$$\begin{aligned}a &= \frac{1}{1 + x^2 + y^2}\left[\left(x^2 - y^2 - 1\right) f_x + 2xy f_y - 2xf\right] \\ b &= \frac{1}{1 + x^2 + y^2}\left[\left(y^2 - x^2 - 1\right) f_y + 2xy f_x - 2yf\right] \\ c &= \frac{1}{1 + x^2 + y^2}\left[2x f_x + 2y f_y - 2f\right].\end{aligned} \tag{22}$$

Throughout this section, it is assumed that the L-minimal surface is parametrized by $u = x + iy$, with u in the unit disk $\mathbb{U}$.

Equations (22) will first be written in terms of complex variables. For this purpose, let

$$v = a + ib, \qquad w = \left(1 + |u|^2\right) v, \qquad C = \left(1 + |u|^2\right) c. \tag{23}$$

Then from (22), the projection of the surface $v = v(u)$ and the height c becomes

$$\begin{aligned}v &= \frac{2\left[u\left(u f_u - f\right) - f_{\overline{u}}\right]}{1 + |u|^2}, \\ c &= \frac{2\left[u f_u - f + \overline{u} f_{\overline{u}}\right]}{1 + |u|^2},\end{aligned} \tag{24}$$

and the coordinates of Ψ (see (2)) are

$$\begin{aligned}w(u) &= 2\left[u\left(u f_u - f\right) - f_{\overline{u}}\right], \\ C &= 2\left[u f_u - f + \overline{u} f_{\overline{u}}\right].\end{aligned} \tag{25}$$

Evidently,

$$\begin{aligned}w_{u\overline{u}} &= 2u f_{u\overline{u}} + 2u^2 f_{uu\overline{u}} - 2 f_{\overline{uu}u} - 2 f_{\overline{u}}, \\ C_{u\overline{u}} &= 2 f_{u\overline{u}} + 2u f_{uu\overline{u}} + 2\overline{u} f_{\overline{uu}u}.\end{aligned} \tag{26}$$

The relations (25) and (26) yield the following general lemma.

Lemma 3. *(a) If f is biharmonic in $\mathbb{U}$, then w and C are biharmonic.*

(b) If $f = (G + \overline{G})/2$ is harmonic and G analytic, then C is harmonic and $w = -u\overline{G} + H$, where the harmonic function H is given by

$$H(u) = u^2 G'(u) - uG(u) - \overline{G'(u)}. \tag{27}$$

Proof. (a) If f is biharmonic in $\mathbb{U}$, then

$$f(u) = |u|^2 g(u) + h(u), \tag{28}$$

where h, g are harmonic. It follows directly by differentiation that

$$\begin{aligned}f_{\overline{u}} &= ug + |u|^2 g_{\overline{u}} + h_{\overline{u}}, \\ u f_{u\overline{u}} &= u\left(g + u g_u + \overline{u} g_{\overline{u}} + h_{\overline{u}u} + u\overline{u} g_{\overline{u}u}\right).\end{aligned} \tag{29}$$

Then

$$\begin{aligned} uf_{u\overline{u}} - f_{\overline{u}} &= u^2 g_u + uh_{\overline{u}\overline{u}} + u^2\overline{u}g_{\overline{u}\overline{u}} - h_{\overline{u}} \\ &= u^2 g_u - h_{\overline{u}} \end{aligned} \tag{30}$$

in light that g and h are harmonic. Since f is biharmonic, (26) shows that $w_{u\overline{u}}$ is harmonic. That f is biharmonic also implies that the leading three terms in the right-hand side of the second equation of (26) are harmonic. Hence $C_{u\overline{u}}$ is harmonic, which proves part (a).

(b) Substituting $f = (G + \overline{G})/2$ into (25) yields

$$w(u) = u^2 G' - uG - u\overline{G} - \overline{G}'. \tag{31}$$

Also the second equation of (25) shows that C is harmonic. □

Lemma 3 gives a structural connection between an L-minimal surface with its projection map, in other words, a connection between the surface $\Psi : (\operatorname{Re} w, \operatorname{Im} w, C)$ and w (see (2), (22), and (25)).

Corollary 4. *(a) If Ψ is an associated L-minimal surface parametrized by the unit disk $|u| < 1$, then the projections w and C are biharmonic.*

(b) If Ψ is an associated L-minimal surface parametrized by the unit disk $|u| < 1$ and the corresponding isotropic surface is given by a harmonic function $z = f(u)$, then $w = -u\overline{G} + H$, where G is analytic, H harmonic satisfying (27), and C is harmonic.

Moving in the opposite direction is the following lemma.

Lemma 5. *If $w(u) = -u\overline{G} + H$, $u \in \mathbb{U}$, where G is analytic and H harmonic is given by (27), then the equation w in (25) has a harmonic solution f satisfying $f = \operatorname{Re} G$.*

Proof. Comparing w as given above and w in (25), it follows that

$$-u\overline{G} + H = 2\left[u\left(uf_u - f\right) - f_{\overline{u}}\right]. \tag{32}$$

Assume that the solution of (26) is harmonic. Differentiating both sides of the above equation leads to

$$w_{u\overline{u}} = -\overline{G}' = 2uf_{u\overline{u}} + 2u^2 f_{uu\overline{u}} - 2f_{\overline{u}\overline{u}u} - 2f_{\overline{u}} = -2f_{\overline{u}}. \tag{33}$$

The result now follows directly by integration, and the resulting f clearly satisfies the conclusion. □

The following corollary is obtained from Lemma 3 and (25).

Corollary 6. *If w is given by Lemma 3(b), then*

$$C(u) = -2\operatorname{Re}\left[\int_0^u uG''(u)\,du\right]. \tag{34}$$

Combining the above lemmas and corollaries results in the following characterization of minimal surfaces with harmonic isotropic maps.

Theorem 7. *Let Φ be a Laguerre surface. A surface Ψ is an associated L-minimal surface with a harmonic isotropic map Φ^i if and only if Ψ is given by*

$$\Psi : (\operatorname{Re} w(u), \operatorname{Im} w(u), C), \tag{35}$$

where w is a biharmonic map given by (27), and C by (34) is harmonic in $\mathbb{U}$.

The i-Gauss curvature of an L-minimal surface in terms of the projection map can be obtained from Lemma 3.

Corollary 8. *Let w be given as in Lemma 3(b). Then the i-Gauss curvature of the L-minimal surface is*

$$K = -\frac{2}{\left|G''(u)\right|^2} = -\frac{2|u|^2}{\left|C_u\right|^2}. \tag{36}$$

Proof. By (9), $K_i = f_{xx}f_{yy} - (f_{xy})^2 = -(f_{xx}^2 + f_{xy}^2) = -2(f_{uu}f_{\overline{u}\overline{u}}) = -|G''(u)|^2/2$. Now $K = 1/K_i$ and Corollary 4 gives both equalities. □

We conclude this section with an estimate for K when G belongs to the class $\mathcal{S}$ consisting of univalent analytic functions F in $\mathbb{U}$ normalized by $F(0) = 0$ and $F'(0) = 1$.

Proposition 9. *Let $G \in \mathcal{S}$ and the corresponding associated L-minimal surface Ψ be given as in Theorem 7. Then*

$$\begin{aligned} -\infty &\leq K(u) \leq -\frac{(1-|u|)^8}{2(2+|u|)^2}, \\ -\infty &\leq K(0) \leq -\frac{1}{8}. \end{aligned} \tag{37}$$

Proof. It is known [16, p. 21] that for $G \in \mathcal{S}$,

$$\begin{aligned} \left|\frac{uG''(u)}{G'(u)}\right| &\leq \frac{4|u| + 2|u|^2}{1-|u|^2}, \\ \left|G'(u)\right| &\leq \frac{1+|u|}{(1-|u|)^3}. \end{aligned} \tag{38}$$

Thus

$$\left|G''(u)\right| \leq \frac{2(2+|u|)}{(1-|u|)^4}, \tag{39}$$

which leads to the desired inequalities. □

4. The Associated L-Surface Is a Graph

This section looks at the case when Ψ is a graph; that is, when Ψ is a nonparametric surface. Interestingly, the graph of the associated L-minimal surface is closely connected to its corresponding projection map w.

Figure 1 makes this relationship evident and gives rise to the following theorem.

Theorem 10. *An associated L-surface parametrized by the unit disk $|u| < 1$ is a graph if and only if w is a univalent biharmonic map.*

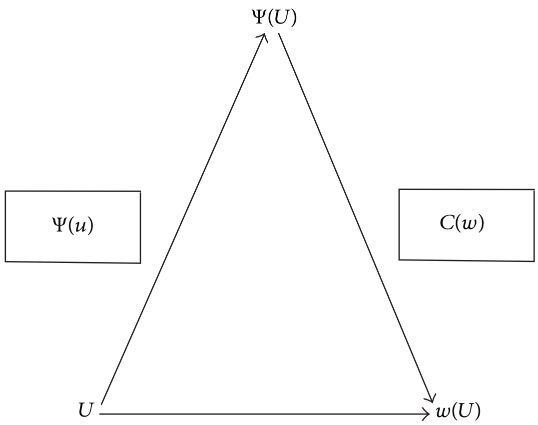

Figure 1: Projection map of a Laguerre surface.

The following lemma gives the derivatives of the function $C = C(w)$ of the graph surface.

Lemma 11. *Let w and f be given as in Lemma 5. If w is univalent with Jacobian $J_w(u) = |-\overline{G}+H_u|^2 - |-u\overline{G}' + H_{\overline{u}}|^2 > 0$ for all u, then*

$$C_w = \frac{\overline{w_u}C_u - \overline{w_{\overline{u}}}C_{\overline{u}}}{J_w(u)}$$
$$C_{\overline{w}} = \frac{w_u C_{\overline{u}} - w_{\overline{u}}C_u}{J_w(u)}. \tag{40}$$

Proof. Differentiating $C = C(w)$ leads to

$$C_u = C_w w_u + C_{\overline{w}}\overline{w}_u,$$
$$C_{\overline{u}} = C_w w_{\overline{u}} + C_{\overline{w}}\overline{w}_{\overline{u}}. \tag{41}$$

Since $\overline{w}_{\overline{u}} = \overline{w_u}$ and $\overline{w}_u = \overline{w_{\overline{u}}}$, it follows that

$$C_u = C_w w_u + C_{\overline{w}}\overline{w_{\overline{u}}}$$
$$C_{\overline{u}} = C_w w_{\overline{u}} + C_{\overline{w}}\overline{w_u}. \tag{42}$$

Solving the linear system gives the desired results. □

Now (27) implies that

$$w(u) = u^2G' - uG - u\overline{G} - \overline{G}', \tag{43}$$

and subsequently,

$$w_{\overline{u}} = -\overline{G}'' - u\overline{G}'$$
$$w_u = 2uG' + u^2G'' - G - uG' - \overline{G} \tag{44}$$
$$= u^2G'' + uG' - 2\operatorname{Re} G.$$

We next present a theorem about the surfaces Ψ and Φ which is a consequence of Landau's theorem for biharmonic maps. This was first proved in [7] and the universal constant was later sharpened in [9, 12]. This theorem will also help provide examples of graphs for L-surfaces.

Theorem 12. *Let Ψ be a surface given by $w(u) = -u\overline{G} + H$, $u \in \mathbb{U}$, where G is analytic and H harmonic is given by (27). If $|G'(u)|$ is bounded by a constant M, $w(0) = 0$ and $G''(0) = 1$, then there are uniform constants $\rho(M) > 0$ and $R(M) > 0$ so that w and $v(u) = w(u)/(1+|u|^2)$ are univalent on the disk $|u| < \rho$, and the image of this disk contains a disk $|w| < R$ on which the surfaces are graphs.*

Proof. The Jacobian of w is given by

$$J_w(u) = |w_u(u)|^2 - |w_{\overline{u}}(u)|^2. \tag{45}$$

It follows from (44) that $J_w(0) = -|G''(0)|^2 = -1$. If $|G'(u)|$ is bounded, then $|G(u)|$ and consequently $|w(u)|$ is bounded. It can now be deduced from Theorem 1 in [7, 9, 12] that there are uniform constants $\rho_1(M) > 0$ and $R_1(M) > 0$ so that w is univalent on the disk $|u| < \rho_1$ whose image contains the disk $|w| < R_1$. Consequently the surface Ψ is a graph above such a disk.

Clearly v is univalent on each circle $|u| = r < \rho_1$. Suppose now that there are u_1 and u_2 with $|u_1| < |u_2| < \rho_1$ so that $v(u_1) = v(u_2)$. Then

$$\frac{|w(u_1) - w(u_2)|}{|u_1 - u_2|} = \frac{|w(u_2)|\left(|u_2|^2 - |u_1|^2\right)}{|u_1 - u_2|\left(1 + |u_2|^2\right)}$$
$$\leq \frac{2|w(u_2)||u_2|}{\left(1 + |u_2|^2\right)} \leq |w(u_2)|. \tag{46}$$

But it was shown in the proof of Theorem 1 in [7, 9, 12] that

$$\frac{|w(u_1) - w(u_2)|}{|u_1 - u_2|} \geq \frac{\pi}{4M} - o(\rho_1). \tag{47}$$

Hence

$$|w(u_2)| \geq \frac{\pi}{4M} - o(\rho_1). \tag{48}$$

As $w(0) = 0$, choose $\rho < \rho_1$ so that $|w(u_2)| < \pi/8M$ and the result follows. □

Corollary 13. *Let w, G, ρ be given as in Theorem 12. Then $w(\rho u)$ is univalent in $\mathbb{U}$ and the corresponding surface Ψ is a graph.*

Interestingly when $G(u) = F(u^2)$, $F \in \mathcal{S}$, a similar result is obtained without imposing the boundedness condition. Recall that $\mathcal{S}$ is the class of univalent analytic functions F normalized by $F(0) = 0$ and $F'(0) = 1$.

Theorem 14. *Let $G(u) = F(u^2)$, $F \in \mathcal{S}$, $C_1(\rho)$, $C_2(\rho)$ polynomials with positive coefficients of degree 2 and 3, respectively (see (51) and (53)), and $R_2(\rho_2)$ described by (56). Then there are two uniform radii $\rho_2 > 0$ and $R_2(\rho_2)$ satisfying*

$$\frac{\rho_2^2\left(C_1\left(\rho_2^2\right) + C_2\left(\rho_2^2\right)\right)}{|1 - \rho_2^2|^4} = 2, \tag{49}$$

so that the corresponding w and v are univalent in $|u| < \rho_2$, and the image of w contains a disk $|w + 2| < R_2$. In this case, the surfaces Ψ and Φ are graphs on $|w + 2| < R_2$.

Proof. Now $G'(u) = 2uF'(u^2)$ and $G''(u) = 2F'(u^2) + 4u^2F''(u^2)$. It follows from (44) that $w_{\overline{u}}(0) = -2$, $w_u(0) = 0$, and $J_w(0) = -4$. The distortion estimates for the class $\mathcal{S}$ [16, p. 21] are

$$|F(z)| \leq \frac{|z|}{(1-|z|)^2}, \qquad |F'(z)| \leq \frac{1+|z|}{(1-|z|)^3}, \qquad |F''(z)| \leq \frac{c}{(1-|z|)^4}. \tag{50}$$

From (44), these inequalities imply that

$$|w_u(u)| \leq \frac{|u|^2 C_1(|u|^2)}{(1-|u|^2)^4}, \tag{51}$$

where $C_1(x)$ can be chosen as a polynomial of degree 2 with positive coefficients and $C_1(0) = 6$.

The distortion inequality also implies that

$$|F'(z) - 1| \leq \frac{4|z| - 3|z|^2 + |z|^3}{(1-|z|)^3}. \tag{52}$$

The latter inequality together with the distortion inequalities imply that

$$\begin{aligned} |w_{\overline{u}}(u) + 2| &= \left|-2\left(F'(u^2) - 1\right) - 4u^2F''(u^2) - 2u^2F'(u^2)\right| \\ &\leq \frac{|u|^2 C_2(|u|^2)}{(1-|u|^2)^4}, \end{aligned} \tag{53}$$

where $C_2(x)$ is taken to be a polynomial with positive coefficients of degree 3 and $C_2(0) = 4c + 10$.

Let u_1, u_2 be two points near 0 in a disk $|u| < \rho$. It follows from (51) and (53) that

$$\begin{aligned} |w(u_1) - w(u_2)| &= \left|\int_{u_1}^{u_2} (w_u(u)\,du + w_{\overline{u}}(u)\,d\overline{u})\right| \\ &= \left|\int_{u_1}^{u_2} (w_u(0)\,du + w_{\overline{u}}(0)\,d\overline{u}) + \int_{u_1}^{u_2} (w_u(u) - w_u(0))\,du + \int_{u_1}^{u_2} (w_{\overline{u}}(u) - w_{\overline{u}}(0))\,d\overline{u}\right| \\ &\geq 2|u_1 - u_2| - \int_{u_1}^{u_2} |w_u(u)|\,|du| - \int_{u_1}^{u_2} |w_{\overline{u}}(u) + 2|\,|d\overline{u}| \\ &\geq 2|u_1 - u_2| - |u_1 - u_2| \frac{\rho^2 C_1(\rho^2)}{(1-\rho^2)^4} - |u_1 - u_2| \frac{\rho^2 C_2(\rho^2)}{(1-\rho^2)^4} \\ &\geq |u_1 - u_2| \left[2 - \frac{\rho^2 (C_1(\rho^2) + C_2(\rho^2))}{|1-\rho^2|^4}\right]. \end{aligned} \tag{54}$$

Now choose ρ_2 so that $\rho^2(C_1(\rho^2) + C_2(\rho^2))/|1-\rho^2|^4 = 2$ to deduce that w is univalent in $|u| < \rho_2$.

Let $\delta > 0$ satisfy

$$\frac{\delta^2 (C_1(\delta^2) + C_2(\delta^2))}{|1-\delta^2|^4} = 1, \tag{55}$$

and let $\gamma_\rho = w(|u| = \rho)$. Then the distance $d(w(0), \partial\gamma_{\rho_2}) \geq d(w(0), \partial\gamma_\delta)$. If we choose $u_2 = 0$ and $u_1 \in \gamma_\delta$ in (54), then

$$d(w(0), \partial\gamma_{\rho_2}) \geq d(w(0), \partial\gamma_\delta) \geq \delta. \tag{56}$$

Thus choose $R_2(\rho_2) = \delta$.

An argument similar to the proof of Theorem 1 in [7, 9, 12] gives the result for v and consequently for Ψ and Φ. □

Corollary 15. *Let $F \in \mathcal{S}$ and ρ_2 be given by Theorem 14. If $G(u) = F((\rho_2 u)^2)$, then the corresponding associated L-minimal surface Ψ is a graph.*

Remark 16. (1) A result similar to Theorem 14 can be obtained for the L-surface $\Phi = \Psi/(1+u^2)$.

(2) Theorem 14 is not true for the class $\mathcal{S}$. The following proposition shows that there is no uniform disk on which the surface Ψ is a graph for all $G \in \mathcal{S}$.

Proposition 17. *Let $\mathcal{F}$ be the set of all convex univalent functions G given by*

$$G(u) = \int_0^u \frac{dz}{(1-xz)^{2t}(1-yz)^{2(1-t)}}, \tag{57}$$

where $0 \leq t \leq 1$, $|x| = 1$ and $|y| = 1$, and let w be the corresponding biharmonic map given by (27).

(a) *There is no uniform disk centered at 0 where $J_w(u) \leq 0$.*

(b) *There is no uniform disk on which w is univalent and, consequently, no uniform disk on which Ψ is a graph.*

Proof. For $G \in \mathcal{F}$,

$$\frac{uG''}{G'} + 1 = t\frac{1+xu}{1-xu} + (1-t)\frac{1+yu}{1-yu}. \tag{58}$$

First we show that $w_{\overline{u}}(u) = 0$, for any $u \in \mathbb{U}\backslash\{0\}$, and for the choices $t = 1/2$, x, y satisfying $yu = \overline{xu}$ with xu, yu being the intersection points between the circles $|z| = |u|$ and $|1-z| = 1$.

These conditions imply that $|1-xu| = |1-yu| = 1$. In this case, (58) gives

$$\begin{aligned}
\frac{uG''}{G'} + |u|^2 &= \frac{1}{2}\frac{1+xu}{1-xu} + \frac{1}{2}\frac{1+yu}{1-yu} - \left(1-|u|^2\right), \\
\operatorname{Re}\left(\frac{uG''}{G'} + |u|^2\right) &= \frac{1}{2}\frac{1-|u|^2}{|1-xu|^2} + \frac{1}{2}\frac{1-|u|^2}{|1-yu|^2} \\
&\quad - \left(1-|u|^2\right) \\
&= \frac{1}{2}\left(1-|u|^2\right) + \frac{1}{2}\left(1-|u|^2\right) \\
&\quad - \left(1-|u|^2\right) = 0, \\
\operatorname{Im}\left(\frac{uG''}{G'} + |u|^2\right) &= 0.
\end{aligned} \tag{59}$$

Hence (44) becomes

$$\begin{aligned}
w_{\overline{u}}(u) &= -\overline{G}'' - u\overline{G}' \\
&= \frac{-\overline{G}'}{\overline{u}}\left(\overline{\frac{uG''}{G'}} + |u|^2\right) \\
&= 0.
\end{aligned} \tag{60}$$

For the above choices of u, x, y, and t, we next show that $w_u \neq 0$. From (44)

$$\begin{aligned}
w_u(u) &= u^2 G'' + uG' - 2\operatorname{Re} G \\
&= uG'\left(\frac{uG''}{G'} + 1\right) - 2\operatorname{Re} G.
\end{aligned} \tag{61}$$

However, (57) and (58) give

$$\begin{aligned}
G'(u) &= \frac{1}{(1-xu)(1-yu)} = \frac{1}{|1-xu|^2}, \\
\frac{uG''}{G'} + 1 &= \operatorname{Re}\frac{1+xu}{1-xu}, \\
\operatorname{Re} G(u) &= \operatorname{Re}\int_0^u \frac{dz}{(1-xz)(1-yz)} = \int_0^1 \frac{\operatorname{Re} u\,dt}{|1-xtu|^2}.
\end{aligned} \tag{62}$$

Since $|1-xu| = 1$, it is geometrically clear that $|1-xtu| \leq 1$ and consequently $\operatorname{Re} G/\operatorname{Re} u > 1$. Hence (61) becomes

$$\begin{aligned}
w_u(u) &= u\frac{1-|u|^2}{|1-xu|^4} - \int_0^1 \frac{2\operatorname{Re} u\,dt}{|1-xtu|^2} \\
&= u\left(1-|u|^2\right) - \int_0^1 \frac{2\operatorname{Re} u\,dt}{|1-xtu|^2}.
\end{aligned} \tag{63}$$

If the last expression is zero, then u should be chosen real satisfying $u = \operatorname{Re} u$ and $1-|u|^2 = \int_0^1 (2dt/|1-xtu|^2) > 2$. Since this is impossible, we conclude, for arbitrary $u \neq 0$ and with the above choices of x, y and $t = 1/2$, that $w_u(u) \neq 0$ and consequently $J_w(u) > 0$. In general, it follows from (44) that $J_w(0) = -|G''(0)|^2$, and this is negative for certain choices of G, especially for $G(u) = u/(1-u)$. Hence there is no uniform disk for the family on which $J_w(u) < 0$. This completes the proof of part (a).

For the proof of part (b), choose $u = -i|u|$, $x = 1$, $y = -1$, and $t = 1/2$. From (57), and with the present choices, we conclude that

$$\begin{aligned}
G'(-i|u|) &= \frac{1}{(1+|u|i)(1-|u|i)} = \frac{1}{1+|u|^2}, \\
\operatorname{Re} G(-i|u|) &= \operatorname{Re}\int_0^{-|u|i} \frac{dz}{(1+z)(1-z)} \\
&= \operatorname{Re}\int_0^1 \frac{-|u|\,i\,dt}{(1+t|u|i)(1-t|u|i)} = 0.
\end{aligned} \tag{64}$$

It is clear from (27) that

$$w(-|u|i) = \frac{-|u|^2}{1+|u|^2} - \frac{1}{1+|u|^2} = w(|u|i). \tag{65}$$

Hence w is not univalent near 0. □

We conclude our exposition with several examples.

Example 18. Let $G(u) = u$. Then $w(u) = -1-|u|^2$ and $|w_{\overline{u}}/w_u| = 1$. Hence the surface Ψ degenerates.

Example 19. Choose $G(u) = u^2$. Then $w(u) = u^3 - |u|^2\overline{u} - 2\overline{u}$. From (44),

$$\left|\frac{w_u}{w_{\overline{u}}}\right| = \left|\frac{3u^2 - \overline{u}^2}{-2-2|u|^2}\right| = \frac{3|u|^2}{2\left(1+|u|^2\right)}\left|1 - \frac{1}{3}\frac{\overline{u}^2}{u^2}\right| < 1, \tag{66}$$

and thus the corresponding w is locally one-to-one in $\mathbb{U}$.

When $u = e^{it}$, $w(u) = u^3 - 3\overline{u} = e^{3it} - 3e^{-it}$,

$$\begin{aligned}
\operatorname{Im}\frac{dw/dt}{w(t)} &= 3\operatorname{Re}\frac{e^{3it} + e^{-it}}{e^{3it} - 3e^{-it}} \\
&= 3\operatorname{Re}\frac{e^{4it} + 1}{e^{4it} - 3} \\
&= -6\frac{1+\cos(4t)}{\left|3 - e^{4it}\right|^2} \leq 0.
\end{aligned} \tag{67}$$

Hence w is univalent on $|u| = 1$, and since $J_w(u) \neq 0$ in $\mathbb{U}$, it must be univalent in $\mathbb{U}$. Figures 2 and 3 show that the associated L-surface Ψ and the corresponding L-minimal surface Φ are total graphs.

Example 20. Let $G(u) = u/(1-xu)$, $|x| = 1$. It follows from (44) that

$$\begin{aligned}
w_u &= \frac{2xu^2}{(1-xu)^3} + \frac{u}{(1-xu)^2} - 2\operatorname{Re}\frac{u}{1-xu}, \\
\overline{w_{\overline{u}}} &= -\frac{2x}{(1-xu)^3} - \frac{\overline{u}}{(1-xu)^2}.
\end{aligned} \tag{68}$$

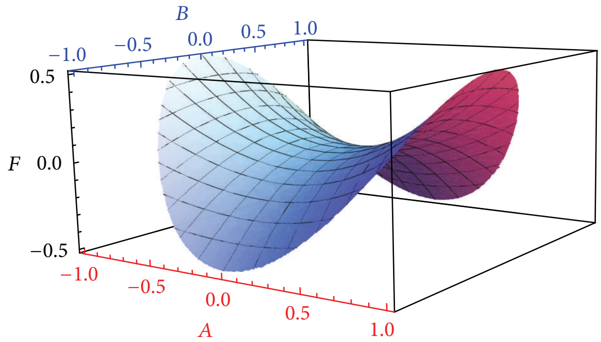

Figure 2: The associated L-surface Ψ when $G(u) = u^2$.

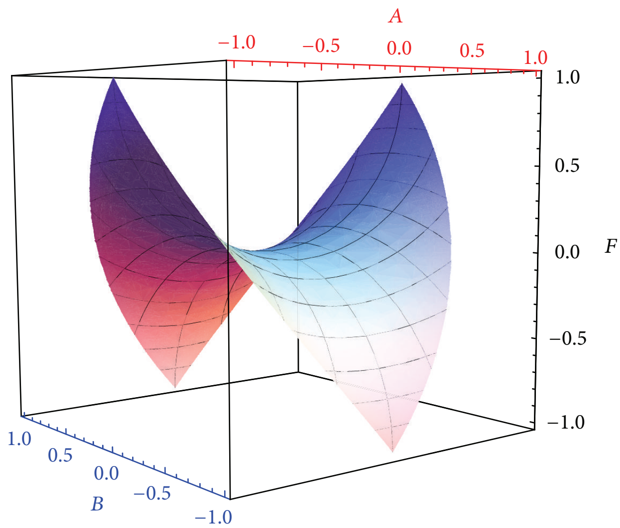

Figure 3: The L-minimal surface Φ when $G(u) = u^2$.

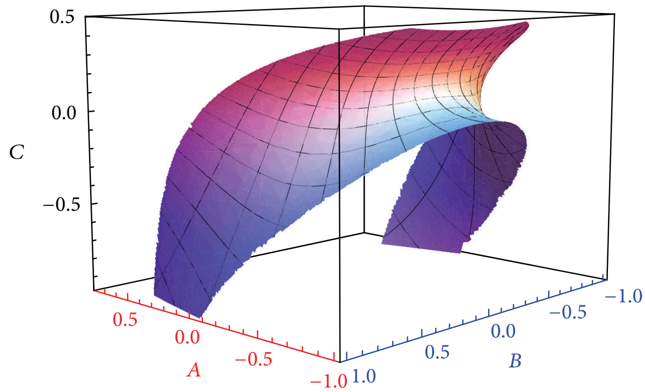

Figure 4: The associated L-surface Ψ when $G(u) = u/(1 - u)$.

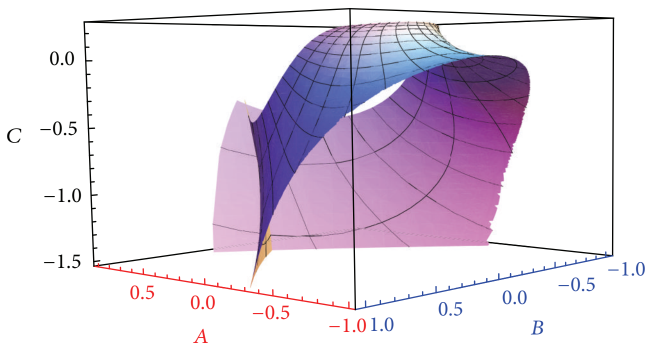

Figure 5: The L-surface Φ when $G(u) = u/(1 - u)$.

Consequently

$$\left|\frac{w_u}{w_{\overline{u}}}\right| \le \frac{|u|^2 + |u| + 2|u|(1 + |u|)^2}{(1 - |u|)(2 + |u|)}. \quad (69)$$

The value of this expression ranges between 0 at 0 and ∞ at $|u| = 1$. Hence placing it less than 1 and solving for $|u|$ give a uniform disk $|u| < 0.32471$ for all x. The corresponding w is then locally univalent, with $J_w(u) < 0$ in $|u| < 0.32471$.

Note that in the case $G(u) = u/(1+u)$, $w_u \to -1$, $w_{\overline{u}} \to 0$ when $u \to 1$. Hence $J_w \to 1$. This implies that w may not be locally univalent in all of $\mathbb{U}$. Figures 4 and 5 show that neither the associated L-surface Ψ nor the corresponding L-surface Φ is a total graph.

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