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Research Article

Uniform Attractor and Approximate Inertial Manifolds for Nonautonomous Long-Short Wave Equations

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Nonautonomous long-short wave equations with quasiperiodic forces are studied. We prove the existence of the uniform attractor for the system by means of energy method, which is widely used to deal with problems who have no continuity (with respect to the initial data) property, as well as to those which Sobolev compact imbedding cannot be applied. Afterwards, we construct an approximate inertial manifold by means of extending phase space method and we estimated the size of the corresponding attracting neighborhood for this manifold.

1. Introduction

In this paper, we investigate the long time behavior of solutions for the following nonautonomous generalized dissipative LS equations with quasiperiodic forces:

$$iu_t + u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u = h_1(x,t), \tag{1}$$

$$n_t + \beta n + |u|_x^2 + f\left(|u|^2\right) = h_2(x,t), \tag{2}$$

with initial conditions and space-periodic boundary conditions as

$$u(x,\tau) = u_\tau(x), \qquad n(x,\tau) = n_\tau(x), \tag{3}$$

$$u(x-D,t) = u(x+D,t), \qquad n(x-D,t) = (x+D,t), \tag{4}$$

where $x \in \Omega = (-D, D)$, $D > 0$ and γ and β are positive constants.

The long-wave short-wave (LS) resonance equations arise in many kinds of physical models (see [1–4]). Due to their rich physical and mathematical properties, the LS equations have drawn much attention. The autonomous situations, including the existence of solutions, the solitary waves and their stability, and the long time behaviors of the solutions, have been deeply researched (see [5–14]).

Recently, the nonautonomous case of LS equations with translation compact forces was studied in [15]. Because of the nonlinear resonance of the equations, it is difficult to prove the continuity of the process $U(t,\tau)$ generated by (1)–(4). Thus, it is hard to construct the uniform attractor directly by constructing a compact uniform absorbing set even if the forces are translation compact, and in [15] only the weakly compact uniform attractor for the system is obtained.

In this paper, we firstly investigate the compact uniform attractor for systems (1)–(4) by employing the energy equations and the energy method presented by Ball (see [16, 17]). The energy method can be concisely understood as the following two steps (e.g., in autonomous cases): (1) construct a weakly compact attractor and (2) prove the strong compactness of the weak attractor, that is, verify that the weak attractor is actually the strong one. To accomplish Step 1, one can construct a bounded (weakly compact) absorbing set and the weak continuity of the system. Step 2 is usually deduced by applying proper energy inequalities together with Lemmas 11 and 12. Obviously, this method is good at solving problems which are not continuous and those that lack

Sobolev embeddings (such as systems defined in unbounded domains).

Besides, approximate inertial manifolds (AIM) for the system is studied afterwards. This manifold is a finite-dimensional smooth surface in a phase space, whose small vicinity attracts all the trajectories at a much higher speed than global attractors. To investigate AIM, by employing the extending phase space method we transfer the nonautonomous system $U_\sigma(t\cdot\tau)$ to an autonomous one $S(t)$, and we get the AIM for $U_\sigma(t\cdot\tau)$ by constructing the AIM for $S(t)$.

The *main result* of this paper contains Theorems 13 and 17. It is summarized by the following.

Main Theorem. *Assume that*

(i) $h_j(x,t)$, $j = 1,2$, *are quasiperiodic forces satisfying Assumption 1;*

(ii) *generalized* $f(s)$ *and* $g(s)$ *are quasilinear functions satisfying* (9) *and* (10).

Then systems (1)–(4) *generate a family of processes* $U_\sigma(t,\tau)$ *in* $E_0 := H^2_{per}(\Omega)\times H^1_{per}(\Omega)$. *Moreover, the family of processes* $U_\sigma(t,\tau)$ *admits a compact uniform attractor* $\mathscr{A}_\Sigma$ *and an AIM in* E_0.

We would like to remark that the existence of the compact uniform attractor for the system does not depend heavily on the quasiperiodicity of the forces. It still holds when the forces are just translation bounded (see Remark 14), that is, it strengthens the result in [15].

This paper is organized as follows. In Section 2, we show the LS equations in details and we deeply introduce the quasiperiodicity conditions. In Section 3, we get the uniform a prior estimates for the solutions. In Section 4, we study the unique existence of the solution. In Section 5 the existence of the uniform attractor for (5)–(8) is obtained by applying weak convergence method. In Section 6, AIM for (5)–(8) is constructed by extending and splitting the phase space and making use of projection operators.

Throughout this paper, we denote by $\|\cdot\|$ the norm of $H = L^2_{\text{per}}(\Omega)$ with usual inner product $(\cdot,\cdot)$, denote by $\|\cdot\|_p$ the norm of $L^p_{\text{per}}(\Omega)$ for all $1\leqslant p\leqslant\infty(\|\cdot\|_2=\|\cdot\|)$, and denote by $\|\cdot\|_X$ the norm of any Banach space X. Besides, notations "$\rightharpoonup$", "$\overset{*}{\rightharpoonup}$", and "$\rightarrow$" denote weakly converges to, weakly star converges to and strongly converges to, respectively. And we denote different constants by the same letter C, and $C(\cdot,\cdot)$ represents that the constant relies only on the parameters that appear in the brackets.

2. Preliminaries

We show the nonautonomous dissipative generalized long-short wave equations with quasiperiodic forces in details as follows:

$$iu_t + u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u = h_1(x,t), \tag{5}$$

$$n_t + \beta n + |u|^2_x + f\left(|u|^2\right) = h_2(x,t), \tag{6}$$

with initial value conditions

$$u(x,\tau) = u_\tau(x), \qquad n(x,\tau) = n_\tau(x), \qquad x\in\Omega = (-D,D), \quad D>0, \tag{7}$$

and periodic boundary value conditions

$$u(x-D,t) = u(x+D,t), \qquad n(x-D,t) = (x+D,t), \tag{8}$$

where $u(x,t)$ is an unknown complex valued vector, $n(x,t)$ is an unknown real valued function, γ,β are positive constants and nonautonomous terms $h_1(x,t)$, and $h_2(x,t)$ are time-depended external forces satisfying quasiperiodicity conditions (see Assumption 1); non-linear terms $f(s)$ and $g(s)$ are given real-valued functions, satisfying

$$\left|f(s)\right| \leqslant c_1\left(s^{p/2}+1\right), \qquad \left|g(s)\right| \leqslant c_2\left(s^{1/2}+1\right), \qquad 0\leqslant s<\infty,\ p<2. \tag{9}$$

$$\left|f^{(k)}(s)\right| \leqslant c_3, \quad \left|g^{(k)}(s)\right| \leqslant c_4, \quad k=1,2, \tag{10}$$

where c_j are given positive constants for $j = 1,2,3,4$.

Let $\mathfrak{I}$ be a topological space, and $\varphi(s)\in\mathfrak{I}$ is a function. The set

$$\mathscr{H}(\varphi) = \overline{\{\varphi(h+s) \mid h\in\mathbb{R}\}} \tag{11}$$

is called the hull of φ in $\mathfrak{I}$, denoted by $\mathscr{H}(\varphi)$. φ is translation compact (resp., translation bounded) if $\mathscr{H}(\varphi)$ is compact (resp., bounded) in $\mathfrak{I}$.

We denote all the translation compact functions in $L^2_{\text{loc}}(\mathbb{R};X)$ by $L^2_c(\mathbb{R};X)$; X is a *Banach* space. Apparently, $\varphi\in L^2_c(\mathbb{R};X)$ implies that φ is translation bounded as follows:

$$\|\varphi\|^2_{L^2_b(\mathbb{R};X)} = \sup_{t\in\mathbb{R}}\int_t^{t+1}\|\varphi\|^2_X ds < \infty. \tag{12}$$

Assumption 1. For $i = 1,2$, we suppose $h_i(x,t)\in C^1(\tau,+\infty; H^1_{\text{per}}(\Omega))$ and it satisfies quasiperiodicity conditions; namely,

$$h_i(x,t) = H_i\left(x,\omega^i(t)\right) = H_i\left(x,\omega^i_1(t),\dots,\omega^i_{k_i}(t)\right), \tag{13}$$

and for all $j\in\{1,2,\dots,k_i\}$,

$$H_i\left(x,\omega^i_1,\omega^i_2,\dots,\omega^i_j+2\pi,\dots,\omega^i_{k_i}\right) = H_i\left(x,\omega^i_1,\omega^i_2,\dots,\omega^i_j,\dots,\omega^i_{k_i}\right), \tag{14}$$

where $\omega^i_j = \alpha^i_j t$, $1\leqslant k_i<\infty$, and $\{\alpha^i_j\}^{k_i}_{j=1}$ are rational and independent; H_i is differentiable to each position and

$$H_i,\ \frac{\partial H_i}{\partial\omega^i_j}\in H^1_{\text{per}}(\Omega). \tag{15}$$

If $h_i(x,t)$ satisfy Assumption 1, we can consider the symbol space $\mathscr{H}(h_i)$ as

$$\mathcal{H}(h_i) = \overline{\left\{H_i\left(\alpha_1^i t + \omega_{01}^i, \alpha_2^i t + \omega_{02}^i, \ldots, \alpha_{k_i}^i t + \omega_{0k_i}^i\right) \mid \left(\omega_{01}^i, \omega_{02}^i, \ldots, \omega_{0k_i}^i\right) =: \omega_0^i \in T^{k_i}\right\}}. \quad (16)$$

Since there is a continuous mapping $T^{k_i} \rightarrow \mathcal{H}(h_i) : \omega_0^i \rightarrow H_i(\alpha_i t + \omega_0^i)$, from [18] we know that the symbol space $\Sigma_0 = \mathcal{H}(h_1) \times \mathcal{H}(h_2)$ can be replaced by $T^k := T^{k_1} \times T^{k_2}$. And, for each $\omega_0 := (\omega_0^1, \omega_0^2) \in T^k$, the translation operator acting on T^k can be defined as

$$\begin{aligned} T(h) : T(h)\,\omega_0 &= [\omega_0 + \alpha h] \\ &= (\omega_0 + \alpha h) \mod (2\pi)^k. \end{aligned} \quad (17)$$

Therefore, T^k is translation compact.

Proposition 2. *Under Assumption 1, we can deduce the following properties:*

(i) *$h_i(x,t)$ is translation bounded in $H^1_{per}(\Omega)$; that is,*

$$\sup_{t\geqslant\tau} \int_t^{t+1} \|h_i(s)\|^2_{H^1_{per}} ds \leqslant \int_\tau^{\tau+T} \|h_i(s)\|^2_{H^1_{per}} ds \leqslant C, \quad (18)$$

where $T = \max\{1, 2\pi \cdot \prod_{j=1}^{k_i} \alpha_j^i\}$, $C = T \cdot \max_{\tau\leqslant t\leqslant \tau+T} \|h_i(t)\|^2_{H^1_{per}}$,

(ii) *for all $y_1 \in \mathcal{H}(h_1)$, $\|y_1\|^2_{L^2_b(\mathbb{R};X)} \leqslant \|h_1\|^2_{L^2_b(\mathbb{R};X)}$,*

(iii) *$h_{it} \in H^1_{per}(\Omega)$, which can be seen directly from* (15) *and the fact that*

$$h_{it} = \sum_{j=1}^{k_i} \frac{\partial H_i}{\partial \omega_j^i} \alpha_j^i. \quad (19)$$

Moreover, h_{it} is translation bounded in $H^1_{per}(\Omega)$. Similarly to (18)*, by the continuity of h_{it}, we can find a constant C, which is independent of t, such that*

$$\|h_{it}\|_{L_b^\infty(\tau,\infty;H^1_{per})} \leqslant C. \quad (20)$$

For brevity, we set $W(x,t) := (u(x,t), n(x,t))$, $Y_0(x,t) := (h_1(x,t), h_2(x,t))$, and let $E_0 := H^2_{\mathrm{per}}(\Omega) \times H^1_{\mathrm{per}}(\Omega)$ with the norm

$$\|W\|_{E_0} = \left\{\|u\|^2_{H^2} + \|n\|^2_{H^1}\right\}^{1/2}. \quad (21)$$

Similarly, we let $\Sigma_0 := \mathcal{H}(h_1) \times \mathcal{H}(h_2)$ and for each $Y = (y_1, y_2) \in \Sigma_0$,

$$\|Y\|_{\Sigma_0} = \left\{\|y_1\|^2_{H^1} + \|y_2\|^2_{H^1}\right\}^{1/2}. \quad (22)$$

Then systems (5)–(8) can be rewritten as

$$\begin{gathered} \partial_t W = A_{\sigma(t)} W, \qquad W|_{t=\tau} = W_\tau, \\ W(x+D,t) = W(x-D,t), \end{gathered} \quad (23)$$

where the symbol $\sigma(s) = Y(x,s)$ or $\omega(s)$ and the symbol space $\Sigma = \Sigma_0$ or T^k.

3. Uniform a Prior Estimates of the Solutions

In this section, we derive uniform *a priori* estimates of the solutions both in time t and in symbols $Y(x,t) = (y_1, y_2)$ which come from the symbol space $\Sigma = \mathcal{H}(h_1) \times \mathcal{H}(h_2)$. First we recall the following interpolation inequality.

Lemma 3. *Let $j, m \in \mathbb{N} \cup \{0\}$, $q, r \in \mathbb{R}^+$, such that $0 \leqslant j < m$, $1 \leqslant q, r \leqslant \infty$. Then we have*

$$\left\|D^j u\right\|_p \leqslant C \left\|D^m u\right\|_r^a \|u\|_q^{1-a}, \quad (24)$$

for $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, where $\Omega \subset \mathbb{R}^1$ and $j/m \leqslant a \leqslant 1$, $1/p = j + a((1/r) - m) + ((1-a)/q)$.

Lemma 4. *Let Assumption 1 hold. If $u_\tau(x) \in L^2_{per}(\Omega)$ and $Y(x,t) \in \Sigma$, then the solutions of problem* (5)–(8) *satisfy*

$$\|u(t)\| \leqslant C_1, \quad \forall t \geqslant t_1, \quad (25)$$

where $C_1 = C(\gamma, h_1)$ and $t_1 = C(\gamma, h_1, R)$, whenever $\|u_\tau\| \leqslant R$.

Proof. Taking the inner product of (5) with u in $L^2_{\mathrm{per}}(\Omega)$ we get that

$$\left(iu_t + u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u, u\right) = \left(y_1(x,t), u\right). \quad (26)$$

Taking the imaginary part of (26), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|u\|^2 = \operatorname{Im}\left(y_1, u\right). \quad (27)$$

By Young inequality and Proposition 2 we have

$$\begin{aligned} \frac{d}{dt}\|u\|^2 + \gamma\|u\|^2 &\leqslant \frac{1}{\gamma}\left\|y_1(x,t)\right\|^2_{L^2_b(\mathbb{R};H_{\mathrm{per}})} \\ &\leqslant \frac{1}{\gamma}\left\|h_1(x,t)\right\|^2_{L^2_b(\mathbb{R};H^1_{\mathrm{per}})}. \end{aligned} \quad (28)$$

Then by Gronwall lemma we can complete the proof. □

In the following, we denote by $\int \cdot\, dx = \int_\Omega \cdot\, dx$, which will not cause no confusions.

Lemma 5. *Under assumptions of* (9) *and* (10) *and Assumption 1, if $W(\tau) \in H^1_{per} \times H_{per}$, then solutions of problems* (5)–(8) *satisfy*

$$\|W(t)\|^2_{H^1_{per}\times H_{per}} \leqslant C_2, \quad \forall t \geqslant t_2, \quad (29)$$

where $C_2 = C(\gamma, \beta, f, g, Y_0, h_{1t})$ and $t_2 = C(\gamma, \beta, f, g, Y_0, h_{1t}, R)$, whenever $\|W_\tau\|_{H^1\times H} \leqslant R$.

Proof. Taking the inner product of (5) with u_t in $H_{\text{per}}(\Omega)$ and taking the real part, we have

$$-\frac{1}{2}\frac{d}{dt}\|u_x\|^2-\frac{1}{2}\int n\frac{d}{dt}|u|^2dx+\operatorname{Re}(i\gamma u,u_t)+\frac{1}{2}\int g\left(|u|^2\right)\frac{d}{dt}|u|^2dx=\operatorname{Re}\left(y_1(x,t),u_t\right). \tag{30}$$

By (6) we know that

$$\begin{aligned}\frac{d}{dt}\int n\frac{d}{dt}|u|^2dx &= \frac{d}{dt}\int n|u|^2dx-\int|u|^2n_tdx\\ &=\frac{d}{dt}\int n|u|^2dx+\int|u|^2|u|_x^2dx+\beta\int n|u|^2dx\\ &\quad+\int f\left(|u|^2\right)|u|^2dx-\int y_2(x,t)|u|^2dx\\ &=\frac{d}{dt}\int n|u|^2dx+\beta\int n|u|^2dx\\ &\quad+\int f\left(|u|^2\right)|u|^2dx-\int y_2|u|^2dx,\end{aligned} \tag{31}$$

which shows that

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|u_x\|^2-\frac{1}{2}\left(\frac{d}{dt}\int n|u|^2dx+\beta\int n|u|^2dx\right.\\ &\qquad\left.+\int f\left(|u|^2\right)|u|^2dx-\int y_2|u|^2dx\right)\\ &\quad+\frac{1}{2}\frac{d}{dt}\int G\left(|u|^2\right)dx+\operatorname{Re}(i\gamma u,u_t)-\frac{d}{dt}\operatorname{Re}(y_1,u)\\ &\quad+\operatorname{Re}\int y_{1t}(x,t)\,\overline{u}\,dx=0,\end{aligned} \tag{32}$$

where $G(s)$ is introduced by

$$G(s)=\int_0^s g(\xi)\,d\xi. \tag{33}$$

Taking the inner product of (5) with u in $H_{\text{per}}(\Omega)$ and taking the real part, we get that

$$\operatorname{Re}(iu_t,u)-\|u_x\|^2-\int n|u|^2dx+\int g\left(|u|^2\right)|u|^2dx-\operatorname{Re}\left(y_1(x,t),u\right)=0. \tag{34}$$

Multiply (34) by γ and add the resulting identity to (32) to get

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\|u_x\|^2-\frac{1}{2}\frac{d}{dt}\int n|u|^2dx-\frac{1}{2}(\beta+2\gamma)\int n|u|^2dx\\ &\quad-\frac{1}{2}\int f\left(|u|^2\right)|u|^2dx+\frac{1}{2}\int y_2(x,t)|u|^2dx\\ &\quad+\frac{1}{2}\frac{d}{dt}\int G\left(|u|^2\right)dx-\gamma\|u_x\|^2+\gamma\int g\left(|u|^2\right)|u|^2dx\\ &\quad-\gamma\operatorname{Re}(a(x,t),u)-\frac{d}{dt}\operatorname{Re}\left(y_1(x,t),u\right)\\ &\quad+\operatorname{Re}\int y_{1t}(x,t)\overline{u}dx=0.\end{aligned} \tag{35}$$

That is,

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx\right.\\ &\qquad\left.+2\operatorname{Re}\int y_1(x,t)\overline{u}dx\right)\\ &\qquad+\gamma\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx\right.\\ &\qquad\qquad\left.+2\operatorname{Re}\int y_1(x,t)\overline{u}dx\right)+\gamma\|u_x\|^2\\ &\quad=-\int f\left(|u|^2\right)|u|^2dx+\int y_2(x,t)|u|^2dx\\ &\qquad-\gamma\int G\left(|u|^2\right)dx-(\gamma+\beta)\int n|u|^2dx\\ &\qquad+2\operatorname{Re}\int y_{1t}(x,t)\,\overline{u}\,dx.\end{aligned} \tag{36}$$

In the following, we denote by C any constants depending only on the data (α,β,f,g) and $C(\cdot,\cdot)$ means it depends not only on (γ,β,f,g) but also on parameters in the brackets. For all $\rho>0$, when t is sufficiently large, by (9), Lemmas 3 and 4 we have

$$\begin{aligned}&\left|-\int f\left(|u|^2\right)|u|^2dx\right|\\ &\quad\leq C\int|u|^{p+2}dx+C\int|u|^2dx\\ &\quad\leq C\int\left(|u|^{2p}+|u|^4\right)dx+C\|u\|^2\\ &\quad\leq C\|u\|_4^4+C\leq C\|u_x\|\,\|u\|^3+C\leq\rho\|u_x\|^2+C(\rho),\\ &\left|\int y_2(x,t)|u|^2dx\right|\\ &\quad\leq\|y_2(x,t)\|^2_{L_b^2(\mathbb{R};H^1_{\text{per}})}+\|u\|_4^4\\ &\quad\leq\|h_2\|^2_{L_b^2(\mathbb{R};H^1_{\text{per}})}+C\|u_x\|\leq\rho\|u_x\|^2+C_2(\rho).\end{aligned} \tag{37}$$

By (9) we deduce that

$$|G(s)| \leqslant \frac{2}{3}c_2 s^{3/2} + c_2 s, \quad \forall s \geqslant 0. \tag{38}$$

And then

$$\begin{aligned}&\left|-\gamma \int_\Omega G\left(|u|^2\right) dx\right| \\ &\quad \leqslant C \int \left(|u|^3 + |u|^2\right) dx \\ &\quad \leqslant C\|u\|_3^3 + C\|u\|^2 \\ &\quad \leqslant C\|u_x\|^{1/2}\|u\|^{5/2} + C \leqslant \rho\|u_x\|^2 + C_3(\rho),\end{aligned} \tag{39}$$

$$\begin{aligned}&\left|-(\gamma+\beta)\int_\Omega n|u|^2 dx\right| \\ &\quad \leqslant \rho\|n\|^2 + C(\rho)\|u\|_4^4 \\ &\quad \leqslant \rho\|n\|^2 + \rho\|u_x\|^2 + C_4(\rho),\end{aligned} \tag{40}$$

$$\begin{aligned}&\left|2\operatorname{Re}\int_\Omega y_{1t}(x,t)\,\overline{u}dx\right| \\ &\quad \leqslant \|y_{1t}\|^2_{L^2_c(\mathbb{R};H^1_{\text{per}})} + \|u\|^2 \\ &\quad \leqslant C\left(\|h_{1t}\|^2_{L^2_c(\mathbb{R};H^1_{\text{per}})}, \|u\|^2\right).\end{aligned} \tag{41}$$

By (36)–(41) we deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2 + \int n|u|^2 dx - \int G\left(|u|^2\right) dx \right.\\ &\quad \left. + 2\operatorname{Re}\int y_1(x,t)\,\overline{u}dx\right) \\ &\quad + \gamma\left(\|u_x\|^2 + \int n|u|^2 dx - \int G\left(|u|^2\right) dx \right.\\ &\quad\quad \left. + 2\operatorname{Re}\int y_1\overline{u}dx\right) + \gamma\|u_x\|^2 \\ &\leqslant \rho\|n\|^2 + 4\rho\|u_x\|^2 + C(\rho) \\ &\quad + C\left(\|h_{1t}\|^2_{L^2_b(\mathbb{R};H^1_{\text{per}})}, \|u\|^2\right).\end{aligned} \tag{42}$$

Similarly we can also deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2 + \int n|u|^2 dx - \int G\left(|u|^2\right) dx \right.\\ &\quad \left. + 2\operatorname{Re}\int y_1\overline{u}dx\right) \\ &\quad + \beta\left(\|u_x\|^2 + \int n|u|^2 dx - \int G\left(|u|^2\right) dx \right.\\ &\quad\quad \left. + 2\operatorname{Re}\int y_1\overline{u}dx\right) + (2\gamma-\beta)\|u_x\|^2 \\ &\leqslant \rho\|n\|^2 + 4\rho\|u_x\|^2 + C(\rho) \\ &\quad + C\left(\|h_{1t}\|^2_{L^2_b(\mathbb{R};H^1_{\text{per}})}, \|u\|^2\right).\end{aligned} \tag{43}$$

Taking the inner product of (6) with n in $H_{\text{per}}(\Omega)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|n\|^2 + \int n|u|_x^2 dx + \beta\|n\|^2 + \int f\left(|u|^2\right) n\,dx \\ &\quad - \int y_2(x,t)\,n\,dx = 0.\end{aligned} \tag{44}$$

By (5) we get that

$$\begin{aligned}&\int n|u|_x^2 dx \\ &\quad = \int n u_x \overline{u}\,dx + \int n u \overline{u}_x dx \\ &\quad = i\int\left(u_t\overline{u}_x - \overline{u}_t u_x\right) dx + 2\operatorname{Re}\int i\gamma u\overline{u}_x dx \\ &\quad\quad - 2\operatorname{Re}\int y_2(x,t)\,\overline{u}_x dx,\end{aligned} \tag{45}$$

$$\begin{aligned}&\frac{d}{dt}\int\left(iu\overline{u}_x - iu_x\overline{u}\right) dx \\ &\quad = i\int\left(u_t\overline{u}_x + u\overline{u}_{xt} - u_{xt}\overline{u} - u_x\overline{u}_t\right) dx \\ &\quad = i\int\left(u_t\overline{u}_x - u_x\overline{u}_t + u_t\overline{u}_x - u_x\overline{u}_t\right) dx \\ &\quad = 2i\int\left(u_t\overline{u}_x - \overline{u}_t u_x\right) dx.\end{aligned} \tag{46}$$

It comes from (44)–(46) that

$$\begin{aligned}&\frac{d}{dt}\|n\|^2 + \frac{d}{dt}\int i\left(u\overline{u}_x - u_x\overline{u}\right) dx + 2\beta\|n\|^2 \\ &\quad + i\gamma\int\left(u\overline{u}_x - u_x\overline{u}\right) dx \\ &\leqslant i\gamma\int\left(u\overline{u}_x - u_x\overline{u}\right) dx - 4\operatorname{Re}\int i\gamma u\overline{u}_x\,dx \\ &\quad + 4\operatorname{Re}\int y_1(x,t)\,\overline{u}_x dx - 2\int f\left(|u|^2\right) n\,dx \\ &\quad + 2\int y_2(x,t)\,n\,dx.\end{aligned} \tag{47}$$

Dealing with the right hand side of inequality (47), by Lemmas 3 and 4, we get

$$\left|i\gamma\int\left(u\overline{u}_x - u_x\overline{u}\right) dx\right| \leqslant 2\gamma\|u\|\|u_x\| \leqslant \rho\|u_x\| + C_1(\rho), \tag{48}$$

$$\left|-4\operatorname{Re}\int i\gamma u\overline{u}_x dx\right| \leqslant 4\gamma\|u\|\|u_x\| \leqslant \rho\|u_x\| + C_2(\rho), \tag{49}$$

$$\begin{aligned}&\left|-2\int f\left(|u|^2\right) n dx\right| \\ &\quad \leqslant C\int |u|^p |n|\,dx + C\int |n|\,dx\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{1}{2}\rho\|n\|^2 + C(\rho)\int |u|^{2p}dx + \frac{1}{2}\rho\|n\|^2 + C(\rho)\\&\leqslant \rho\|n\|^2 + C(\rho)\|u_x\|^{p-1}\|u\|^{p+1} + C(\rho)\\&\leqslant \rho\|n\|^2 + \rho\|u_x\|^2 + C_3(\rho),\end{aligned} \tag{50}$$

$$\begin{aligned}&\left|4\,\mathrm{Re}\int y_2\bar{u}_x dx\right|\\&\quad\leqslant 4\|y_1\|_{L^2_b(\mathbb{R};H^1_{\mathrm{per}})}\|u_x\| \leqslant \rho\|u_x\| + C_4\left(\rho, \|h_1\|_{L^2_b(\mathbb{R};H^1_{\mathrm{per}})}\right),\end{aligned} \tag{51}$$

$$\begin{aligned}&\left|2\int y_2 n\,dx\right|\\&\quad\leqslant 2\|y_2\|_{L^2_b(\mathbb{R};H^1_{\mathrm{per}})}\|n\| \leqslant \rho\|n\|^2 + C_5\left(\rho, \|h_2\|_{L^2_b(\mathbb{R};H^1_{\mathrm{per}})}\right).\end{aligned} \tag{52}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}\left(\|n\|^2 + i\int(u\bar{u}_x - u_x\bar{u})\,dx\right) + 2\beta\|n\|^2\\&\quad + i\gamma\int(u\bar{u}_x - u_x\bar{u})\,dx\\&\quad\leqslant 2\rho\|n\|^2 + 4\rho\|u_x\|^2 + C\left(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right).\end{aligned} \tag{53}$$

Analogously, we can also deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\|n\|^2 + i\int(u\bar{u}_x - u_x\bar{u})\,dx\right) + 2\beta\|n\|^2\\&\quad + i\beta\int(u\bar{u}_x - u_x\bar{u})\,dx\\&\quad\leqslant 2\rho\|n\|^2 + 4\rho\|u_x\|^2 + C\left(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right).\end{aligned} \tag{54}$$

Set $\epsilon = \min\{\gamma, \beta\}$, and

$$\begin{aligned}E &= \|u_x\|^2 + \|n\|^2 + \int n|u|^2dx - \int G(|u|^2)\,dx\\&\quad + 2\,\mathrm{Re}\int y_1\bar{u}dx + i\int(u\bar{u}_x - u_x\bar{u})\,dx.\end{aligned} \tag{55}$$

Then by (42) and (53) and (43) and (54) we can respectively deduce that

$$\begin{aligned}&\frac{d}{dt}E + \gamma E + \gamma\|u_x\|^2 + \beta\|n\|^2\\&\quad\leqslant 8\rho\|u_x\|^2 + 3\rho\|n\|^2 + C\left(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right),\\&\frac{d}{dt}E + \beta E + \gamma\|u_x\|^2 + \beta\|n\|^2\\&\quad\leqslant 8\rho\|u_x\|^2 + 3\rho\|n\|^2 + C\left(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right),\end{aligned} \tag{56}$$

which shows that if we set $\rho \leqslant \min\{\alpha/8, \beta/3\}$, we can deduce that

$$\frac{d}{dt}E + \epsilon E \leqslant C_0, \quad \forall t \geqslant t_0, \tag{57}$$

where $C_0 = C(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}, \|h_{1t}(x,t)\|_{L^2_b(\mathbb{R};\Sigma)})$. By Gronwall lemma we have that

$$E(t) \leqslant E(t_0)e^{-\gamma(t-t_0)} + \frac{C_0}{\epsilon}, \quad \forall t \geqslant t_0. \tag{58}$$

Similarly to (39), (40), (51), and (48), for $t \geqslant t_0$ we have

$$\begin{aligned}&\left|\int n|u|^2dx - \int G(|u|^2)\,dx + 2\,\mathrm{Re}\int y_1\bar{u}\,dx\right.\\&\quad\left. + i\int(u\bar{u}_x - u_x\bar{u})\,dx\right|\\&\quad\leqslant \rho\|n\|^2 + \rho\|u_x\|^2 + C\left(\rho, \|h_1(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right).\end{aligned} \tag{59}$$

And then

$$\begin{aligned}|E(t_0)| &\leqslant \|u_x(t_0)\|^2 + \|n(t_0)\|^2\\&\quad + \left|\int n(t_0)|u(t_0)|^2dx - \int G(|u(t_0)|^2)\,dx\right.\\&\quad + 2\,\mathrm{Re}\int y_1(t_0)\bar{u}(t_0)\,dx\\&\quad\left. + i\int(u(t_0)\bar{u}_x(t_0) - u_x(t_0)\bar{u}(t_0))\,dx\right|\\&\leqslant C(R),\end{aligned} \tag{60}$$

where $C(R) = C(\rho, \|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}, \|h_{1t}(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}, R)$ when $\|W_\tau\|_{H^1\times H} \leqslant R$. Then by (58) we infer that

$$\begin{aligned}E(t) &\leqslant C(R)e^{-\gamma(t-t_0)} + \frac{C_0}{\epsilon}, \quad \forall t \geqslant t_0,\\&\leqslant \frac{2C_0}{\epsilon}, \quad \forall t \geqslant t_*,\end{aligned} \tag{61}$$

where $t_* = \inf\{t \mid t \geqslant t_0 \text{ and } C(R)e^{-\epsilon(t_*-t_0)} \leqslant C_0/\epsilon\}$. By (55), (59) and (61) we infer that

$$\|u_x(t)\|^2 + \|n(t)\|^2 \leqslant \rho\|n\|^2 + \rho\|u_x\|^2 + C_0. \tag{62}$$

Choosing $\rho = \min\{\gamma/8, \beta/3, 1/2\}$, we have

$$\|u_x\|^2 + \|n(t)\|^2 \leqslant C\left(\|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}, \|h_{1t}(x,t)\|_{L^2_b(\mathbb{R};\Sigma)}\right), \quad \forall t \geqslant t_*, \tag{63}$$

which concludes the proof by using Lemma 4. □

Lemma 6. *Under assumptions of Lemma 5, if $W(\tau) \in E_0 = H^2_{per}(\Omega) \times H^1_{per}(\Omega)$, then solutions of problems* (5)–(8) *satisfy*

$$\|W(t)\|^2_{E_0} \leqslant C_2, \quad \forall t \geqslant t_3, \tag{64}$$

where $C_2 = C(\gamma, \beta, f, g, Y_0, a_{0t})$ and $t_3 = C(\gamma, \beta, f, g, Y_0, a_{0t}, R)$, whenever $\|W_\tau\|_{E_0} \leqslant R$.

Proof. Taking the real part of the inner product of (5) with u_{xxt} in $H_{\text{per}}(\Omega)$, we have

$$\frac{d}{dt}\|u_{xx}\|^2 - \operatorname{Re}\int n\bar{u}_{xxt}dx + \operatorname{Re}(i\gamma u, u_{xxt}) + \operatorname{Re}\int g(|u|^2)u\bar{u}_{xxt}dx - \operatorname{Re}\int y_1(x,t)\bar{u}_{xxt}dx = 0. \quad (65)$$

By (5) and (6), we have

$$\begin{aligned} -\operatorname{Re}\int n\bar{u}_{xxt}dx &= -\frac{d}{dt}\int \operatorname{Re}(n\bar{u}_{xx})dx + \operatorname{Re}\int n_t\bar{u}_x x dx + \operatorname{Re}\int nu_t\bar{u}_{xx}dx \\ &= -\frac{d}{dt}\int \operatorname{Re}(n\bar{u}_{xx})dx - \operatorname{Re}\int u\bar{u}_{xx}\left(|u|_x^2 + \beta n + f(|u|^2) - y_2\right)dx \\ &\quad + \operatorname{Re}\int n\bar{u}_{xx}\left(-inu - \gamma u + ig(|u|^2)u - iy_1\right)dx. \end{aligned} \quad (66)$$

Since

$$\operatorname{Re}(iu, u_{xxt}) = \operatorname{Re}\int iu\bar{u}_{txx}\,dx = -\operatorname{Re}\int iu_t\bar{u}_{xx}dx, \quad (67)$$

we know that

$$\begin{aligned} \operatorname{Re}(i\gamma u, u_{xxt}) &= \gamma\|u_{xx}\|^2 - \gamma\operatorname{Re}\int n u\bar{u}_{xx}dx \\ &\quad + \gamma\operatorname{Re}\int g(|u|^2)u\bar{u}_{xx}dx - \gamma\operatorname{Re}\int y_1\bar{u}_{xx}dx. \end{aligned} \quad (68)$$

Multiply (5) by $\bar{u}$ and take the real part, we find that

$$|u|_t^2 = 2\operatorname{Re}(iu_{xx}\bar{u}) - 2\gamma|u|^2 - 2\operatorname{Re}(iy_1\bar{u}). \quad (69)$$

Therefore,

$$\begin{aligned} \operatorname{Re}\int g(|u|^2)u\bar{u}_{xxt}dx &= -\int g'(|u|^2)|u|_x^2\operatorname{Re}(u\bar{u}_{xt})dx - \int g(|u|^2)\frac{1}{2}\frac{d}{dt}|u_x|^2dx \\ &= -\int g'(|u|^2)|u|_x^2\operatorname{Re}(u\bar{u}_{xt})dx - \frac{1}{2}\frac{d}{dt}\int g(|u|^2)|u_x|^2dx \\ &\quad + \int g'(|u|^2)|u_x|^2\left(\operatorname{Re}(iu_{xx}\bar{u}) - \gamma|u|^2 - \operatorname{Re}(iy_1\bar{u})\right)dx. \end{aligned} \quad (70)$$

Now we deal with (70) to get (78). Due to equalities

$$\begin{aligned} |u|_x^2 &= 2\operatorname{Re}(u\bar{u}_x), \\ \frac{d}{dt}\operatorname{Re}(u\bar{u}_x) &= \operatorname{Re}(u_t\bar{u}_x) + \operatorname{Re}(u\bar{u}_{xt}), \end{aligned} \quad (71)$$

we deduce that

$$\begin{aligned} &\int g'(|u|^2)|u|_x^2\operatorname{Re}(u\bar{u}_{xt})dx \\ &= \frac{d}{dt}\int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u\bar{u}_x)dx \\ &\quad - \int g''(|u|^2)|u|_t^2 2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u\bar{u}_x)dx \\ &\quad - \int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)_t\operatorname{Re}(u\bar{u}_x)dx \\ &\quad - \int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u_t\bar{u}_x)dx. \end{aligned} \quad (72)$$

We take care of terms in (72) as follows

$$\begin{aligned} &\int g''(|u|^2)|u|_t^2 2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u\bar{u}_x)dx \\ &= 4\int g''(|u|^2)(\operatorname{Re}(u\bar{u}_x))^2 \\ &\quad \times\left(\operatorname{Re}(iu_{xx}\bar{u}) - \gamma|u|^2 - \operatorname{Re}(iy_1\bar{u})\right)dx, \end{aligned} \quad (73)$$

$$\begin{aligned} &\int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)_t\operatorname{Re}(u\bar{u}_x)dx \\ &= \int g'(|u|^2)2\operatorname{Re}(u_t\bar{u}_x)\operatorname{Re}(u\bar{u}_x)dx \\ &\quad + \int g'(|u|^2)2\operatorname{Re}(u\bar{u}_{xt})\operatorname{Re}(u\bar{u}_x)dx \\ &= 2\int g'(|u|^2)\operatorname{Re}(u\bar{u}_x) \\ &\quad \times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g(|u|^2)u - y_1\right)\right)dx \\ &\quad + \int g'(|u|^2)|u|_x^2\operatorname{Re}(u\bar{u}_{xt})dx, \end{aligned} \quad (74)$$

$$\begin{aligned} &\int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u_t\bar{u}_x)dx \\ &= 2\int g'(|u|^2)\operatorname{Re}(u\bar{u}_x) \\ &\quad \times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g(|u|^2)u - y_1\right)\right)dx. \end{aligned} \quad (75)$$

It follows from (72)–(75) that

$$\begin{aligned} &-\int g'(|u|^2)|u|_x^2\operatorname{Re}(u\bar{u}_{xt})dx \\ &= -\frac{d}{dt}\int g'(|u|^2)2\operatorname{Re}(u\bar{u}_x)\operatorname{Re}(u\bar{u}_x)dx \\ &\quad + 4\int g''(|u|^2)(\operatorname{Re}(u\bar{u}_x))^2 \\ &\quad \times\left(\operatorname{Re}(iu_{xx}\bar{u}) - \gamma|u|^2 - \operatorname{Re}(iy_1\bar{u})\right)dx \end{aligned}$$

$$+ 4 \int g'\left(|u|^2\right) \operatorname{Re}\left(u\bar{u}_x\right) \times \operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u - y_1\right)\right) dx + \int g'\left(|u|^2\right)|u|_x^2 \operatorname{Re}\left(u\bar{u}_{xt}\right) dx. \tag{76}$$

And then

$$\begin{aligned} - \int g'\left(|u|^2\right)|u|_x^2 \operatorname{Re}\left(u\bar{u}_{xt}\right) dx &= -\frac{d}{dt}\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)\operatorname{Re}\left(u\bar{u}_x\right) dx \\ &+ 2\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 \times \left(\operatorname{Re}\left(iu_{xx}\bar{u}\right) - \gamma|u|^2 - \operatorname{Re}\left(iy_1\bar{u}\right)\right) dx \\ &+ 2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right) \times \operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u - y_1\right)\right) dx. \end{aligned} \tag{77}$$

From (70) and (77) we have

$$\begin{aligned} \operatorname{Re}\int g\left(|u|^2\right) u\bar{u}_{xxt} dx &= -\frac{d}{dt}\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)\operatorname{Re}\left(u\bar{u}_x\right) dx \\ &- \frac{1}{2}\frac{d}{dt}\int g\left(|u|^2\right)\left|u_x\right|^2 dx \\ &+ 2\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 \times \left(\operatorname{Re}\left(iu_{xx}\bar{u}\right) - \gamma|u|^2 - \operatorname{Re}\left(iy_1\bar{u}\right)\right) dx \\ &+ 2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right) \times \operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u - y_1\right)\right) dx \\ &+ \int g'\left(|u|^2\right)\left|u_x\right|^2 \times \left(\operatorname{Re}\left(iu_{xx}\bar{u}\right) - \gamma|u|^2 - \operatorname{Re}\left(iy_1\bar{u}\right)\right) dx. \end{aligned} \tag{78}$$

By (65), (66), (68), and (78) we conclude that

$$\begin{aligned} &\frac{d}{dt}\left(\left\|u_{xx}\right\|^2 - 2\operatorname{Re}\int nu\bar{u}_{xx} - 2\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 - \int g\left(|u|^2\right)|u|_x^2 - 2\operatorname{Re}\int y_1\bar{u}_{xx}\right) \\ &+ 2\gamma\left(\left\|u_{xx}\right\|^2 - 2\operatorname{Re}\int nu\bar{u}_{xx} - 2\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 - \int g\left(|u|^2\right)\left|u_x\right|^2 - 2\operatorname{Re}\int y_1\bar{u}_{xx}\right) \\ &+ 2\gamma\int nu\bar{u}_{xx} dx + 4\gamma\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 dx \\ &+ 2\gamma\int g\left(|u|^2\right)\left|u_x\right|^2 dx + 2\gamma\operatorname{Re}\int y_1\bar{u}_{xx} dx \\ &- 2\operatorname{Re}\int u\bar{u}_{xx}\left(|u|_x^2 + \beta n + f\left(|u|^2\right) - y_2\right) dx \\ &+ 2\operatorname{Re}\int n\bar{u}_{xx}\left(-inu - \gamma u + ig\left(|u|^2\right) - iy_1\right) dx \\ &+ 2\gamma\operatorname{Re}\int g\left(|u|^2\right)\bar{u}u_{xx} dx \\ &+ 4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right) - \gamma|u|^2 - \operatorname{Re}\left(iy_1\bar{u}\right)\right) dx \\ &+ 4\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right) \times \operatorname{Re}\left(i\bar{u}_x\left(u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u - y_1\right)\right) dx \\ &+ 2\int g'\left(|u|^2\right)\left|u_x\right|^2 \times \left(\operatorname{Re}\left(iu_{xx}\bar{u}\right) - \gamma|u|^2 - \operatorname{Re}\left(iy_1\bar{u}\right)\right) dx \\ &+ 2\operatorname{Re}\int y_{1t}\bar{u}_{xx} dx = 0, \end{aligned} \tag{79}$$

where $\int \cdot = \int \cdot dx$.

For later purpose, we let

$$F\left(u, n, y_1\right) = -2\operatorname{Re}\int nu\bar{u}_{xx} dx - 2\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 dx - \int g\left(|u|^2\right)\left|u_x\right|^2 dx - 2\operatorname{Re}\int y_1\bar{u}_{xx} dx, \tag{80}$$

$$- G\left(u, n, y_1, y_2\right) = 2\gamma\int nu\bar{u}_{xx} dx + 4\gamma\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 dx$$

$$
\begin{aligned}
&+2\gamma\int g\left(|u|^2\right)|u_x|^2dx+2\gamma\operatorname{Re}\int y_1\overline{u}_{xx}dx\\
&-2\operatorname{Re}\int u\overline{u}_{xx}\left(|u|_x^2+\beta n+f\left(|u|^2\right)-y_2\right)dx\\
&+2\operatorname{Re}\int n\overline{u}_{xx}\left(-inu-\gamma u+ig\left(|u|^2\right)-iy_1\right)dx\\
&+2\gamma\operatorname{Re}\int g\left(|u|^2\right)\overline{u}u_{xx}dx\\
&+4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\overline{u}_x\right)\right)^2\left(\operatorname{Re}\left(iu_{xx}\overline{u}\right)-\gamma|u|^2\right.\\
&\qquad\left.-\operatorname{Re}\left(iy_1\overline{u}\right)\right)dx\\
&+4\int g'\left(|u|^2\right)\operatorname{Re}\left(u\overline{u}_x\right)\\
&\qquad\times\operatorname{Re}\left(i\overline{u}_x\left(u_{xx}-nu+i\gamma u\right.\right.\\
&\qquad\qquad\left.\left.+g\left(|u|^2\right)u-y_1\right)\right)dx\\
&+2\int g'\left(|u|^2\right)|u_x|^2\\
&\qquad\times\left(\operatorname{Re}\left(iu_{xx}\overline{u}\right)-\gamma|u|^2-\operatorname{Re}\left(iy_1\overline{u}\right)\right)dx\\
&+2\operatorname{Re}\int y_{1t}\overline{u}_{xx}dx.
\end{aligned}
\tag{81}
$$

Then from (79) we have

$$\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+2\gamma\left(\|u_{xx}\|^2+F\right)=G \tag{82}$$

or

$$\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+\gamma\left(\|u_{xx}\|^2+F\right)+\alpha\|u_{xx}\|^2=G-\gamma F. \tag{83}$$

By Lemma 5 and Agmon inequality we have

$$\|u(t)\|_{H^1}^2+\|u(t)\|_\infty^2+\|n(t)\|_H^2\leqslant 2C_2,\qquad\forall t\geqslant t_2. \tag{84}$$

In the following, we denote $C=C(\alpha,\beta,f,g,Y_0,a_{0t})$. By Lemma 3 and (84) we estimate the size of $|G-\gamma F|$ to get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+\gamma\left(\|u_{xx}\|^2+F\right)+\gamma\|u_{xx}\|^2\\
&\leqslant C\int|n|^2|u_{xx}|dx+C\int|u_x|^2|u_{xx}|dx\\
&\quad+C\int|u_x|^2|nu|dx+C\\
&\leqslant C\|u_{xx}\|\|n\|_4^2\\
&\quad+C\|u_{xx}\|\|u_x\|_4^2+C\|u\|_{L^\infty}\|n\|\|u_x\|_4^2+C\\
&\leqslant C\|u_{xx}\|\|n_x\|^{1/2}\|n\|^{3/2}+C\|u_{xx}\|^{3/2}\|u_x\|^{3/2}\\
&\quad+C\|u\|_{L^\infty}\|n\|\|u_{xx}\|^{7/4}\|u\|^{1/4}+C\\
&\leqslant C\|u_{xx}\|\|n_x\|^{1/2}+C\|u_{xx}\|^{7/4}+C\\
&\leqslant\frac{\gamma}{2}\|u_{xx}\|^2+\frac{\beta}{2}\|n_x\|^2+C.
\end{aligned}
\tag{85}
$$

Taking the inner product of (6) with n_{xx} in $H_{\text{per}}(\Omega)$, we see that

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|n_x\|^2+\int|u|_x^2n_{xx}dx-\beta\|n_x\|^2\\
&\quad+\int f\left(|u|^2\right)n_{xx}dx-\int y_2n_{xx}dx=0.
\end{aligned}
\tag{86}
$$

Since

$$
\begin{aligned}
\int|u|_x^2n_{xx}dx&=2\int\operatorname{Re}\left(u\overline{u}_xn_{xx}\right)dx\\
&=-2\int\operatorname{Re}\left(u\overline{u}_{xx}n_x+|u_x|^2n_x\right)dx,
\end{aligned}
\tag{87}
$$

by (86) we can deduce that

$$
\begin{aligned}
&\frac{d}{dt}\|n_x\|^2+4\int\operatorname{Re}\left(u\overline{u}_{xx}n_x\right)dx+4\int|u_x|^2n_xdx+2\beta\|n_x\|^2\\
&\quad+2\int f'\left(|u|^2\right)\left(u_x\overline{u}+u\overline{u}_x\right)n_xdx-2\int y_{2x}n_xdx=0.
\end{aligned}
\tag{88}
$$

From (5) we know that

$$
\begin{aligned}
&iu_{tx}+u_{xxx}-n_xu-nu_x+i\gamma u_x+g'\left(|u|^2\right)|u|_x^2u\\
&\quad+g\left(|u|^2\right)u_x-y_{1x}(x,t)=0.
\end{aligned}
\tag{89}
$$

Taking the real part of the inner product to (89) with u_{xx} in $H_{\text{per}}(\Omega)$, we have

$$
\begin{aligned}
&\operatorname{Re}\int iu_{tx}\overline{u}_{xx}-\operatorname{Re}\int n_xu\overline{u}_{xx}dx-\operatorname{Re}\int nu_x\overline{u}_{xx}dx\\
&\quad+\operatorname{Re}\int i\gamma u_x\overline{u}_{xx}dx+\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\overline{u}_{xx}dx\\
&\quad+\operatorname{Re}\int g\left(|u|^2\right)u_x\overline{u}_{xx}dx-\operatorname{Re}\int y_{1x}\overline{u}_{xx}dx=0.
\end{aligned}
\tag{90}
$$

Because of

$$\frac{d}{dt}\operatorname{Re}\int iu_x\overline{u}_{xx}dx=2\operatorname{Re}\int iu_{tx}\overline{u}_{xx}dx, \tag{91}$$

it holds that

$$\frac{1}{2}\frac{d}{dt}\operatorname{Re}\int iu_x\bar{u}_{xx}dx-\operatorname{Re}\int n_xu\bar{u}_{xx}dx-\operatorname{Re}\int nu_x\bar{u}_{xx}dx$$
$$+\operatorname{Re}\int i\gamma u_x\bar{u}_{xx}dx+\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\bar{u}_{xx}dx$$
$$+\operatorname{Re}\int g\left(|u|^2\right)u_x\bar{u}_{xx}dx-\operatorname{Re}\int y_{1x}\bar{u}_{xx}dx=0. \tag{92}$$

By (92) and (88), we find that

$$\frac{d}{dt}\|n_x\|^2+2\frac{d}{dt}\operatorname{Re}\int iu_x\bar{u}_{xx}dx+4\int|u_x|^2n_xdx$$
$$+2\beta\|n_x\|^2+2\int f'\left(|u|^2\right)|u|_x^2n_xdx-2\int y_{2x}n_xdx$$
$$-4\operatorname{Re}\int nu_x\bar{u}_{xx}dx+4\gamma\operatorname{Re}\int iu_x\bar{u}_{xx}dx \tag{93}$$
$$+4\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\bar{u}_{xx}dx$$
$$+4\operatorname{Re}\int g\left(|u|^2\right)u_x\bar{u}_{xx}dx-4\operatorname{Re}\int y_{1x}\bar{u}_{xx}dx=0.$$

That is,

$$\frac{d}{dt}\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx\right)$$
$$+2\beta\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx\right)$$
$$=4\beta\operatorname{Re}\int iu_x\bar{u}_{xx}dx-4\int|u_x|^2n_xdx$$
$$-2\int f'\left(|u|^2\right)|u|_x^2n_xdx+2\int y_{2x}n_xdx \tag{94}$$
$$+4\operatorname{Re}\int nu_x\bar{u}_{xx}dx-4\gamma\operatorname{Re}\int iu_x\bar{u}_{xx}dx$$
$$-4\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\bar{u}_{xx}dx$$
$$-4\operatorname{Re}\int g\left(|u|^2\right)u_x\bar{u}_{xx}dx+4\operatorname{Re}\int y_{1x}\bar{u}_{xx}dx.$$

For later use, we let

$$F_1(u,n)=\|n_x\|^2+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx, \tag{95}$$

$$G_1(u,n,y_1,y_2)=\text{the right hand side of }(94). \tag{96}$$

Then identity (94) as being equivalent to

$$\frac{d}{dt}F_1+2\beta F_1=G_1. \tag{97}$$

Similarly to (85) we estimate each term in (94) and then we get

$$\frac{d}{dt}\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx\right)$$
$$+\beta\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx\right)+\beta\|n_x\|^2$$
$$\leqslant 2\beta\operatorname{Re}\int iu_x\bar{u}_{xx}dx+C\|u_{xx}\|+C\|u_x\|_4^2\|n_x\|$$
$$+C\|n_x\|+C\|u_x\|_\infty\|n\|\|u_{xx}\| \tag{98}$$
$$\leqslant C\|u_{xx}\|+C\|u_x\|^{3/2}\|u_{xx}\|^{1/2}\|n_x\|+C\|n_x\|$$
$$+C\|u\|^{1/4}\|u_{xx}\|^{3/4}\|n\|\|u_{xx}\|$$
$$\leqslant\frac{\gamma}{2}\|u_{xx}\|^2+\frac{\beta}{2}\|n_x\|^2+C.$$

Let $\epsilon=\min\{\gamma,\beta\}$, and

$$E=\|u_{xx}\|^2+\|n_x\|^2+F+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx. \tag{99}$$

By (85) and (98) we deduce that

$$\frac{d}{dt}E+\epsilon E\leqslant C,\quad \forall t\geqslant t_2, \tag{100}$$

which has the same form with (57) in the proof of Lemma 5. Similarly to the study of (57) we can derive that

$$E(t_2)\leqslant C(R_2),\quad E(t)\leqslant\frac{2C}{\epsilon},\quad \forall t\geqslant t_{2*}, \tag{101}$$

where $t_{2*}=\inf\{t\mid t\geqslant t_{2*},C(R_2)e^{-\epsilon(t_{2*}-t_0)}\leqslant C_0/\epsilon\}$ and $C(R_2)=C(\gamma,\beta,f,g,Y_0,h_{1t},R_2)$ when $\|W_\tau\|_{H^2\times H^1}\leqslant R_2$. By (80) we deduce that

$$\left|F+2\operatorname{Re}\int iu_x\bar{u}_{xx}dx\right|$$
$$\leqslant 2\|u\|_\infty\|n\|\|u_{xx}\|$$
$$+C\|u\|_\infty^2\|u_x\|^2+\|u\|_\infty\|u_x\|^2 \tag{102}$$
$$+\|h_1\|_{L_b^2(\mathbb{R};\Sigma)}\|u_{xx}\|+2\|u_x\|\|u_{xx}\|+C$$
$$\leqslant C\|u_{xx}\|+C\leqslant\frac{1}{2}\|u_{xx}\|^2+C,$$

and then by (99), (101), and (102), we deduce that

$$\|u_{xx}\|^2+2\|n_x\|^2\leqslant C,\quad \forall t\geqslant t_{2*}, \tag{103}$$

which concludes the proof by Lemma 5. □

To study the AIM for the system, we construct the following higher order estimate.

Lemma 7. *Under the assumptions of* (9) *and* (10) *and Assumption 1, for each* $W_\tau \in H^3_{per}(\Omega) \times H^1_{per}(\Omega)$, *solutions of* (5)–(8) *satisfy*

$$\sup_{\tau\leqslant t\leqslant T}\|u_{xt}\| + \sup_{\tau\leqslant t\leqslant T}\|u\|_{H^3_{per}} \leqslant C, \quad \forall T > \tau,$$
$$\|n_{tx}\| \leqslant C_1, \quad \forall t > t_0, \tag{104}$$

where $C = C(T, \|u_\tau\|_{H^3_{per}}, \|n_\tau\|_{H^1_{per}}, C_0), C_1 = C(C_0)$.

Proof. Taking the partial derivative of (6) with respect to x, by (9) and (10) we have

$$\begin{aligned}\|n_{tx}\| &= \left\|-\beta n_x - |u|^2_{xx} - f'\left(|u|^2\right)|u|^2_x + h_{2x}\right\| \\ &\leqslant C\|n\|_{H^1_{\text{per}}} + C\|u\|_{H^2_{\text{per}}} \\ &\quad + \|h_2\|_{L^\infty_b(\tau,\infty;H^1_{\text{per}})} \leqslant C_1.\end{aligned} \tag{105}$$

Taking the partial derivatives of (5) with respect to x and t, we have that

$$\begin{aligned}&iu_{ttx} + u_{xxxt} - (n_{xt}u + n_tu_x + n_xu_t + nu_{xt}) + i\gamma u_{xt} \\ &\quad + g''\left(|u|^2\right)|u|^2_x|u|^2_t u + g'\left(|u|^2\right)|u|^2_{xt}u \\ &\quad + g'\left(|u|^2\right)|u|^2_t u_x + g'\left(|u|^2\right)|u|^2_x u_t + g\left(|u|^2\right)u_{xt} = h_{1xt}.\end{aligned} \tag{106}$$

Taking the inner product of (106) with u_{xt} and taking the imaginary part, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{xt}\|^2 + \gamma\|u_{xt}\|^2 \\ &\quad\leqslant \int\left|(n_{xt}u + n_tu_x + n_xu_t)\,\overline{u}_{xt}\right|dx \\ &\qquad + \int\Big|g''\left(|u|^2\right)u\overline{u}_{xt} \\ &\qquad\qquad \times\left(u_xu_t\overline{u}^2 + \overline{u}_xu_t|u|^2 + u_x\overline{u}_t|u|^2 + \overline{u}_x\overline{u}_t^2\right)\Big|dx \\ &\qquad + \int\Big|g'\left(|u|^2\right)u\overline{u}_{xt} \\ &\qquad\qquad \times\left(u_{tx}\overline{u} + \overline{u}_xu_t + u_x\overline{u}_t + u\overline{u}_{tx}\right)\Big|dx \\ &\qquad + \int\left|g'\left(|u|^2\right)u_x\overline{u}_{xt}\left(u_t\overline{u} + \overline{u}_tu\right)\right|dx \\ &\qquad + \int\left|g'\left(|u|^2\right)u_t\overline{u}_{xt}\left(u_x\overline{u} + \overline{u}_xu\right)\right|dx \\ &\qquad + \int\left|h_{1xt}\overline{u}_{xt}\right|dx.\end{aligned} \tag{107}$$

Since

$$\begin{aligned}\int\left|n_xu_t\overline{u}_{xt}\right|dx &\leqslant \|u_t\|_{L^\infty(\Omega)}\int\left|n_x\overline{u}_{xt}\right|dx \\ &\leqslant \|u_t\|_{H^1_{\text{per}}}\|n\|_{H^1_{\text{per}}}\|u_{xt}\| \\ &\leqslant C\|u_{xt}\|^2 + C, \\ \int\left|g'\left(|u|^2\right)u\overline{u}_{xt}u_{tx}\overline{u}\right|dx &\leqslant C\|u\|^2_{L^\infty(\Omega)}\|u_{xt}\|^2 \leqslant C\|u_{xt}\|^2,\end{aligned} \tag{108}$$

dealing with each term in (107) in the same way, we deduce that

$$\frac{d}{dt}\|u_{xt}\|^2 \leqslant C\|u_{xt}\|^2 + C. \tag{109}$$

By Gronwall lemma and (5) it follows that

$$\sup_{\tau\leqslant t\leqslant T}\|u_{xt}\| + \sup_{\tau\leqslant t\leqslant T}\|u\|_{H^3_{\text{per}}} \leqslant C, \quad \forall T > \tau, \tag{110}$$

where $C = C(T, \|u_\tau\|_{H^3}, \|n_\tau\|_{H^1})$, which completes the proof. □

4. Unique Existence of the Solution

In this section, we show the unique existence theorem of the solutions. Since uniform *a priori* estimates have been established in the former section, one can readily get the existence of the solution by *Galërkin*'s method (see [9, 14, 16, 19]) or operator semigroup method (see [6]). We show the theorem and prove it briefly for readers' convenience.

Theorem 8. *Under assumptions of Lemma 6, for each* $W_\tau \in E_0$, *systems* (5)–(8) *has a unique global solution* $W(x,t) \in L^\infty(\tau, T; E_0)$, *for all* $T > \tau$.

Proof. We prove this theorem briefly by two steps.

Step 1. Existence. By *Galërkin*'s method, we apply the following approximate solution

$$W^l(x,t) = \sum_{j=1}^{l} w^l_j(t)\,\eta_j(x) \tag{111}$$

to approach $W(x,t)$ and the solution of the problems (5)–(8), where $\{\eta_j\}_{j=1}^\infty$ is a orthogonal basis of $H(\Omega)$ satisfying $-\Delta\eta_j = \lambda_j\eta_j$ $(j = 1, 2, \ldots)$. And $W^l(x,t)$ satisfies, $j = 1, 2, \ldots, l$,

$$\begin{gathered}\left(iu^l_t + u^l_{xx} - n^lu^l + i\alpha u^l + g\left(\left|u^l\right|^2\right)u^l - a, \eta_j\right) = 0, \\ \left(n^l_t + \beta n^l + \left|u^l\right|^2_x + f\left(\left|u^l\right|^2\right) - b, \eta_j\right) = 0, \\ \left(W^l(x,\tau), \eta_j\right) = \left(W_\tau, \eta_j\right), \qquad W^l|_{\partial\Omega} = 0.\end{gathered} \tag{112}$$

It is easy to see that system (112) is an initial boundary value problem of ODE for the unknown coefficients (u^l_j, n^1_j);

the solution of which is known to be unique. Like [19], by the *a priori* estimates in Section 3 we know that $\{W^l\}_{l=1}^{\infty}$ converges (weakly star) to a $W(x,t)$ which solves (5)–(8).

Step 2. Uniqueness. Suppose W_1 and W_2 are two solutions of the problems (5)–(8). Let $W = W_1 - W_2$; then $W(x,t)$ satisfies

$$\begin{gathered} iu_t + u_{xx} - n_1u_1 + n_2u_2 + i\alpha u \\ + g\left(|u_1|^2\right)u_1 - g\left(|u_2|^2\right)u_2 = 0, \\ n_t + \beta n + |u_1|_x^2 - |u_2|_x^2 + f\left(|u_1|^2\right) - f\left(|u_2^2|\right) = 0, \\ W|_{t=\tau} = 0, \qquad W|_{\partial\Omega} = 0, \end{gathered} \tag{113}$$

which has nothing to do with symbols. Similarly to [19–21], we can deduce that $\|W\| = 0$, which concludes the proof. □

Similarly to [16], by Lemma 7 we note that systems (5)–(8) has an unique global smooth solution in $H^3_{\text{per}}(\Omega) \times H^1_{\text{per}}(\Omega)$, and, moreover, it follows the following lemma.

Lemma 9. *Under the assumptions of Lemma 7, for each $W_\tau \in H^3_{per}(\Omega) \times H^1_{per}(\Omega)$, problems (5)–(8) have a unique global solution $W(x,t) \in L^\infty(\tau,T;H^3_{per} \times H^1_{per})$, for all $T > \tau$, satisfying*

$$\|W(t)\|^2_{H^3_{per}\times H^1_{per}} \leqslant C, \quad \forall t \geqslant t_1, \tag{114}$$

where C relies on the data and t_1 relies on the data and R whenever $\|W_\tau\|_{H^3_{per}\times H^1_{per}} \leqslant R$.

5. Existence of the Compact Uniform Attractor

In this section, we derive the existence of the compact uniform attractor for the system applying Ball's idea (see [16, 17]). That is, first we construct the weak uniform attractor (the convergences are taken in the sense of weak topology), and then we show that the weak uniform attractor is actually the strong one.

First we recall the following facts. Each solution trajectory for systems (5)–(8) satisfie

$$\frac{d}{dt}\left(\|u_{xx}\|^2 + F\right) + 2\alpha\left(\|u_{xx}\|^2 + F\right) = G, \tag{115}$$

$$\frac{d}{dt}\left(\|n_x\|^2 + F_1\right) + 2\beta\left(\|n_x\|^2 + F_1\right) = G_1, \tag{116}$$

where F, G, F_1, and G_1 are given by (80), (81), (95), and (96), respectively. Moreover, by the uniform boundedness and the compactness embedding $H^k(\Omega) \hookrightarrow H^{k-1}(\Omega)$ (for all $k \in \mathbb{Z}_+$) we have that F, G and F_1, G_1 are all weakly continuous in $E_0 \times \Sigma$.

Since we have uniformly estimated the size of solutions by Lemma 6 and shown the unique existence of the solution by Theorem 8, following the method of [8, 15] we have the following theorem.

Theorem 10. *Under assumptions of Theorem 8, the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ generated by systems (5)–(8) is weakly $(E_0 \times \Sigma, E_0)$-continuous and it admits a weakly compact uniform attractor $\mathscr{A}_\Sigma$ in $E_0 = H^2_{per}(\Omega) \times H^1_{per}(\Omega)$.*

Proof. Since by Lemma 6 we know there exists a bounded uniform absorbing set, it suffices to prove that $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is weakly $(E_0\times\Sigma, E_0)$-continuous and the existence of the weak uniform attractor follows.

For any fixed $t_1 \geqslant \tau \in \mathbb{R}$, let

$$(W_{\tau k}, \sigma_k) \rightharpoonup (W_\tau, \sigma) \quad \text{in } E_0 \times \Sigma, \tag{117}$$

we will complete the proof if we deduce that

$$W_{\sigma_k}(t_1) \rightharpoonup W_\sigma(t_1) \quad \text{in } E_0, \tag{118}$$

where $W_{\sigma_k}(t_1) = (u_k(t_1), n_k(t_1)) = U_{\sigma_k}(t_1,\tau)W_{\tau k}$, and $W_\sigma(t_1) = (u(t_1), n(t_1)) = U_\sigma(t_1,\tau)W_\tau$.

By (117) and Theorem 8 we get the boundedness

$$\|W_{\tau k}\|_{E_0} \leqslant C, \tag{119}$$

$$\sup_{t\in[\tau,T]}\left\|W_{\sigma_k}(t)\right\|_{E_0} \leqslant C. \tag{120}$$

By Agmon inequality $\|v\|_\infty \leqslant C\|v\|_{H^1}$ we see that

$$\left\|W_{\sigma_k}(t)\right\|_\infty \leqslant C, \quad \forall \tau \leqslant t \leqslant T. \tag{121}$$

Note that

$$iu_{kt} = -u_{kxx} + n_ku_k - i\gamma u_k - g\left(|u_k|^2\right)u_k + y_{1k}(x,t), \tag{122}$$

$$n_{kt} = -|u_k|_x^2 - \beta n_k - f\left(|u_k|^2\right) + y_{2k}(x,t), \tag{123}$$

and $\sigma_k = (y_{1k}(x,t), y_{2k}(x,t)) \in \Sigma$. By (120) and (121) we find that $\partial_t W_{\sigma_k}(t) \in L^\infty(\tau,T;H)$ and

$$\left\|\partial_t W_{\sigma_k}(t)\right\|_{L^\infty(\tau,T;H)} \leqslant C. \tag{124}$$

Due to Theorem 8 and (124) we know that there exists $\widetilde{W}(t) \triangleq (\widetilde{u}(t), \widetilde{n}(t)) \in L^\infty(\tau,T;E_0)$ and subsequences of $\{W_{\sigma_k}(t)\}$, which are still denoted by $\{W_{\sigma_k}(t)\}$, such that

$$W_{\sigma_k}(t) \overset{*}{\rightharpoonup} \widetilde{W}(t) \quad \text{in } L^\infty(\tau,T;E_0), \tag{125}$$

$$\partial_t W_{\sigma_k}(t) \overset{*}{\rightharpoonup} \partial_t\widetilde{W}(t) \quad \text{in } L^\infty(\tau,T;H). \tag{126}$$

Besides, for all $t_1 \in [\tau.T]$, by (120) we know that there exists $W^0 \triangleq (u^0, n^0) \in E_0$ such that

$$W_{\sigma k}(t_1) \rightharpoonup W^0 \quad \text{in } E_0. \tag{127}$$

By (125) and a compactness embedding theorem, we claim that

$$u_k(t) \longrightarrow \widetilde{u}(t) \quad \text{strongly in } L^2(0,T;H). \tag{128}$$

In the following, we will show that $\widetilde{W}(t)$ is a solution of problems (5)–(8).

For all $v \in H$, $\forall \psi \in C_0^\infty(\tau, T)$, by (122) we find that

$$\int_\tau^T (iu_{kt}, \psi(t) v)\, dt + \int_\tau^T (u_{kxx}, \psi(t) v)\, dt - \int_\tau^T (n_k u_k, \psi(t) v)\, dt + \int_\tau^T (i\gamma u_k, \psi(t) v)\, dt + \int_\tau^T \left(g\left(|u_k|^2\right) u_k, \psi(t) v\right) dt - \int_\tau^T (y_{1k}(x,t), \psi(t) v)\, dt = 0. \tag{129}$$

Due to

$$\int_\tau^T (n_k u_k, \psi(t) v)\, dt - \int_\tau^T (\widetilde{n}\widetilde{u}, \psi(t) v)\, dt = \int_\tau^T ((u_k - \widetilde{u}) n_k, \psi(t) v)\, dt + \int_\tau^T (\widetilde{u}(n_k - \widetilde{n}), \psi(t) v)\, dt, \tag{130}$$

and by (121), (128), and (125)

$$\begin{aligned} \int_\tau^T ((u_k - \widetilde{u}) n_k, \psi(t) v)\, dt &\leqslant \sup_{0\leqslant t\leqslant T} \|n_k(t)\|_\infty \|\psi(t) v\|_{L^2(0,T;H)} \times \|u_k - \widetilde{u}\|_{L^2(0,T;H)} \longrightarrow 0, \\ \int_\tau^T (\widetilde{u}(n_k - \widetilde{n}), \psi(t) v)\, dt &= \int_\tau^T \left((n_k - \widetilde{n}), \psi(t) v\overline{\widetilde{u}}\right) dt \longrightarrow 0, \end{aligned} \tag{131}$$

we have

$$\int_\tau^T (n_k u_k, \psi(t) v)\, dt \longrightarrow \int_\tau^T (\widetilde{n}\widetilde{u}, v)\, \psi(t)\, dt. \tag{132}$$

Taking care of other terms of (129) in similar methods and taking the limit, we have

$$\int_\tau^T (i\widetilde{u}_t, v)\, \psi(t)\, dt + \int_\tau^T (\widetilde{u}_{xx}, v)\, \psi(t)\, dt - \int_\tau^T (\widetilde{n}\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\gamma\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T \left(g\left(|\widetilde{u}|^2\right)\widetilde{u}, v\right) \psi(t)\, dt - \int_\tau^T (y_1(x,t), v)\, \psi(t)\, dt = 0. \tag{133}$$

Therefore, in the sense of distributions it holds that

$$i\widetilde{u}_t + \widetilde{u}_{xx} - \widetilde{u}\widetilde{n} + i\gamma\widetilde{u} + g\left(|\widetilde{u}|^2\right)\widetilde{u} = y_1(x,t), \tag{134}$$

which shows that $(\widetilde{u}, \widetilde{n}, y_1(t))$ satisfies (5).

For all $v \in H$ and for all $\psi \in C_0^\infty(\tau, T)$ with $\psi(T) = 0$, $\psi(\tau) = 1$, by (122) we find that

$$-\int_\tau^T (iu_k, v)\, \psi'(t)\, dt + \int_\tau^T (u_{kxx}, v)\, \psi(t)\, dt - \int_\tau^T (n_k u_k, v)\, \psi(t)\, dt + \int_\tau^T (i\gamma u_k, v)\, \psi(t)\, dt + \int_\tau^T \left(g\left(|u_k|^2\right) u_k, v\right) \psi(t)\, dt - \int_\tau^T (y_{1k}(x,t), v)\, \psi(t)\, dt = i(u_k(\tau), v). \tag{135}$$

Assumption (117) implies that

$$u_k(\tau) = u_{\tau k} \rightharpoonup u_\tau \quad \text{in } H. \tag{136}$$

Then taking the limit of (135), by (136) we have

$$-\int_\tau^T (i\widetilde{u}, v)\, \psi'(t)\, dt + \int_\tau^T (\widetilde{u}_{xx}, v)\, \psi(t)\, dt - \int_\tau^T (\widetilde{n}\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\gamma\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T \left(g\left(|\widetilde{u}|^2\right)\widetilde{u}, v\right) \psi(t)\, dt - \int_\tau^T (y_1(x,t), v)\, \psi(t)\, dt = i(u_\tau, v). \tag{137}$$

While from (134) we know that

$$-\int_\tau^T (i\widetilde{u}, v)\, \psi'(t)\, dt + \int_\tau^T (\widetilde{u}_{xx}, v)\, \psi(t)\, dt - \int_\tau^T (\widetilde{n}\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\gamma\widetilde{u}, v)\, \psi(t)\, dt + \int_\tau^T \left(g\left(|\widetilde{u}|^2\right)\widetilde{u}, v\right) \psi(t)\, dt - \int_\tau^T (y_1(x,t), v)\, \psi(t)\, dt = i(\widetilde{u}(\tau), v). \tag{138}$$

It come from (137) and (138) that

$$(u_\tau, v) = (\widetilde{u}(\tau), v), \qquad \forall v \in H, \tag{139}$$

and then

$$\widetilde{u}(\tau) = u_\tau. \tag{140}$$

Equations (134) and (140) imply that

$$\widetilde{u}(t) = u(t). \tag{141}$$

For all $v \in H$, $\forall \psi \in C_0^\infty(\tau, t_1)$, with $\psi(\tau) = 0$, $\psi(t_1) = 1$. Then repeating the procedure of the proofs of (135)–(138), by (127) we deduce that

$$u^0 = \widetilde{u}(t_1). \tag{142}$$

It comes from (127), (141), and (142) that

$$u_k(t_1) \rightharpoonup u(t_1) \quad \text{in } H^2_{\text{per}}(\Omega). \tag{143}$$

Similarly, we can also deduce that

$$n_k(t_1) \rightharpoonup n(t_1) \quad \text{in } H^1_{\text{per}}(\Omega), \tag{144}$$

which together with (143) proves (118) and then the theorem. □

To prove the strong compactness of the attractor $\mathscr{A}_\Sigma$, we recall the following two lemmas.

Lemma 11. *Let $(X, \|\cdot\|_X)$ be a uniform convex Banach space (particularly, a Hilbert space) and let $\{x_k\}_{k\geqslant 0}$ be a sequence in X. If $x_k \rightharpoonup x_0$ and $\|x_k\|_X \to \|x_0\|_X$, then $x_k \to x_0$.*

Lemma 12. *Let $\{x_k\}_{k\geqslant 0}$ be a sequence in B^* space X. If $x_k \rightharpoonup x_0$, then*

$$\sup_{k\geqslant 1}\|x_k\|_X < \infty, \qquad \|x_0\|_X \leqslant \liminf_{k\to\infty}\|x_k\|_X. \tag{145}$$

Theorem 13. *Under assumptions of Theorem 8, the weak uniform attractor $\mathscr{A}_\Sigma$ in Theorem 10 is actually the strong one for the system in E_0.*

Proof. Since a point (w, m) belongs to the weak uniform attractor $\mathscr{A}_\Sigma$ if and only if there exist two sequences $\{w^0_k, m^0_k\}_{k=1}^\infty$ and $\{t_k\}_{k=1}^\infty$ such that for all $\sigma(t) \in \Sigma$, it uniformly holds that

$$U_\sigma(t_k, \tau)\left(w^0_k, m^0_k\right) \rightharpoonup (w, m) \quad \text{in } E_0,\ k \longrightarrow \infty, \tag{146}$$

where $t_k \to \infty$ as $k \to \infty$. The theorem is concluded if the weak convergence is strong.

For each $S > \tau$ fixed, since $t_k \to \infty$, we can consider it as $S < t_k - 2\tau$, $k \in N_+$. By Lemma 6 we know $U_\sigma(t_k - S, \tau)(w^0_k, m^0_k)$ is bounded in E_0, and then there exists a $(v, p) \in E_0$ and a subsequence of $U_\sigma(t_k - S, \tau)(w^0_k, m^0_k)$, which is still denoted by $U_\sigma(t_k - S, \tau)(w^0_k, m^0_k)$, such that

$$U_\sigma(t_k - S, \tau)\left(w^0_k, m^0_k\right) \rightharpoonup (v, p) \quad \text{in } E_0. \tag{147}$$

Let

$$\begin{aligned}
(w_k(t), m_k(t)) &= U_{T(t_k-S-\tau)\sigma}(t, \tau)\, U_\sigma(t_k - S, \tau)\left(w^0_k, m^0_k\right) \\
&= U_\sigma(t + t_k - S - \tau, t_k - S)\, U_\sigma(t_k - S, \tau)\left(w^0_k, m^0_k\right) \\
&= U_\sigma(t + t_k - S - \tau, \tau)\left(w^0_k, m^0_k\right),
\end{aligned} \tag{148}$$

where $T(\cdot)$ is the translation operator on Σ. Since $\sigma(t)$ is quasiperiodic, there exists a $\sigma^* \in \Sigma$ such that

$$T(t_k - S - \tau)\sigma \overset{*}{\rightharpoonup} \sigma^* \quad \text{in } \Sigma. \tag{149}$$

Therefore, by (147) and (148) and the weak $(E\times\Sigma)$-continuity of $U_{\sigma\in\Sigma}(t, \tau)$ we see that

$$(w_k(t), m_k(t)) \rightharpoonup U_{\sigma^*}(t, \tau)(v, p) \quad \text{in } E_0,\ \forall t > \tau, \tag{150}$$

and by taking $t = S + \tau$,

$$(w, m) = U_{\sigma^*}(S + \tau, \tau)(v, p). \tag{151}$$

From the first equality of (148) we can consider $(w_k(t), m_k(t))$ as the solution trajectory, starting at $U_\sigma(t_k - S, \tau)(w^0_k, m^0_k)$, created by $U_{T(t_k-S-\tau)\sigma}(t, \tau)$. Hence, by (115) and from the boundedness that $\|U_\sigma(t_k - S, \tau)w^0_k\|^2_{H^2_{\text{per}}} \leqslant C$ we find that

$$\begin{aligned}
&\|w_k(t)\|^2_{H^2_{\text{per}}} + F(w_k(t), m_k(t)) \\
&\quad = e^{-2\alpha(t-\tau)}\Big(\left\|U_\sigma(t_k - S, \tau)w^0_k\right\|^2_{H^2_{\text{per}}} \\
&\qquad + F\left(U_\sigma(t_k - S, \tau)w^0_k, U_\sigma(t_k - S, \tau)m^0_k\right)\Big) \\
&\qquad + \int_\tau^t e^{-2\alpha(t-\tau)}G(w_k(s), m_k(s))\,ds \\
&\quad \leqslant e^{-2\alpha(t-\tau)}\left(C + F\left(U_\sigma(t_k - S, \tau)w^0_k, U_\sigma(t_k - S, \tau)m^0_k\right)\right) \\
&\qquad + \int_\tau^t e^{-2\alpha(t-\tau)}G(w_k(s), m_k(s))\,ds.
\end{aligned} \tag{152}$$

Since F and G are weakly continuous in E_0, by taking $t = S + \tau$ in (152), from (148), (147), (150), (151), and the Lebesgue dominated convergence theorem we get that

$$\begin{aligned}
&\limsup_{k\to\infty}\left\|U_\sigma(t_k, \tau)w^0_k\right\|^2_{H^2_{\text{per}}} + F\left(U_{\sigma^*}(S + 2\tau, \tau)(v, p)\right) \\
&\quad \leqslant e^{-2\alpha S}(C + F(v, p)) \\
&\qquad + \int_\tau^{S+\tau} e^{-2\alpha S}G\left(U_{\sigma^*}(s + \tau, \tau)(v, p)\right)ds.
\end{aligned} \tag{153}$$

While $(w, m) = U_{\sigma^*}(S+\tau, \tau)(v, p)$, we can consider (w, m) as the solution at $S + \tau$ corresponding to the initial data (v, p) and the symbol σ^*. Similarly to (152) we have

$$\begin{aligned}
&\|w\|^2_{H^2_{\text{per}}} + F(w, m) \\
&\quad = e^{-2\alpha S}\left(\|v\|^2_{H^2_{\text{per}}} + F(v, p)\right) \\
&\qquad + \int_\tau^{S+\tau} e^{-2\alpha S}G\left(U_{\sigma^*}(s + \tau, \tau)(v, p)\right)ds.
\end{aligned} \tag{154}$$

Deducting (154) from (153), we see that

$$\begin{aligned}
&\limsup_{k\to\infty}\left\|U_\sigma(t_k, \tau)w^0_k\right\|^2_{H^2_{\text{per}}} \\
&\quad \leqslant \|w\|^2_{H^2_{\text{per}}} + Ce^{-2\alpha S} - e^{-2\alpha S}\|v\|^2_{H^2_{\text{per}}} \\
&\quad \leqslant \|w\|^2_{H^2_{\text{per}}} + Ce^{-2\alpha S}.
\end{aligned} \tag{155}$$

Since S is fixed arbitrarily, let $S \to \infty$; we conclude that

$$\limsup_{k\to\infty}\left\|U_\sigma(t_k,\tau)w_k^0\right\|^2_{H^2_{\text{per}}} \leqslant \|w\|^2_{H^2_{\text{per}}}. \quad (156)$$

On the other hand, by Lemma 12, the weak convergence $U_\sigma(t_k,\tau)w_k^0 \rightharpoonup w$ implies that

$$\liminf_{k\to\infty}\left\|U_\sigma(t_k,\tau)w_k^0\right\|^2_{H^2_{\text{per}}} \geqslant \|w\|^2_{H^2_{\text{per}}}. \quad (157)$$

It follows from the previous two inequalities that

$$\lim_{k\to\infty}\left\|U_\sigma(t_k,\tau)w_k^0\right\|^2_{H^2_{\text{per}}} = \|w\|^2_{H^2_{\text{per}}}. \quad (158)$$

Similarly to the previous arguments, by using (116) we can derive that

$$\lim_{k\to\infty}\left\|U_\sigma(t_k,\tau)m_k^0\right\|^2_{H^2_{\text{per}}} = \|m\|^2_{H^2_{\text{per}}}. \quad (159)$$

By (146), (158), and (159) and Lemma 11, we conclude that $U_\sigma(t_k,\tau)(w_k^0,m_k^0) \to (w,m)$ in E_0, which completes the proof. □

Remark 14. We remark that up to this point the quasiperiodicity of the forces is not essentially necessary. We have actually used the uniform boundedness and the weak compactness of the symbol space in H, which can be totally satisfied by translation bounded external forces. In other words, if $h_i(x,t)(i=1,2)$ are relaxed to be translation bounded: $h_i \in L_b^2(\tau,\infty;H^1_{\text{per}}(\Omega))$, then all the results here still hold.

6. Approximate Inertial Manifolds for (5)–(8)

6.1. Extending and Splitting the Phase Space. From Theorem 13 we know the systems (5)–(8) create a family of processes $\{U_{\sigma\in\Sigma}(t,\tau)\}$, which admit a compact uniform attractor in E_0. Then from phase plane extension formula in [18], there is a semigroup $\{S(t)\}$, where

$$S(t)(W_0,\omega_0) = \left(U_{\omega_0}(t,0)W_0, T(t)\omega_0\right), \quad W_0\in E_0,\ \omega_0\in T^k,\ t\geqslant 0, \quad (160)$$

which is created by the following autonomous system:

$$iu_t + u_{xx} - nu + i\gamma u + g\left(|u|^2\right)u = H_1(x,\omega_1(t)), \quad (161)$$

$$n_t + \beta n + |u|_x^2 + f\left(|u|^2\right) = H_2(x,\omega_2(t)), \quad (162)$$

$$\frac{\partial\omega_i(t)}{\partial t} = \alpha^i \quad (i=1,2), \quad (163)$$

$$(W(x,0),\omega(x,0)) = (W_0(x),\omega_0(x)). \quad (164)$$

Let $B(u,n)=un$ be a bilinear operator: $H^2\times H^1 \to H^1$, $H(u)=|u|_x^2$ a nonlinear operator: $H^2 \to H^1$, and $A=-\partial_{xx}$.

Equations (161)–(163) can be transformed into the following abstract differential form

$$\begin{gathered} iu_t - Au - B(n,u) + i\gamma u + g\left(|u|^2\right)u = H_1(x,\omega_1(t)), \\ n_t + H(u) + \beta n + f\left(|u|^2\right) = H_2(x,\omega_2(t)), \\ \frac{\partial\omega_i(t)}{\partial t} = \alpha^i \quad (i=1,2). \end{gathered} \quad (165)$$

Since A is an unbounded self-conjugate compact operator, there is a complete orthogonal set $\{\eta_j\}_{j=1}^\infty$ of eigenfunctions of A such that $A\eta_j = \lambda_j\eta_j$, and

$$0\leqslant\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_j \longrightarrow +\infty, \quad j\longrightarrow+\infty. \quad (166)$$

For all $m>0$, we let $P: H \to H_m := \text{span}\{\eta_1,\dots,\eta_m\}$ be a projective operator and let $Q := I-P : H \to H_m^\perp$. Taking the projection of (165) we get

$$i\frac{dy}{dt} - Ay - PB(u,n) + i\gamma y + P\left(g\left(|u|^2\right)u\right) = PH_1(x,\omega_1(x)), \quad (167)$$

$$i\frac{dz}{dt} - Az - QB(u,n) + i\gamma z + Q\left(g\left(|u|^2\right)u\right) = QH_1(x,\omega_1(x)), \quad (168)$$

$$\frac{dp}{dt} + PH(u) + \beta p + Pf\left(|u|^2\right) = PH_2(x,\omega_2(t)), \quad (169)$$

$$\frac{dq}{dt} + QH(u) + \beta q + Qf\left(|u|^2\right) = QH_2(x,\omega_2(t)), \quad (170)$$

where $y=Pu$, $z=Qu$, $p=Pn$, and $q=Qn$.

By Parseval's formula we can get the following propo sition.

Proposition 15. *For all $v\in H$, there is the following expansion:*

$$v = \sum_{j=1}^\infty \left\langle v,\eta_j\right\rangle\eta_j, \quad (171)$$

and if $v\in H^1$,

$$\|v\|^2 = \sum_{j=1}^\infty\left|\left\langle v,\eta_j\right\rangle\right|^2, \qquad \|\nabla v\|^2 = \sum_{j=1}^\infty\lambda_j\left|\left\langle v,\eta_j\right\rangle\right|^2. \quad (172)$$

Moreover, because of (166)*, it holds that*

$$\|\nabla(Qv)\|^2 \geqslant \lambda_{m+1}\|Qv\|^2, \qquad \|\nabla(Pv)\|^2\leqslant\lambda_m\|Pv\|^2. \quad (173)$$

From Lemma 9, Proposition 15 and Agmon inequality $\|v\|_\infty \leqslant \|v\|_{H^1}$, we can deduce the following lemma.

Lemma 16. *If assumptions of Lemma 9 are satisfied, the solution* (u,n) *satisfies*

$$\begin{gathered}\|u\|_{H^3}, \|n\|_{H^1}, \|n\|_\infty, \|u\|_\infty \leqslant C, \\ \|Az\|, \left\|A^{1/2}z\right\|, \|z\|, \|z_t\|, \|q\|, \|q_t\| \leqslant C\lambda_{m+1}^{-1/2}, \\ \forall t \geqslant t_1.\end{gathered} \tag{174}$$

6.2. Constructing the AIM. Now we are in the position to show the AIM for (5)–(8).

Set $\Phi : H_m \times H_m \times T^k \to H_m^\perp \times H_m^\perp$,

$$\Phi(y,p,\omega) = (\psi_1,\psi_2), \quad \forall (y,p,\omega) \in H_m \times H_m \times T^k, \tag{175}$$

where (ψ_1,ψ_2) satisfies

$$\begin{aligned} &-A\psi_1 - QB(y,p) + i\gamma\psi_1 + Q\left(g\left(|y|^2\right)y\right) \\ &\quad = QH_1(x,\omega_1(t)), \end{aligned} \tag{176}$$

$$QH(y) + \beta\psi_2 + Qf\left(|y|^2\right) = QH_2(x,\omega_2(t)). \tag{177}$$

Let $\Pi_1 : H \times H \times T^k \to H \times H, \Pi_2 : H \times H \times T^k \to T^k$ be orthogonal projection mappings.

The following theorem shows that Graph(Φ), the graph of mapping Φ, is just an AIM for the autonomous system (165) and Π_1 Graph(Φ) is for problems (5)–(8), which concludes this paper.

Theorem 17. *Under assumptions of Lemma 9, it holds that*

$$\operatorname{dist}_{H^2\times H^1}((W,\omega), \operatorname{Graph}(\Phi)) \leqslant C\lambda_{m+1}^{-1/2}, \quad t \geqslant t_1, \tag{178}$$

where Graph(Φ) $= ((y,p,\omega),(\psi_1,\psi_2))$ *is the graph of* Φ *and* C *depends only on the data. Moreover,*

$$\operatorname{dist}_{H^2\times H^1}\left(W(t), \prod_1 \operatorname{Graph}(\Phi)\right) \leqslant C\lambda_{m+1}^{-1/2}, \quad t \geqslant t_1, \tag{179}$$

which shows that Π_1 Graph(Φ) *is an approximate inertial manifold for problems* (5)–(8).

Proof. From (176) and (168) we deduce that

$$\begin{aligned} A\psi_1 - Az &= Q(B(u,n) - B(y,p)) + i\gamma(\psi_1 - z) \\ &\quad + Q\left(g\left(|u|^2\right)u\right) - Q\left(g\left(|y|^2\right)y\right) - iz_t \\ &= Q(B(u,n) - B(y,n) + B(y,n) - B(y,p)) \\ &\quad + i\gamma(\psi_1 - z) + g\left(|u|^2\right)z - iz_t \\ &= Q(B(z,n) + B(y,q)) + i\gamma(\psi_1 - z) \\ &\quad + g\left(|u|^2\right)z - iz_t \\ &= Q(nz + qy) + i\gamma(\psi_1 - z) + g\left(|u|^2\right)z - iz_t, \end{aligned} \tag{180}$$

which implies that

$$\begin{aligned} \|A\psi_1 - Az\| &\leqslant \|n\|_\infty \|z\| + \|y\|_\infty \|q\| \\ &\quad + \gamma\|\psi_1 - z\| + \left\|g(|u|^2)\right\|_\infty \|z\| + \|z_t\|. \end{aligned} \tag{181}$$

Since

$$\|\psi_1 - z\| \leqslant \lambda_{m+1}^{-1}\|A\psi_1 - Az\|, \tag{182}$$

from (181) and Lemma 16, we get

$$\|A\psi_1 - Az\| \leqslant C\lambda_{m+1}^{-1/2} + \gamma\lambda_{m+1}^{-1}\|A\psi_1 - Az\|. \tag{183}$$

Because $\lambda_j \to +\infty$ $(j \to \infty)$, there exists $m_0 > 0$ such that for all $m > m_0$,

$$\|A\psi_1 - Az\| \leqslant C\lambda_{m+1}^{-1/2}, \quad t \geqslant t_1. \tag{184}$$

From (177) and (170) we see that

$$\begin{aligned} \beta\psi_2 - \beta q &= Q(H(u) - H(y)) \\ &\quad + Q\left(f\left(|u|^2\right) - f\left(|y|^2\right)\right) + q_t. \end{aligned} \tag{185}$$

While $u = Pu + Qu = y \oplus z$,

$$\begin{aligned} H(u) - H(y) &= |u|_x^2 - |y|_x^2 = |z|_x^2, \\ f\left(|u|^2\right) - f\left(|y|^2\right) &= f'\left(|\zeta|^2\right)\left(|u|^2 - |y|^2\right) \\ &= f'\left(|\zeta|^2\right)|z|^2, \end{aligned} \tag{186}$$

where $\zeta \in H^2$ and $|y|^2 \leqslant |\zeta|^2 \leqslant |u|^2$. Then from (185), we see that

$$\begin{aligned} &\left\|A^{1/2}(\psi_2 - q)\right\| \\ &\quad \leqslant C\left\|A|z|^2\right\| + C\left\|A^{1/2}\left(f'\left(|\zeta|^2\right)|z|^2\right)\right\| + C\left\|A^{1/2}q_t\right\|. \end{aligned} \tag{187}$$

By Lemma 16 we can deduce that

$$\begin{aligned} &\left\|A^{1/2}\left(f'\left(|\zeta|^2\right)|z|^2\right)\right\| \\ &\quad \leqslant \left\|\left(A^{1/2}f'\left(|\zeta|^2\right)\right)|z|^2\right\| + C\left\|A^{1/2}|z|^2\right\| \\ &\quad \leqslant C\left\||z|^2A^{1/2}|\zeta|^2\right\| + C\|z_x\|_\infty\|z\| \\ &\quad \leqslant C\left\||z|^2\right\| + C\|z\| \leqslant C\lambda_{m+1}^{-1/2}. \end{aligned} \tag{188}$$

While from Lemma 6 we know that

$$\begin{gathered} \|Az\| \leqslant C\lambda_{m+1}^{-1/2}, \qquad \left\|A^{1/2}z\right\| \leqslant C\lambda_{m+1}^{-1/2}, \\ \left\|A^{1/2}q_t\right\| \leqslant C\lambda_{m+1}^{-1/2}. \end{gathered} \tag{189}$$

Therefore,

$$\left\|A|z|^2\right\| \leqslant C\|z\|_\infty\|Az\| + C\left\|A^{1/2}z\right\|_\infty\left\|A^{1/2}z\right\| \leqslant C\lambda_{m+1}^{-1/2}. \tag{190}$$

By (187), (188), (189), and (190), we get

$$\left\| A^{1/2} \left(\psi_2 - q \right) \right\| \leqslant C \lambda_{m+1}^{-1/2}. \tag{191}$$

Then from (184) and (191) we can conclude that

$$\operatorname{dist}_{H^2 \times H^1} \left((W, \omega), \operatorname{Graph}(\Phi) \right) \leqslant C \lambda_{m+1}^{-1/2}, \qquad t \geqslant t_1. \tag{192}$$

The estimate (179) follows from

$$\begin{aligned} & \operatorname{dist}\left(W(t), \prod_1 \operatorname{Graph}(\Phi) \right) \\ & \quad = \operatorname{dist}\left(U_\omega(t, \tau) W_\tau, \prod_1 \operatorname{Graph}(\Phi) \right) \\ & \quad = \operatorname{dist}\left(\prod_1 S(t) \left(W_0, \omega_0 \right), \prod_1 \operatorname{Graph}(\Phi) \right) \\ & \quad \leqslant \operatorname{dist}\left(\prod_1 S(t) \left(W_0, \omega_0 \right), \prod_1 \operatorname{Graph}(\Phi) \right) \\ & \qquad + \operatorname{dist}\left(\prod_2 S(t) \left(W_0, \omega_0 \right), \prod_2 \operatorname{Graph}(\Phi) \right) \\ & \quad = \operatorname{dist}\left(S(t) \left(W_0, \omega_0 \right), \operatorname{Graph}(\Phi) \right) \\ & \quad = \operatorname{dist}\left((W, \omega), \operatorname{Graph}(\Phi) \right) \leqslant C \lambda_{m+1}^{-1/2}, \end{aligned} \tag{193}$$

and we complete the proof. □

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