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Research Article

Note on a q-Contour Integral Formula of Gasper-Rahman

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We use the q-Chu-Vandermonde formula and transformation technique to derive a more general q-integral equation given by Gasper and Rahman, which involves the Cauchy polynomial. In addition, some applications of the general formula are presented in this paper.

1. Introduction and Main Result

It is well known that the q-integral is an important branch of q-series theory. There are many techniques to achieve the ends; for instance, combinatorics method (cf. [1]), analysis methods (cf. [2–4]), and method of transformation (cf. [5–7]) are usually used. In 1989, Gasper and Rahman applied some analysis techniques to derive the following q-contour integral formula (cf. [8, Equation (3.17)]):

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^{m_0}; q)_{m_0} \times \prod_{i=1}^{r}(a_i z; q)_{m_i}\frac{dz}{z} = \frac{(\gamma/\alpha, \alpha q/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty}(q/b\gamma q^{m_0}; q)_{m_0}(b\alpha)^{\sum_{i=0}^{r} m_i} \times \prod_{i=1}^{r}(a_i/b; q)_{m_i}. \tag{1}$$

Inspired by [7, 8], we employ the above equation and transformation technique to derive a more general q-contour integral equation. The main result of this paper is stated as follows.

Theorem 1. *If $m_0, m_1, \ldots, m_r$, and h are nonnegative integers and $q = a\gamma q^{\sum_{i=0}^{r} m_i}$, then*

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^{m_0}; q)_{m_0} \times \prod_{i=1}^{r}(a_i z; q)_{m_i}\prod_{j=1}^{h+1}P_{n_j}\left(1/z; d_j\right)\frac{dz}{z} = \frac{\prod_{l=0}^{h}(\alpha d_l; q)_{n_l}}{\alpha^{\sum_{i=0}^{h} n_i}}\frac{(\gamma/\alpha, q\alpha/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty} \times (q/b\gamma q^{m_0}; q)_{m_0}(b\alpha)^{\sum_{i=0}^{r} m_i} \times \prod_{j=1}^{r}\left(a_j/b; q\right)_{m_j} \times \prod_{i=0}^{h}\sum_{k_i=0}^{n_i}\frac{\left(q^{-n_i}, q^{A_i}b\alpha, q^{A_i+n_{i+1}}d_{i+1}\alpha, \ldots, q^{A_i+n_h}d_h\alpha; q\right)_{k_i}}{\left(q, q^{A_i}d_i\alpha, q^{A_i}d_{i+1}\alpha, \ldots, q^{A_i}d_h\alpha; q\right)_{k_i}} \times q^{k_i(1-\sum_{j=i+1}^{h} n_j)}, \tag{2}$$

provided that $|\gamma/\alpha| < 1$ and C is a deformation of the unit circle so that the poles of $1/(az, bz; q)_\infty$ lie outside the contour and the origin and the poles of $1/(\alpha/z; q)_\infty$ lie inside the contour.

Where $P_n(a;b)$ denotes the Cauchy polynomial defined as (7), one denotes that $A_i = \sum_{j=0}^{i-1} k_j$, and when $i = 0$, one sets one $A_0 = \sum_{j=0}^{i-1} k_j = 0$.

2. Notations and Lemmas

We adopt the custom notations given in [9]. It is supposed that $0 < |q| < 1$ in this paper. We use N to denote the set of all nonnegative integers.

For any complex parameter a, the q-shifted factorials are defined as

$$(a;q)_0 = 1, (a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad n = 1,2,\ldots,$$
$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right). \tag{3}$$

For brevity, we also use the following notation:

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n. \tag{4}$$

The q-binomial coefficient and the q-binomial theorem are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$
$$\sum_{n=0}^{\infty}\frac{(a;q)_n x^n}{(q;q)_n} = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{5}$$

The basic hypergeometric series ${}_s\Phi_t$ is given by

$${}_s\Phi_t\left(\begin{matrix} a_1,a_2,\ldots,a_s \\ b_1,b_2,\ldots,b_t \end{matrix};q,x\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2\ldots,a_s;q)_k}{(q,b_1,\ldots,b_t;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+t-s} x^k. \tag{6}$$

In this paper, we denote that $\binom{n}{2} = n(n-1)/2$ and $k, m, n, s, t \in N$.

Let a, b be any complex variables; then, the Cauchy polynomial $P_n(a;b)$ is defined as

$$P_0(a;b) = 1, \quad P_n(a;b) = (a-b)(a-bq)\cdots\left(a-bq^{n-1}\right), \quad n \geq 1. \tag{7}$$

Recall that q-Chu-Vandermonde's identity (cf. [9, page 14, Equation (1.5.3)]) is given as follows:

$${}_2\Phi_1\left(\begin{matrix} q^{-n}, & a \\ & f \end{matrix};q,q\right) = \frac{a^n(f/a;q)_n}{(f;q)_n}. \tag{8}$$

As we know, it is one of the fundamental formulas in the basic hypergeometric series. Some applications of it were introduced in [5, 10, 11]. We will apply this identity to start our proof in the following. Since we assume that the integrals are the same established condition as the theorem, we omit the condition in the following.

Lemma 2. *One has*

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0} \times \prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{P_n(1/z;f)}{z}dz$$
$$= \frac{(f\alpha;q)_n}{\alpha^n}\frac{(\gamma/\alpha,\alpha q/\gamma,qb/a;q)_\infty}{(a\alpha,q/a\alpha,b\alpha;q)_\infty}(q/b\gamma q^{m_0};q)_{m_0} \times (b\alpha)^{\sum_{i=0}^{r} m_i}\prod_{i=1}^{r}(a_i/b;q)_{m_i} \times \sum_{k=0}^{n}\frac{(q^{-n},b\alpha;q)_k\left(a\gamma q^{\sum_{i=0}^{r} m_i}\right)^k}{(q,f\alpha;q)_k}. \tag{9}$$

Proof. We rewrite (8) as follows:

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q,f;q)_k}q^k\frac{1}{(aq^k;q)_\infty} = \frac{a^n(f/a;q)_n}{(f;q)_n(a;q)_\infty}. \tag{10}$$

Replacing (a,c) by $(\alpha/z, f\alpha)$, respectively, we have

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q,f\alpha;q)_k}q^k\frac{1}{(q^k\alpha/z;q)_\infty} = \frac{\alpha^n}{(f\alpha;q)_n}\frac{P_n(1/z;f)}{(\alpha/z;q)_\infty}. \tag{11}$$

Both sides of (11) multiply by

$$\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{1}{z}. \tag{12}$$

Then, we have

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,f\alpha;q)_k}\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(q^k\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0} \times \prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{1}{z}$$
$$= \frac{\alpha^n}{(f\alpha;q)_n}\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0} \times \prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{P_n(1/z;f)}{z}. \tag{13}$$

Taking the q-integral on both sides of (13) with respect to variable z, we get

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,f\alpha;q)\,k}\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(q^k\alpha/z,az,bz;q)_\infty} \times (qz/\gamma q^{m_0};q)_{m_0}\prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{1}{z}dz$$
$$= \frac{\alpha^n}{(f\alpha;q)_n}\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty} \times (qz/\gamma q^{m_0};q)_{m_0}\prod_{i=1}^{r}(a_i z;q)_{m_i}\frac{P_n(1/z;f)}{z}dz. \tag{14}$$

Employing (1) to the left side of (14), we have the desired result after some simplification. □

On the other hand, if we multiply (13) by $P_{n_1}(1/z;g)$, we have

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,f\alpha;q)_k}\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(q^k\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\times\prod_{i=1}^{r}(a_iz;q)_{m_i}\frac{P_{n_1}(1/z;g)}{z}$$
$$=\frac{\alpha^n}{(f\alpha;q)_n}\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\times\prod_{i=1}^{r}(a_iz;q)_{m_i}\frac{P_{n_1}(1/z;g)P_n(1/z;f)}{z}. \quad (15)$$

Taking the q-integral on both sides of (15) with respect to variable z, we use (9) in the resulting equation. After simple rearrangements, noting that $q=a\gamma q^{\sum_{i=0}^{r}m_i}$, we get the following.

Lemma 3. *One has*

$$\frac{1}{2\pi i}\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\times\prod_{i=1}^{r}(a_iz;q)_{m_i}\frac{P_{n_1}(1/z;g)P_n(1/z;f)}{z}dz$$
$$=\frac{(g\alpha;q)_{n_1}(f\alpha;q)_n}{\alpha^{n_1}\alpha^n}\frac{(\gamma/\alpha,q\alpha/\gamma,qb/a;q)_\infty}{(a\alpha,q/a\alpha,b\alpha;q)_\infty}\times(q/b\gamma q^{m_0};q)_{m_0}(b\alpha)^{\sum_{i=0}^{r}m_i}\prod_{i=1}^{r}(a_i/b;q)_{m_i}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},b\alpha,q^{n_1}g\alpha;q)_k q^{k(1-n_1)}}{(q,f\alpha,g\alpha;q)_k}\times\sum_{k_1=0}^{n_1}\frac{(q^{-n_1},b\alpha q^k;q)_{k_1}q^{k_1}}{(q,g\alpha q^k;q)_{k_1}}. \quad (16)$$

Both sides of (11) multiply by

$$\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\prod_{i=1}^{r}(a_iz;q)_{m_i}\times\prod_{j=1}^{h}P_{n_j}(1/z;d_j)\frac{1}{z}. \quad (17)$$

Then, taking the q-integral on both sides of the result equation with respect to variable z, we find the following.

Lemma 4. *On has*

$$\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\prod_{i=1}^{r}(a_iz;q)_{m_i}\times\prod_{j=1}^{h+1}P_{n_j}(1/z;d_j)\frac{dz}{z}$$
$$=\frac{(\alpha d_{h+1};q)_{n_{h+1}}}{\alpha^{n_{h+1}}}\times\sum_{k=0}^{n_{h+1}}\frac{(q^{-n_{h+1}};q)_k q^k}{(q,\alpha d_{h+1};q)_k}\times\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(q^k\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\times\prod_{i=1}^{r}(a_iz;q)_{m_i}\prod_{j=1}^{h}P_{n_j}(1/z;d_j)\frac{dz}{z}, \quad (18)$$

where (n_{h+1},d_{h+1}) *denote* (n,f)*, respectively.*

3. Proof and Some Applications

Now, we return to the proof of Theorem 1.

The following result can be easily derived from (16) and (18):

$$\int_C\frac{(\gamma/z,bqz,qz/\gamma;q)_\infty}{(\alpha/z,az,bz;q)_\infty}(qz/\gamma q^{m_0};q)_{m_0}\times\prod_{i=1}^{r}(a_iz;q)_{m_i}\prod_{j=1}^{3}P_{n_j}(1/z;d_j)\frac{dz}{z}$$
$$=\frac{\prod_{i=1}^{3}(\alpha d_i;q)_{n_i}}{\alpha^{\sum_{i=1}^{3}n_i}}\frac{(\gamma/\alpha,q\alpha/\gamma,qb/a;q)_\infty}{(a\alpha,q/a\alpha,b\alpha;q)_\infty}\times(q/b\gamma q^{m_0};q)_{m_0}(b\alpha)^{\sum_{i=0}^{r}m_i}\prod_{j=1}^{r}(a_j/b;q)_{m_j}$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},b\alpha,q^{n_1}d_1\alpha,q^{n_2}d_2\alpha;q)_k}{(q,f\alpha,d_1\alpha,d_2\alpha;q)_k}q^{k(1-n_1-n_2)}\times\sum_{k_1=0}^{n_1}\frac{(q^{-n_1},q^kb\alpha,q^{n_2+k}d_2\alpha;q)_{k_1}}{(q,q^kd_1\alpha,q^kd_2\alpha;q)_{k_1}}q^{k_1(1-n_2)}\times\sum_{k_2=0}^{n_2}\frac{(q^{-n_2},q^{k+k_1}b\alpha;q)_{k_2}}{(q,q^{k+k_1}d_2\alpha;q)_{k_2}}q^{k_2}. \quad (19)$$

Letting $n=n_0$, $k=k_0$, and $f=d_0$ and combining (19) with (18), by induction, similar proof can be performed to get the desired result.

Taking $n_1=n_2=\cdots=n_{h+1}=0$ in (2), the theorem goes back to formula (1). Putting $n_1=\cdots=n_h=0$ in (2), we have the following.

Corollary 5. *One has*

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^{m_0}; q)_{m_0} \times \prod_{i=1}^{r}(a_i z; q)_{m_i}\frac{P_n(1/z; d_0)}{z}dz = \frac{(\gamma/\alpha, \alpha q/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty}(q/b\gamma q^{m_0}; q)_{m_0}(b\alpha)^{\sum_{i=0}^{r} m_i} \times \prod_{i=1}^{r}(a_i/b; q)_{m_i}(d_0/b; q)_n b^n. \tag{20}$$

Letting $n_2 = \cdots = n_h = 0$ in (2), we get the following.

Corollary 6. *One has*

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^{m_0}; q)_{m_0} \times \prod_{i=1}^{r}(a_i z; q)_{m_i}\frac{P_{n_1}(1/z; d_1)P_n(1/z; d_0)}{z}dz = \frac{(\gamma/\alpha, \alpha q/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty}(q/b\gamma q^{m_0}; q)_{m_0}(b\alpha)^{\sum_{i=0}^{r} m_i} \times \prod_{i=1}^{r}(a_i/b; q)_{m_i}(d_0/b; q)_n(d_1/b; q)_{n_1} b^{n+n_1}. \tag{21}$$

Combining (21) with (18), by induction and applying (2), we can conclude the following.

Theorem 7. *One has*

$$\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^{m_0}; q)_{m_0} \times \prod_{i=1}^{r}(a_i z; q)_{m_i}\prod_{j=0}^{h}P_{n_j}(1/z; d_j)\frac{dz}{z} = \frac{(\gamma/\alpha, q\alpha/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty}(q/b\gamma q^{m_0}; q)_{m_0}(b\alpha)^{\sum_{i=0}^{r} m_i} b^{\sum_{i=0}^{h} n_i} \times \prod_{i=1}^{r}(a_i/b; q)_{m_i}\prod_{j=0}^{h}(d_j/b; q)_{n_j}. \tag{22}$$

Comparing (2) and (22), we have the following interesting identity.

Corollary 8. *If $m_0, m_1, \ldots, m_r$, and h are nonnegative integers, then*

$$\sum_{k_0=0}^{n_0}\frac{(q^{-n_0}, b\alpha, q^{n_1}d_1\alpha, \ldots, q^{n_h}d_h\alpha; q)_{k_0}}{(q, d_0\alpha, d_1\alpha, \ldots, d_h\alpha; q)_{k_0}}q^{k_0(1-\sum_{j=1}^{h} n_j)} \times \prod_{i=1}^{h}\sum_{k_i=0}^{n_i}\frac{(q^{-n_i}, q^{A_i}b\alpha, q^{A_i+n_{i+1}}d_{i+1}\alpha, \ldots, q^{A_i+n_h}d_h\alpha; q)_{k_i}}{(q, q^{A_i}d_i\alpha, q^{A_i}d_{i+1}\alpha, \ldots, q^{A_i}d_h\alpha; q)_{k_i}} \times q^{k_i(1-\sum_{j=i+1}^{h} n_j)} = \prod_{i=0}^{h}\frac{(d_i/b; q)_{n_i}}{(d_i\alpha; q)_{n_i}}(b\alpha)^{n_0+n_1+\cdots+n_h}. \tag{23}$$

Taking $h = 1$ and $d_0 = d_1 = qb$ in (23), we have

$$\sum_{k_0=0}^{n_0}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}\frac{(b\alpha, q^{n_1+1}b\alpha; q)_{k_0}}{(qb\alpha, qb\alpha; q)_{k_0}}(-1)^{k_0}q^{\binom{k_0+1}{2}-k_0(n_0+n_1)} \times \sum_{k_1=0}^{n_1}\begin{bmatrix} n_1 \\ k_1 \end{bmatrix}\frac{(q^{k_0}b\alpha; q)_{k_1}}{(q^{k_0+1}b\alpha; q)_{k_1}}(-1)^{k_1}q^{\binom{k_1+1}{2}-k_1 n_1} = \frac{(q; q)_{n_0}(q; q)_{n_1}}{(qb\alpha; q)_{n_0}(qb\alpha; q)_{n_1}}(b\alpha)^{n_0+n_1}. \tag{24}$$

Setting $b\alpha = q$, then letting $q \to 1$ in the above identity, we have the following.

Corollary 9. *If $n_0, n_1 \in N$, then*

$$\sum_{k_0=0}^{n_0}\binom{n_0}{k_0}\frac{(n_1+2)_{k_0}}{(2)_{k_0}}(-1)^{k_0}\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}\frac{1}{k_0+k_1+1}(-1)^{k_1} = \frac{1}{(n_0+1)(n_1+1)}, \tag{25}$$

where $(a)_0 = 1$ *and* $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$, $n \in N$.

Taking $h = 2$ and $d_0 = d_1 = d_2 = qb$ in (23), we have

$$\sum_{k=0}^{n_0}\begin{bmatrix} n_0 \\ k_0 \end{bmatrix}\frac{(b\alpha, q^{n_1+1}b\alpha, q^{n_2+1}b\alpha; q)_{k_0}}{(qb\alpha, qb\alpha, qb\alpha; q)_{k_0}}(-1)^{k_0} \times q^{\binom{k_0+1}{2}-k_0(n_0+n_1+n_2)}$$

$$\times \sum_{k_1=0}^{n_1} \begin{bmatrix} n_1 \\ k_1 \end{bmatrix} \frac{\left(q^{k_0} b\alpha, q^{n_2+k_0+1} b\alpha; q\right)_{k_1}}{\left(q^{k_0+1} b\alpha, q^{k_0+1} b\alpha; q\right)_{k_1}} (-1)^{k_1}$$
$$\times q^{\binom{k_1+1}{2} - k_1(n_1+n_2)}$$
$$\times \sum_{k_2=0}^{n_2} \begin{bmatrix} n_2 \\ k_2 \end{bmatrix} \frac{\left(q^{k_0+k_1} b\alpha; q\right)_{k_2}}{\left(q^{k_0+k_1+1} b\alpha; q\right)_{k_2}} (-1)^{k_2}$$
$$\times q^{\binom{k_2+1}{2} - k_2 n_2}$$
$$= \frac{(q;q)_{n_0} (q;q)_{n_1} (q;q)_{n_2}}{(qb\alpha;q)_{n_0} (qb\alpha;q)_{n_1} (qb\alpha;q)_{n_2}} (b\alpha)^{n_0+n_1+n_2}. \tag{26}$$

Setting $b\alpha = q$, then letting $q \to 1$ in the above identity, we have the following.

Corollary 10. *If* $n_0, n_1, n_2 \in N$, *then*

$$\sum_{k_0=0}^{n_0} \binom{n_0}{k_0} \frac{(n_1+2)_{k_0} (n_2+2)_{k_0}}{(2)_{k_0} (2)_{k_0}} (-1)^{k_0}$$
$$\times \sum_{k_1=0}^{n_1} \binom{n_1}{k_1} \frac{(n_2+k_0+2)_{k_1}}{(k_0+2)_{k_1}} (-1)^{k_1}$$
$$\times \sum_{k_2=0}^{n_2} \binom{n_2}{k_2} \frac{1}{k_0+k_1+k_2+1} (-1)^{k_2}$$
$$= \frac{1}{(n_0+1)(n_1+1)(n_2+1)}, \tag{27}$$

where $(a)_0 = 1$ *and* $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geq 1$, $n \in N$.

More general, we have the following identity.

Corollary 11. *If* $h, n_0, n_1, \ldots, n_h \in N$, *then*

$$\sum_{k_0,\ldots,k_h} \prod_{i=0}^{h-1} \binom{n_i}{k_i} \frac{(A_i + n_{i+1} + 2)_{k_i} \cdots (A_i + n_h + 2)_{k_i}}{(A_i + 2)_{k_i} \cdots (A_i + 2)_{k_i}}$$
$$\times \frac{(-1)^{k_0+\cdots+k_h}}{A_h + k_h + 2} = \prod_{i=0}^{h} \frac{1}{(n_i+1)}, \tag{28}$$

where $0 \leq k_i \leq n_i$, $i = 0, \ldots, h$.

Both sides of (20) multiply by $1/(q;q)_n$; then, summing n from 0 to ∞ and using the q-binomial theorem, we find the following.

Corollary 12. *If* $\max\{|1/z|, |b|\} < 1$, *then*

$$\frac{1}{2\pi i} \int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, 1/z, az, bz; q)_\infty} (qz/\gamma q^{m_0}; q)_{m_0}$$
$$\times \prod_{i=1}^{r} (a_i z; q)_{m_i} \frac{dz}{z}$$
$$= \frac{(\gamma/\alpha, \alpha q/\gamma, qb/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha, b; q)_\infty} (q/b\gamma q^{m_0}; q)_{m_0} (b\alpha)^{\sum_{i=0}^{r} m_i}$$
$$\times \prod_{i=1}^{r} (a_i/b; q)_{m_i}. \tag{29}$$

Remark 13. If $n_1 = n_2 = \cdots = n_h = 0$, identity (23) becomes the q-Chu-Vandermonde formula.

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