Research Article

Robust Fault Detection of Linear Uncertain Time-Delay Systems Using Unknown Input Observers

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This paper deals with the problem of fault detection for linear uncertain time-delay systems. The proposed method for Luenberger observers is developed for unknown input observers (UIOs), and a novel procedure for the design of residual based on UIOs is presented. The design procedure is carried out based on the H_∞ model matching approach which minimizes the difference between generated residuals by the optimal observer and those by the designed observer in the presence of uncertainties. The optimal observer is designed for the ideal system and works so that the fault effect is maximized while the exogenous disturbances and noise effects are minimized. This observer can give disturbance decoupling in the presence of noise and uncertainties for linear uncertain time-delay systems. The developed method is applied to a numerical example, and the simulation results show that the proposed approach is able to detect faults reliably in the presence of modeling errors, disturbances, and noise.

1. Introduction

Fault detection and isolation (FDI) is an essential and challenging problem in many industrial applications. Among the various reported methods, much attention has been paid to model based approaches in the field of control engineering in recent years. For example, fault detection problem for discrete-time Markov jump systems and switched systems is investigated in [1, 2], respectively. The problem of fault reconstruction for a class of descriptor linear systems using sliding mode observers is presented in [3]. The sliding mode observers have been designed such that the actuator fault can be reconstructed using output measurements. The data-driven scheme for FDI is presented in [4] which exploits an adaptive residual generator and a bank of isolation observers. The designed scheme obtains observer parameters without identification of complete process model.

However, model based approaches are based on some idealized assumptions, one of which is that the mathematical model of the plant is a faithful replica of the plant dynamics [5]. As the mathematical model of a plant hardly represents its complete behavior, due to the existence of model uncertainty, noise, and unknown disturbances, it is essential to design a fault diagnosis system to take these effects into consideration. Motivated by the abovementioned issues, a robust fault detection scheme is exploited to design fault detection systems so that high sensitivity to faults as well as low sensitivity to uncertainties and perturbation can be obtained. Optimization techniques are widely used to solve this problem. One of the commonly used approaches to design such FDI scheme is representing the design procedure by H_∞ and H_- performance indexes. The main advantage of this approach is that it can be solved by linear matrix inequality (LMI) [6]. In [7, 8], a two-step FDI design methodology is presented.

In this methodology, the optimal fault detection filter (FDF) has firstly been designed, neglecting the existence of model uncertainty. Next, the FDF, which is used as residual generator, has been obtained via H_∞ model matching technique. The same approach is used in [9]; however, different dynamics is considered for the FDF.

Time delay is an inherent characteristic of many industrial systems; therefore, robust FDI problem for LTI systems with time delay received great attention over recent decades, and numerous articles have been presented. One approach is to solve the formulated design procedure using the eigenstructure assignment approach in which the residual signal is thoroughly decoupled from delay-free unknown input. Then the effect of the unknown input is minimized using H_∞ norm [10]. In the presence of uncertainty, the same approach as delay-free case can be employed to obtain a robust FDI system. Indeed, solving the H_∞ model matching problem results in achieving an FDF which acts the same as the optimal one [11–13]. Although the same approach is considered in these references, solving procedures are completely different owing to the difference between the dynamic of the filter and system. In [14], the problem of robust FDF design for the class of linear systems has been investigated. The system is subjected to mixed neutral and discrete time-varying delays and some nonlinear perturbations. The Luenberger type observer has been utilized to design FDF such that the residual signals effectively show fault occurrence.

Another approach commonly used to robust FDI scheme is to employ the unknown input observers (UIO), in which the residual is designed to be sensitive to faults but insensitive to unknown disturbances. Although the UIO has been widely used in estimation problems in both time delay and delay-free systems [15–18], there are few references that handle the problem of designing robust FDI [19, 20]. In [20], a design procedure has been proposed for delay-free systems so that perturbations and exogenous signals have less effect on the residual signal and the fault has a detectable effect on the residual; however, the problem has not been presented in H_∞ model matching technique. Motivated by this consideration, a robust FDF design using UIO for uncertain systems with time delay is presented and solved using H_∞ model matching approach. In contrast to our previous work [18], we are concerned to design a robust FDF for the case in which the dynamic characteristic of fault signal is known. For this purpose, a two-step design procedure is developed. In the first step, the optimal fault detection based on UIO is designed for the system without uncertainty. Next, the UIO-based fault detection filter is approached to optimal one in the sense of H_∞ norm. It is demonstrated through simulation that the presented fault detection observer is robust against uncertainty and sensitive enough to the faults.

Notation. The notations used throughout the paper are fairly standard. **I** and **0** represent identity matrix and zero matrix; the superscript "*T*" stands for matrix transposition. $\| \cdot \|$ refers to the Euclidean vector norm or the induced matrix 2-norm. diag$\{\cdot\}$ represents a block diagonal matrix.

The notation $P > 0$ means that P is real symmetric and positive definite; the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix.

2. Problem Statement

Many different industrial systems such as mechanical, electrical, meteorological, chemical, economic, and biological systems include time delay. In many studies linearized model of these systems around point of operation is considered. However, there are always some discrepancies between the real dynamics of the system and linearized model. These differences arise from systems uncertainty, as a consequence of neglecting dynamics, and changes in system parameters. Therefore, the following linear uncertain system with additive disturbances and time delay is considered to represent the described model:

$$\dot{x}(t) = (A + \Delta A(t)) x(t) + (A_d + \Delta A_d(t)) x(t - \tau)$$
$$+ (B + \Delta B(t)) u(t) + Ed(t) + F_x f(t) + Rn(t), \quad (1)$$
$$y(t) = Cx(t) + F_y f(t) + Dn(t),$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $y(t) \in \mathfrak{R}^p$ is the output vector, $u(t) \in \mathfrak{R}^q$ is the input vector, $d(t) \in \mathfrak{R}^m$ is an unknown scalar function representing the disturbance that belongs to $L_2^m(0, \infty)$, $f(t) \in \mathfrak{R}^f$ denotes the faults, and $n(t) \in \mathfrak{R}^r$ represents the noise. Note that $\Delta A(t)$, $\Delta B(t)$, and $\Delta A_d(t)$ are the norm bounded time-varying uncertainties of the matrices A, B, and A_d, respectively, and $\tau \geq 0$ is a constant delay. It is assumed that the characteristics of uncertainty matrices belong to

$$\Omega_1 = \left\{ \Delta A(t) \mid \Delta A(t) = M_1 \Sigma_1(t) N_1, \ \Sigma_1^T(t) \Sigma_1(t) \leq I \right\},$$
$$\Omega_2 = \left\{ \Delta B(t) \mid \Delta B(t) = M_2 \Sigma_2(t) N_2, \ \Sigma_2^T(t) \Sigma_2(t) \leq I \right\},$$
$$\Omega_3 = \left\{ \Delta A_d(t) \mid \Delta A_d(t) = M_3 \Sigma_3(t) N_3, \ \Sigma_3^T(t) \Sigma_3(t) \leq I \right\},$$
$$(2)$$

where M_i and N_i are predefined matrices. It is supposed that all over the paper the dimensions of matrices are compatible if they are not explicitly mentioned.

The dynamic characteristic of fault signal can be described by [21]

$$\dot{\theta}(t) = A_\theta \theta(t), \quad t \geq t_f,$$
$$\theta(t) = 0, \quad t \in [0, t_f],$$
$$\theta(t_f) = \theta_0, \quad (3)$$
$$f(t) = F_\theta \theta(t),$$

where t_f is the time when a fault occurs and A_θ and F_θ are known matrices with appropriate dimensions. The initial time, t_f, and initial state, θ_0, are supposed to be unknown. The dynamic equation (3) represents any fault with known

dynamic characteristic and unknown amplitude and phase [21].

Unknown input observer for the class of time-delay system (1) is considered as [22]

$$\dot{z}(t) = Fz + Gz(t - \tau) + Hu(t) + K_{1z}y(t) + K_{2z}y(t - \tau),$$

$$\hat{x}(t) = z(t) + L_1 y(t), \tag{4}$$

where $\hat{x}(t)$ is the estimated state vector. The dynamic of $\hat{x}(t)$ is governed by

$$\dot{\hat{x}}(t) = F\hat{x} + G\hat{x}(t - \tau) + Hu(t) + L_1\dot{y}(t)$$
$$+ L_2 y(t) + L_3 y(t - \tau), \tag{5}$$

where F, G, H, and L_1 are the observer matrices and $L_2 = K_{1z} - FL_1, L_3 = K_{2z} - GL_1$.

The observer matrices will be designed such that the disturbance and input are decoupled from the estimation error defined by $e(t) = x(t) - \hat{x}(t)$. When UIO-based filter defined by (4) is applied to the system described in (1), the state estimation error will be

$$\dot{e}(t) = Fe(t) + Ge(t - \tau) + ((I - L_1C)A - L_2C - F)x(t)$$

$$+ ((I - L_1C)A_d - L_3C - G)x(t - \tau)$$

$$+ ((I - L_1C)B - H)u(t)$$

$$+ (I - L_1C)Ed(t)$$

$$+ ((I - L_1C)F_xF_\theta - L_2F_yF_\theta - L_1F_yF_\theta A_\theta)\theta(t)$$

$$- L_3F_yF_\theta\theta(t - \tau) + ((R - L_1CR) - L_2D)n(t)$$

$$- L_3Dn(t - \tau) - L_1D\dot{n}(t)$$

$$+ (I - L_1C)\Delta A(t)x(t) + (I - L_1C)\Delta A_d(t)x(t - \tau)$$

$$+ (I - L_1C)\Delta B(t)u(t). \tag{6}$$

In the absence of uncertainties and faults, it is shown that the observer, defined by (4), is UIO for the predefined system by (1) if the following conditions are satisfied [22].

Condition 1:

$$\dot{e}(t) = Fe(t) + Ge(t - \tau) \text{ is asymptotically stable.} \tag{7}$$

Condition 2:

$$F = (I - L_1C)A - L_2C. \tag{8}$$

Condition 3:

$$G = (I - L_1C)A_d - L_3C. \tag{9}$$

Condition 4:

$$H = (I - L_1C)B. \tag{10}$$

Condition 5:

$$(I - L_1C)E = \mathbf{0}, \tag{11}$$

where $\mathbf{0}$ denotes a null matrix with compatible dimension. Using these relationships, and considering definitions in (12), the state estimation error dynamic (6) is transformed to (13):

$$T = (I - L_1C),$$

$$\overline{F} = [(I - L_1C)F_xF_\theta - L_2F_yF_\theta - L_1F_yF_\theta A_\theta \quad -L_3F_yF_\theta],$$

$$\overline{R} = [-L_1D \quad (I - L_1C)R - L_2D \quad -L_3D],$$

$$\overline{\theta} = [\theta^T(t) \quad \theta^T(t - \tau)]^T,$$

$$\overline{n} = [\dot{n}^T(t) \quad n^T(t) \quad n^T(t - \tau)]^T, \tag{12}$$

$$\dot{e}(t) = Fe(t) + Ge(t - \tau) + \overline{F}\,\overline{\theta} + \overline{R}\,\overline{n}$$

$$+ T\Delta A(t)x(t) + T\Delta A_d(t)x(t - \tau) \tag{13}$$

$$+ T\Delta B(t)u(t).$$

In order to use the UIO for fault detection purposes, a residual signal should be defined. Difference between measured output and estimated output is usually considered as a residual signal. In current work, a more general form for residual reference signal is considered as follows:

$$\hat{y}(t) = C\hat{x}(t),$$

$$r(t) = V_1(y(t) - \hat{y}(t)) + V_2(y(t - \tau) - \hat{y}(t - \tau))$$

$$= V_1Ce(t) + V_2Ce(t - \tau) + K_1\overline{\theta}(t) + K_2\overline{n}(t), \tag{14}$$

$$K_1 = [V_1F_yF_\theta \quad V_2F_yF_\theta],$$

$$K_2 = [\mathbf{0} \quad V_1D \quad V_2D].$$

The goal of robust fault detection problem is to minimize the performance index defined in (15) for all classes of model uncertainties belonging to Ω_i. In general, this performance index is minimized using H_∞ model matching approach which minimizes the difference between the residual signal ($r(t)$) and reference residual signal ($r_f(t)$)

in the presence of the worst case disturbance signals. This performance index has been minimized by the following steps.

Step 1. The ideal residual signal generator system has been designed for the system without uncertainty defined in (16). The residual signal shows the maximum sensitivity to the fault signal while it has the minimum sensitivity to disturbance, noise, and unknown inputs.

Step 2. The residual signal generator system has been designed such that the performance index (17) is minimized and the overall system (19) is asymptotically stable:

$$J_r = \min_{(\Delta A, \Delta B, \Delta A_d) \in \Omega_i} \frac{\|G_{r[d\,\bar{n}]}\|_\infty}{\|G_{r\bar{f}}\|_\infty}, \tag{15}$$

$$\dot{e}_f(t) = F^* e_f(t) + G^* e_f(t-\tau) + \overline{F}^* \overline{\theta}(t) + \overline{R}^* \overline{n}(t),$$

$$r_f(t) = V_1^* C e_f(t) + V_2^* C e_f(t-\tau) + K_1^* \overline{\theta}(t) + K_2^* \overline{n}(t), \tag{16}$$

$$J_{r_e} = \sup_{(\Delta A, \Delta B, \Delta A_d) \in \Omega_i} \frac{\|r_e\|_2}{\|\omega\|_2} < \gamma, \tag{17}$$

where

$$r_e(t) = r(t) - r_f(t),$$

$$\omega = \begin{bmatrix} u^T & \overline{\theta}^T & d^T & \overline{n}^T \end{bmatrix}^T, \tag{18}$$

$$\dot{\zeta}(t) = \left(\widetilde{A} + \Delta\widetilde{A}\right)\zeta(t) + \left(\widetilde{A}_d + \Delta\widetilde{A}_d\right)\zeta(t-\tau)$$

$$+ \left(\widetilde{B}_{\omega_1} + \Delta\widetilde{B}_{\omega_1}\right)\omega(t), \tag{19}$$

$$r_e(t) = \widetilde{C}_1\zeta(t) + \widetilde{C}_2\zeta(t-\tau) + \widetilde{D}\omega(t),$$

where

$$\zeta(t) = \begin{bmatrix} e^T(t) & e_f^T(t) & x^T(t) \end{bmatrix}^T,$$

$$\widetilde{A} = \begin{bmatrix} F & 0 & 0 \\ 0 & F^* & 0 \\ 0 & 0 & A \end{bmatrix}, \qquad \widetilde{A}_d = \begin{bmatrix} G & 0 & 0 \\ 0 & G^* & 0 \\ 0 & 0 & A_d \end{bmatrix},$$

$$\widetilde{B}_{\omega_1} = \begin{bmatrix} 0 & \overline{F} & 0 & \overline{R} \\ 0 & \overline{F}^* & 0 & \overline{R}^* \\ B & K_3 & E & K_4 \end{bmatrix}, \qquad \Delta\widetilde{A} = \begin{bmatrix} 0 & 0 & T\Delta A \\ 0 & 0 & 0 \\ 0 & 0 & \Delta A \end{bmatrix},$$

$$\Delta\widetilde{A}_d = \begin{bmatrix} 0 & 0 & T\Delta A_d \\ 0 & 0 & 0 \\ 0 & 0 & \Delta A_d \end{bmatrix}, \qquad \Delta\widetilde{B}_{\omega_1} = \begin{bmatrix} T\Delta B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta B & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{C}_1 = \begin{bmatrix} V_1 C & -V_1^* C & 0 \end{bmatrix}, \qquad \widetilde{C}_2 = \begin{bmatrix} V_2 C & -V_2^* C & 0 \end{bmatrix},$$

$$\widetilde{D} = \begin{bmatrix} 0 & K_1 - K_1^* & 0 & K_2 - K_2^* \end{bmatrix},$$

$$K_3 = \begin{bmatrix} F_x F_\theta & 0 \end{bmatrix}, \qquad K_4 = \begin{bmatrix} 0 & R & 0 \end{bmatrix}. \tag{20}$$

Furthermore, using (2) it is easy to see that $\Delta\widetilde{A}, \Delta\widetilde{A}_d, \Delta\widetilde{B}_{\omega_1}$ can be expressed by

$$\Delta\widetilde{A} = \widetilde{M}_1 \Sigma_1(t)\widetilde{N}_1 = \begin{bmatrix} TM_1 \\ 0 \\ M_1 \end{bmatrix} \Sigma_1(t) \begin{bmatrix} 0 & 0 & N_1 \end{bmatrix},$$

$$\Delta\widetilde{A}_d = \widetilde{M}_2 \Sigma_2(t)\widetilde{N}_2 = \begin{bmatrix} TM_2 \\ 0 \\ M_2 \end{bmatrix} \Sigma_2(t) \begin{bmatrix} 0 & 0 & N_2 \end{bmatrix}, \tag{21}$$

$$\Delta\widetilde{B}_{\omega_1} = \widetilde{M}_3 \Sigma_3(t)\widetilde{N}_3 = \begin{bmatrix} TM_3 \\ 0 \\ M_3 \end{bmatrix} \Sigma_3(t) \begin{bmatrix} N_3 & 0 & 0 & 0 \end{bmatrix}.$$

Before developing theorems that are utilized in designed procedure, the following lemmas, which are useful to prove the theorems, are introduced.

Lemma 1 (see [22]). *Condition 5 is solvable if and only if the following relation holds:*

$$\text{rank}(CE) = m, \quad m \le p. \tag{22}$$

The general solution of condition 5 can be calculated by

$$L_1 = E(CE)^+ + Y\left[I - CE(CE)^+\right]$$

$$= \Theta_1 + Y\Theta_2, \tag{23}$$

where Y is an arbitrary matrix with an appropriate dimension.

Lemma 2. *Suppose that M, N, and $\Sigma(t)$ are compatible and $\Sigma^T(t)\Sigma(t) \le I$; then there exists a scalar $\varepsilon > 0$ such that the following equation holds:*

$$M\Sigma N + (M\Sigma N)^T \le \varepsilon MM^T + \varepsilon^{-1} N^T N. \tag{24}$$

(a) *Reference Model Selection (Step 1).* Reference model selection is an important key to the design of robust fault detection filter for linear uncertain time-delay systems. To this end, the analogues procedure as that for fault detection in [6] is extended for delay systems considering the UIO as the

fault detection filter. According to (16), the reference residual signal can be written as sum of two signals, $r_{f\bar{n}}(t)$ and $r_{f\bar{\theta}}(t)$. The reference model should be chosen such that the effect of exogenous signals on the reference residual signal is minimized while the effect of fault signal is maximized. These two tasks are described mathematically by

$$\left\| T\left(r_{f\bar{n}}, \bar{n}\right)\right\|_{\infty} \leq \alpha,$$
$$\left\| T\left(r_{f\bar{f}}, \bar{f}\right)\right\|_{-} \geq \beta, \tag{25}$$

where $T(\cdot, \cdot)$ is the transfer function between two signals. The following two theorems provide conditions which ensure the asymptotic stability of (16). They also provide the conditions that increase the sensitivity of the residual signal from faults and decrease the sensitivity of residual signal from noise.

Theorem 3. *For given $\alpha > 0$, if there exist symmetric positive matrices P, Q, V_1^*, V_2^*, Φ_1^*, Φ_2^*, and Φ_3^* such that the following LMI holds:*

$$\begin{bmatrix} PF^* + F^{*T}P + Q & PG^* & P\overline{R}^* & C^T V_1^{*T} \\ * & -Q & 0 & C^T V_2^{*T} \\ * & * & -\alpha^2 I & K_2^{*T} \\ * & * & * & -I \end{bmatrix} < 0, \tag{26}$$

where

$$PF^* = P\left(A - \Theta_1 CA\right) - \Phi_1^*\left(\Theta_2 CA\right) - \Phi_2^* C,$$

$$PG^* = P\left(A_d - \Theta_1 CA_d\right) - \Phi_1^*\left(\Theta_2 CA_d\right) - \Phi_3^* C,$$

$$P\overline{R}^* = \left[-P\left(\Theta_1 D\right) - \Phi_1^*\left(\Theta_2 D\right) \quad P\left(R - \Theta_1 CR\right) - \Phi_1^*\left(\Psi_2 CR\right) - \Phi_2^* D \quad -\Phi_3^* D\right], \tag{27}$$

$$K_2^* = \left[\mathbf{0} \quad V_1^* D \quad V_2^* D\right],$$

then the system (28) is asymptotically stable and $\left\|T(r_{f\bar{n}}, \bar{n})\right\|_{\infty} \leq \alpha$. Furthermore, the UIO matrices are obtained from conditions 2 to 5, and $Y^* = P^{-1}\Phi_1$, $L_2 = P^{-1}\Phi_2$, $L_3 = P^{-1}\Phi_3$,

$$\dot{e}_{f\bar{n}}(t) = F^* e_{f\bar{n}}(t) + G^* e_{f\bar{n}}(t - \tau) + \overline{R}^* \bar{n}(t),$$
$$r_{f\bar{n}}(t) = V_1^* C e_{f\bar{n}}(t) + V_2^* C e_{f\bar{n}}(t - \tau) + K_2^* \bar{n}(t). \tag{28}$$

Proof. Condition $\|T(r_{f\bar{n}}, \bar{n})\|_{\infty} \leq \alpha$ is equivalent to $J_{r_{f\bar{n}}}$: $\int_0^\infty (r_{f\bar{n}}^T(t) r_{f\bar{n}}(t) - \alpha^2 \bar{n}^T(t)\bar{n}(t))dt \geq 0$. Now Consider the Lyapunov-Krasovskii function which is defined as $V(t) = e_{f\bar{n}}^T(t)Pe_{f\bar{n}}(t) + \int_{t-\tau}^t e_{f\bar{n}}^T(s)Qe_{f\bar{n}}(s)ds$. Then we have

$$J_{r_{f\bar{n}}} = \int_0^\infty \left(r_{f\bar{n}}^T(t) r_{f\bar{n}}(t) - \alpha^2 \bar{n}^T(t)\bar{n}(t) + \dot{V}(t)\right) dt + V(0) - V(\infty). \tag{29}$$

Assume $r_{f\bar{n}}(t) = \mathbf{0}$ for $t \in [-\tau, 0]$. Since $V(\infty) > 0$, it can be concluded that

$$J_{r_{f\bar{n}}} \leq \int_0^\infty \left(r_{f\bar{n}}^T(t) r_{f\bar{n}}(t) - \alpha^2 \bar{n}^T(t)\bar{n}(t) + \dot{V}(t)\right) dt. \tag{30}$$

Taking derivative from $V(t)$ and considering (28) yield

$$J_{r_{f\bar{n}}} \leq \int_{t-\tau}^t \begin{bmatrix} e_{f\bar{n}}(t) \\ e_{f\bar{n}}(t-\tau) \\ \bar{n}(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} PF^* + F^{*T}P + Q + C^T V_1^{*T} V_1^* C & PG^* + C^T V_1^{*T} V_2^* C & P\overline{R}^* + C^T V_1^{*T} K_2^* \\ * & C^T V_2^{*T} V_2^* C - Q & C^T V_2^{*T} K_2^* \\ * & * & -\alpha^2 I + K_2^{*T} K_2^* \end{bmatrix}}_{\Xi} \begin{bmatrix} e_{f\bar{n}}(t) \\ e_{f\bar{n}}(t-\tau) \\ \bar{n}(t) \end{bmatrix}. \tag{31}$$

Hence, $\Xi < 0$ implies $J_{r_{f\bar{n}}} < 0$. Using the Schur complement theorem (26) is concluded from (31). Indeed, the inequality (26), without considering (27), includes nonlinear terms of PY^*, PL_2^*, and PL_3^* which lead the LMI to be infeasible. To overcome this problem, define $\Phi_1^* = PY^*$,

$\Phi_2^* = PL_2^*$, and $\Phi_3^* = PL_3^*$. Using conditions 2, 3 and (23) it can be shown that (27) makes the LMI feasible. It completes the proof. □

Theorem 4. *For given $\beta > 0$, if there exists symmetric positive matrices P, Q, V_1^*, V_2^*, Φ_1^*, Φ_2^*, and Φ_3^* such that the following LMI holds:*

$$
\begin{bmatrix}
PF^* + F^{*T}P - Q + 2\varphi_1\left(V_1^*, V_{1c}^n\right) & -PG^* & -P\overline{F}^* & C^T V_1^{*T} \\
* & -Q + 2\varphi_2\left(V_2^*, V_{2c}^n\right) & \mathbf{0} & C^T V_2^{*T} \\
* & * & \beta^2 F_\theta^T F_\theta + 2\varphi_3\left(K_1^*, K_1^n\right) & K_1^{*T} \\
* & * & * & -I
\end{bmatrix} < 0,
\tag{32}
$$

where

$$
PF^* = P\left(A - \Theta_1 CA\right) - \Phi_1^*\left(\Theta_2 CA\right) - \Phi_2^* C,
$$

$$
PG^* = P\left(A_d - \Theta_1 CA_d\right) - \Phi_1^*\left(\Theta_2 CA_d\right) - \Phi_3^* C,
$$

$$
P\overline{F}^* = \left[P\left(F_x F_\theta - \Theta_1 C F_x F_\theta\right) - \Phi_1^*\left(\Theta_2 C F_x F_\theta\right) - \Phi_2^* F_y F_\theta - P\left(\Psi_1 F_y F_\theta A_\theta\right) - \Phi_1^*\left(\Psi_2 F_y F_\theta A_\theta\right) \quad -\Phi_3^* F_y F_\theta \right],
\tag{33}
$$

$$
K_1^* = \left[V_1^* F_y F_\theta \quad V_2^* F_y F_\theta \right],
$$

$$
V_{1c}^n = V_1^{*n-1} C, \quad V_{2c}^n = V_2^{*n-1} C, \quad K_{11}^n = V_1^{*n-1} F_y F_\theta, \; K_{12}^n = V_2^{*n-1} F_y F_\theta, \quad \text{for } n = 1, 2, \ldots,
$$

$$
K_1^n = \left[K_{11}^n \quad K_{12}^n \right],
$$

$$
\varphi_1\left(V_1^*, V_{1c}^n\right) = \left(V_{1c}^n\right)^T V_{1c}^n - \left(V_{1c}^n\right)^T V_1^* C - C^T V_1^{*T} V_{1c}^n,
\tag{34}
$$

$$
\varphi_2\left(V_2^*, V_{2c}^n\right) = \left(V_{2c}^n\right)^T V_{2c}^n - \left(V_{2c}^n\right)^T V_2^* C - C^T V_2^{*T} V_{2c}^n,
$$

$$
\varphi_3\left(K_1^*, K_1^n\right) = \left(K_1^n\right)^T K_1^n - \left(K_1^n\right)^T V_1^* F_y F_\theta - F_\theta^T F_y^T V_1^{*T} K_1^n,
$$

then the system (35) is asymptotically stable, and $\|T(r_{f\bar{\theta}}, \overline{f})\|_- \geq \beta$. Moreover, the UIO matrices are obtained from conditions 2 to 5, and $Y^* = P^{-1}\Phi_1$, $L_2 = P^{-1}\Phi_2$, $L_3 = P^{-1}\Phi_3$,

$$
\dot{e}_{f\bar{\theta}}(t) = F^* e_{f\bar{\theta}}(t) + G^* e_{f\bar{\theta}}(t - \tau) + \overline{F}^* \overline{\theta}(t),
\tag{35}
$$

$$
r_{f\bar{\theta}}(t) = V_1^* C e_{f\bar{\theta}}(t) + V_2^* C e_{f\bar{\theta}}(t - \tau) + K_2^* \overline{\theta}(t).
$$

Proof. Condition $\|T(r_{f\bar{f}}, \overline{f})\|_- \geq \beta$ is equivalent to $J_{r_{f\bar{f}}}$: $\int_0^\infty (r_{f\bar{\theta}}^T(t) r_{f\bar{\theta}}(t) - \beta^2 \overline{f}^T(t)\overline{f}(t))dt \geq 0$. Now consider the Lyapunov-Krasovskii function which is defined as $V(t) =$

$e_{f\bar{\theta}}^T(t) P e_{f\bar{\theta}}(t) + \int_{t-\tau}^t e_{f\bar{\theta}}^T(s) Q e_{f\bar{\theta}}(s)ds$. Then we have

$$
J_{r_{f\bar{\theta}}} = \int_0^\infty \left(r_{f\bar{\theta}}^T(t) r_{f\bar{\theta}}(t) - \beta^2 \overline{f}^T(t) \times \overline{f}(t) - \dot{V}(t) \right) dt - V(0) + V(\infty).
\tag{36}
$$

Assume $r_{f\bar{\theta}}(t) = \mathbf{0}$ for $t \in [-\tau, 0]$. Since $V(\infty) > 0$, we have

$$
J_{r_{f\bar{\theta}}} \geq \int_0^\infty \left(r_{f\bar{\theta}}^T(t) r_{f\bar{\theta}}(t) - \beta^2 \overline{f}^T(t) \overline{f}(t) - \dot{V}(t) \right) dt. \tag{37}
$$

Taking derivative from $V(t)$ and considering (35) yield

$$
J_{r_{f\bar{\theta}}} > \int_{t-\tau}^t \begin{bmatrix} e_{f\bar{\theta}}(t) \\ e_{f\bar{\theta}}(t - \tau) \\ \overline{\theta}(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} -PF^* - F^{*T}P - Q + C^T V_1^{*T} V_1^* C & -PG^* + C^T V_1^{*T} V_2^* C & -P\overline{F}^* + C^T V_1^{*T} K_1^* \\ * & C^T V_2^{*T} V_2^* C + Q & C^T V_2^{*T} K_1^* \\ * & * & -\beta^2 F_\theta^T F_\theta + K_1^{*T} K_1^* \end{bmatrix}}_{\Xi} \begin{bmatrix} e_{f\bar{\theta}}(t) \\ e_{f\bar{\theta}}(t - \tau) \\ \overline{\theta}(t) \end{bmatrix}.
\tag{38}
$$

Hence, $\Xi > 0$ implies $J_{r_{f\theta}} > 0$. $\Xi > 0$ is equivalent to

$$\begin{bmatrix} PF^* + F^{*^T}P + Q - C^TV_1^{*T}V_1^*C & -PG^* + C^TV_1^{*T}V_2^*C & -P\overline{F}^* + C^TV_1^{*T}K_1^* \\ * & -Q - C^TV_2^{*T}V_2^*C & C^TV_2^{*T}K_1^* \\ * & * & \beta^2 F_\theta^T F_\theta - K_1^{*T}K_1^* \end{bmatrix} < 0. \tag{39}$$

Then, (39) can be written as

$$\begin{bmatrix} PF^* + F^{*^T}P + Q - 2C^TV_1^{*T}V_1^*C & -PG^* + C^TV_1^{*T}V_2^*C & -P\overline{F}^* + C^TV_1^{*T}K_1^* \\ * & -Q - 2C^TV_2^{*T}V_2^*C & C^TV_2^{*T}K_1^* \\ * & * & \beta^2 F_\theta^T F_\theta - 2K_1^{*T}K_1^* \end{bmatrix} + \begin{bmatrix} C^TV_1^{*T} & C^TV_2^{*T} & K_1^{*T} \end{bmatrix} I \begin{bmatrix} V_1^*C \\ V_2^*C \\ K_1^* \end{bmatrix} < 0. \tag{40}$$

Making use of Lemma 2 with $\varepsilon = 1$, we have

$$-C^TV_1^{*T}V_1^*C \le (V_{1c}^n)^T V_{1c}^n - (V_{1c}^n)^T V_1^*C - C^TV_1^{*T}V_{1c}^n,$$

$$-C^TV_2^{*T}V_2^*C \le (V_{2c}^n)^T V_{2c}^n - (V_{2c}^n)^T V_2^*C - C^TV_2^{*T}V_{2c}^n, \tag{41}$$

$$-K_1^{*T}K_1^* \le (K_1^n)^T K_1^n - (K_1^n)^T V_1^* F_y F_\theta - F_\theta^T F_y^T V_1^{*T} K_1^n.$$

Applying Schur complement to (40) and changing variables $\varphi_1(V_1^*, V_{1c}^n)$, $\varphi_2(V_2^*, V_{2c}^n)$, and $\varphi_3(K_1^*, K_1^n)$, the LMI (32) is obtained. To overcome the infeasibility of (33), the same variables as those selected in Theorem 3 are used. It completes the proof. □

Corollary 5. *The system is asymptotically stable and satisfies* (16) *if there exists symmetric positive matrices* P, Q, V_1^*, V_2^*, Φ_1^*, Φ_2^*, *and* Φ_3^* *such that the LMIs* (26) *and* (32) *hold.*

Remark 6. It is desired to obtain a reference residual system which has maximum sensitivity to the fault as well as the minimum sensitivity to the exogenous signal. This aim can be formulated by performance index defined by inf α/β. To this end, an iterative optimization method presented in [6] is developed for the proposed structure. The procedures of this method are as follows.

(1) Choose appropriate values of α and β.

(2) Solve the LMI (26), and find a feasible solution for P, Q, V_1^*, V_2^*, Φ_1^*, Φ_2^*, and Φ_3^* matrices.

(3) Set $V_{1c}^n = V_1^{*n-1}C$, $V_{2c}^n = V_2^{*n-1}C$, $K_{11}^n = V_1^{*n-1}F_yF_\theta$, and $K_{12}^n = V_2^{*n-1}F_yF_\theta$. Then, solve (26) and (32) by increasing n to find a feasible solution for P, Q, V_1^*, V_2^*, Φ_1^*, Φ_2^*, and Φ_3^*.

(4) Increase β and decrease α and go to step 2. Continue this procedure until the feasible solution cannot be found for LMIs (26) and (32).

(b) *Robust UIO Design (Step 2).* As mentioned before, the residual signal generator system is obtained by minimizing (17). To this end, Theorem 8 is presented which guarantees that the overall system (19) is asymptotically stable and performance index (17) is minimized. Before presenting Theorem 8, the following theorem is presented which helps prove Theorem 8.

Theorem 7. *For a given* $\gamma > 0$ *and the system* (37), *if there exist symmetric positive matrices* P, Q *and constants* $\varepsilon_1, \varepsilon_2$, *and* ε_3 *such that the LMI* (44) *holds, then the system* (37) *is asymptotically stable and* $\|v(t)\|_2 \le \gamma\|u(t)\|_2$:

$$\dot{\chi}(t) = (\widetilde{A} + \Delta\widetilde{A})\chi(t) + (\widetilde{A}_d + \Delta\widetilde{A}_d)\chi(t-\tau) + (\widetilde{B}_u + \Delta\widetilde{B}_u)u(t), \tag{42}$$

$$v(t) = \widetilde{C}_1\chi(t) + \widetilde{C}_2\chi(t-\tau) + \widetilde{D}u(t), \tag{43}$$

$$\begin{bmatrix} P\widetilde{A} + \widetilde{A}^TP + Q + \varepsilon_1^{-1}\widetilde{N}_1^T\widetilde{N}_1 & P\widetilde{A}_d & P\widetilde{B}_u & \widetilde{C}_1^T & P\widetilde{M}_1 & P\widetilde{M}_2 & P\widetilde{M}_3 \\ * & -Q + \varepsilon_2^{-1}\widetilde{N}_2^T\widetilde{N}_2 & 0 & \widetilde{C}_2^T & 0 & 0 & 0 \\ * & * & -\gamma^2 I + \varepsilon_3^{-1}\widetilde{N}_3^T\widetilde{N}_3 & \widetilde{D}^T & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_1^{-1}I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_2^{-1}I & 0 \\ * & * & * & * & * & * & -\varepsilon_3^{-1}I \end{bmatrix} < 0. \tag{44}$$

Proof. Define the following Lyapunov-Krasovskii function:

$$V(t) = \chi^T(t) P \chi(t) + \int_{t-\tau}^t \chi^T(s) Q \chi(s)\, ds. \tag{45}$$

The performance index $\|v(t)\|_2 \leq \gamma \|u(t)\|_2$ can be written as

$$J_v = \int_0^\infty \left(v^T(t) v(t) - \gamma^2 u^T(t) u(t) + \dot{V}(t) \right) dt$$
$$+ V(0) - V(\infty). \tag{46}$$

Assume $\chi(t) = \mathbf{0}$ for $t \in [-\tau, 0]$. Since $V(\infty) > 0$, we have

$$J_v \leq \int_0^\infty \left(v^T(t) v(t) - \gamma^2 u^T(t) u(t) + \dot{V}(t) \right) dt. \tag{47}$$

Taking derivative from (45) and considering (37) yield

$$J_{r_e} \leq \int_0^\infty \begin{bmatrix} \chi(t) \\ \chi(t-\tau) \\ u(t) \end{bmatrix}^T \underbrace{\begin{bmatrix} P(\widetilde{A} + \Delta\widetilde{A}) + (\widetilde{A} + \Delta\widetilde{A})^T P + Q + \widetilde{C}_1^T \widetilde{C}_1 & P(\widetilde{A}_d + \Delta\widetilde{A}_d) + \widetilde{C}_1^T \widetilde{C}_2 & P(\widetilde{B}_u + \Delta\widetilde{B}_u) + \widetilde{C}_1^T \widetilde{D} \\ * & -Q + \widetilde{C}_2^T \widetilde{C}_2 & \mathbf{0} \\ * & * & -\gamma^2 I + \widetilde{D}^T \widetilde{D} \end{bmatrix}}_{\Xi}$$
$$\times \begin{bmatrix} \chi(t) \\ \chi(t-\tau) \\ u(t) \end{bmatrix} dt. \tag{48}$$

$\Xi < 0$ implies $J_{r_e} < 0$. The $\Xi < 0$ can be written as

$$\begin{bmatrix} P\widetilde{A} + \widetilde{A}^T P + Q & P\widetilde{A}_d & P\widetilde{B}_u \\ * & -Q & \mathbf{0} \\ * & * & -\gamma^2 I \end{bmatrix}$$
$$+ \begin{bmatrix} \widetilde{C}_1^T \\ \widetilde{C}_2^T \\ \widetilde{D}^T \end{bmatrix} I \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 & \widetilde{D} \end{bmatrix}$$
$$+ \begin{bmatrix} P\Delta\widetilde{A} + \Delta\widetilde{A}^T P & P\Delta\widetilde{A}_d & P\Delta\widetilde{B}_u \\ * & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} \end{bmatrix} < 0. \tag{49}$$

Using Lemma 2, one can write the following inequality:

$$\begin{bmatrix} P\Delta\widetilde{A} + \Delta\widetilde{A}^T P & P\Delta\widetilde{A}_d & P\Delta\widetilde{B}_u \\ * & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} \end{bmatrix}$$
$$\leq \begin{bmatrix} P\widetilde{M}_1 & P\widetilde{M}_2 & P\widetilde{M}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \varepsilon_1 I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon_2 I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varepsilon_3 I \end{bmatrix}$$
$$\times \begin{bmatrix} P\widetilde{M}_1 & P\widetilde{M}_2 & P\widetilde{M}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T + \begin{bmatrix} \widetilde{N}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{N}_2^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{N}_3^T \end{bmatrix}$$
$$\times \begin{bmatrix} \varepsilon_1 I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon_2 I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \varepsilon_3 I \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{N}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{N}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{N}_3 \end{bmatrix}. \tag{50}$$

Considering (49), (50) and using Schur complement (48) lead to (44). It completes the proof. □

Theorem 8. *For a given $\gamma > 0$, if there exist symmetric positive matrices P_1, P_2, P_3, Q_1, Q_2, and Q_3, matrices Φ_1, Φ_2, and Φ_3, and constants ε_1, ε_2, and ε_3 such that LMI $[s_{ij}]_{14\times14} < 0$ holds, then the overall system (19) is asymptotically stable, and $J_{r_e} < \gamma$. The observer matrices are calculated by considering (4), (5), and conditions 2 to 5, and $Y = P_1^{-1}\Phi_1$, $L_2 = P_1^{-1}\Phi_2$, and $L_3 = P_1^{-1}\Phi_3$. The LMI coefficients are defined as*

$$s_{1,1} = P_1 F + (P_1 F)^T + Q_1, \quad s_{1,4} = P_1 G, \quad s_{1,8} = P_1 \overline{F},$$

$$s_{1,10} = P_1 \overline{R}, \quad s_{1,11} = C^T V_1^T, \quad s_{1,12} = P_1 (TM_1),$$

$$s_{1,13} = P_1 (TM_2), \quad s_{1,14} = P_1 (TM_3),$$

$$s_{2,2} = P_2 F^* + (P_2 F^*)^T + Q_2, \quad s_{2,5} = P_2 G^*, \quad s_{2,8} = P_2 \overline{F}^*,$$

$$s_{2,10} = P_2 \overline{R}^*, \quad s_{2,11} = -C^T V_1^{*T},$$

$$s_{3,3} = P_3 A + (P_3 A)^T + Q_3 + \varepsilon_1^{-1} N_1^T N_1, \quad s_{3,6} = P_3 A_d,$$

$$s_{3,7} = P_3 B, \quad s_{3,8} = P_3 K_3, \quad s_{3,9} = P_3 E, \quad s_{3,10} = P_3 K_4,$$

$$s_{3,12} = P_3 (M_1), \quad s_{3,13} = P_3 (M_2), \quad s_{3,14} = P_3 (M_3),$$

$$s_{4,4} = -Q_1, \quad s_{4,11} = C^T V_2^T,$$

$$s_{5,5} = -Q_2, \quad s_{5,11} = C^T V_2^{*T}, \quad s_{6,6} = -Q_3 + \varepsilon_2^{-1} N_2^T N_2,$$

$$s_{7,7} = -\gamma I + \varepsilon_3^{-1} N_3^T N_3, \quad s_{8,8} = -\gamma I, \quad s_{8,11} = K_1^T - K_1^{*T},$$

$$s_{9,9} = -\gamma I, \quad s_{10,10,} = -\gamma I, \quad s_{10,11} = K_2^T - K_2^{*T},$$

$$s_{11,11} = -I, \quad s_{12,12} = -\varepsilon_1^{-1} I,$$

$$s_{13,13} = -\varepsilon_2^{-1}I, \quad s_{14,14} = -\varepsilon_3^{-1}I,$$

$$\text{otherwise} \quad s_{i,j} = \mathbf{0},$$

(51)

where

$$P_1 T = P_1 \left(I - \Theta_1 C\right) - \Phi_1 \left(\Theta_2 C\right),$$

$$P_1 F = P_1 \left(A - \Theta_1 CA\right) - \Phi_1 \left(\Theta_2 CA\right) - \Phi_2 C,$$

$$P_1 G = P_1 \left(A_d - \Theta_1 CA_d\right) - \Phi_1 \left(\Theta_2 CA_d\right) - \Phi_3 C,$$

$$P_1 \overline{F} = \left[P_1 \left(F_x F_\theta - \Theta_I CF_x F_\theta\right) - \Phi_1 \left(\Theta_2 CF_x F_\theta\right) - \Phi_2 F_y F_\theta - P_1 \left(\Theta_1 F_y F_\theta A_\theta\right) - \Phi_1 \left(\Theta_2 F_y F_\theta A_\theta\right) \ -\Phi_3 F_y F_\theta\right],$$

$$P_1 \overline{R} = \left[P_1 \left(R - \Theta_1 CR\right) - \Phi_1 \left(\Theta_2 CR\right) - \Phi_2 D \ -\Phi_3 D \ -P_1 \left(\Theta_1 D\right) - \Phi_1 \left(\Theta_2 D\right)\right].$$

(52)

Proof. In Theorem 7 assume that $P = \mathrm{diag}(P_1 \ P_2 \ P_3)$ and $Q = \mathrm{diag}(Q_1 \ Q_2 \ Q_3)$. Then, using system dynamic (19) it is straight forward to see that s_{ij} are the same as (37). Without considering (52), the inequality (51) includes nonlinear terms of PY, PL_2, and PL_3 which lead the LMI to be infeasible. To overcome this problem, define $\Phi_1 = P_1 Y$, $\Phi_2 = P_1 L_2$, and $\Phi_3 = P_1 L_3$. Using conditions 2, 3 and (23) it can be seen that (52) makes the obtained LMI feasible. It completes the proof. $\square$

Remark 9. It should be noted that the present work differs from [18] in the following perspectives.

(a) The results in [18] are obtained without considering dynamic characteristic for fault; however, the current results are achieved by considering dynamic characteristic that is modeled by (3). Hence, the design procedure in [18] is not applicable for the current case.

(b) The residual signal $r(t)$ is constructed based on (14) which uses both estimation error and delay in estimation error; however, the residual signal in [18] is constructed using estimation error. Since two design parameters V_1 and V_2 appear in the LMIs, the obtained LMIs are more flexible.

Remark 10. After designing the FDI system, residual evaluation methods and appropriate level of threshold should be selected to take a decision about the occurrence of fault. According to (13) and (14), the residual signal for fault-free system $r^0(t)$ satisfies the following equation:

$$\left\|r^0\left(t\right)\right\|_2 = \left\|r_{\bar{n}}\left(t\right) + r_u\left(t\right)\right\|_2$$

$$\leq \left\|r_{\bar{n}}\left(t\right)\right\|_2 + \left\|r_u\left(t\right)\right\|_2 \leq J_{\mathrm{th},\bar{n}} + J_{\mathrm{th},u},$$

(53)

where

$$J_{\mathrm{th},\bar{n}} = \sup_{(\Delta A, \Delta B, \Delta A_d)\, \in \Omega_i} \left\|r_{\bar{n}}\left(t\right)\right\|_2,$$

$$J_{\mathrm{th},u} = \sup_{(\Delta A, \Delta B, \Delta A_d)\, \in \Omega_i} \left\|r_u\left(t\right)\right\|_2.$$

(54)

$J_{\mathrm{th},\bar{n}}$ can be computed offline, and under the assumption that $\bar{n} \in L_2$ we have $\sup_{(\Delta A, \Delta B, \Delta A_d)\in\Omega_i} \|r_{\bar{n}}(t)\|_2 = M_{\bar{n}}$. Since the signal u is supposed to be known online, the value of $J_{\mathrm{th},u}$ can be determined online by $J_{\mathrm{th},u} = \gamma_u \|u(t)\|_2$, where $\gamma_u = \sup_{(\Delta A, \Delta B, \Delta A_d)\in\Omega_i} (\|r_u(t)\|_2)/(\|u(t)\|_2)$. γ_u can be computed by Theorem 7. Therefore, the threshold value can be evaluated by

$$J_{\mathrm{th}} = M_{\bar{n}} + \gamma_u \|u(t)\|_2.$$

(55)

Since values of (53) and (54) increase by passing the time and, consequently, need more memory in real application, one can use root-mean-square (RMS) norm of $r(t)$, defined in (56), to detect the fault signals:

$$\|r\left(t\right)\|_2^T = \int_{t_1}^{t_2} r^T\left(t\right) r\left(t\right) dt, \quad T = t_2 - t_1,$$

(56)

where T is designed parameter.

3. Simulation Results

The main objective of this section is to investigate the effectiveness of the designed UIO. To this end, a numerical example is used and simulation results are presented. Consider a system which is defined by (1) with the following matrices:

$$A = \begin{bmatrix} -3.8 & 1.5 & -0.5 \\ 0.5 & -3 & 1 \\ -0.3 & 0.7 & -2.4 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.4 & 0.1 & -0.2 \\ 0.1 & -0.8 & 0.2 \\ 0.7 & -0.1 & 0.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.4 \end{bmatrix}, \quad F_x = \begin{bmatrix} 0.6 \\ -0.5 \\ 0.4 \end{bmatrix}, \quad E = \begin{bmatrix} -0.4 \\ 0.1 \\ -0.3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_y = \begin{bmatrix} 0.2 \\ 0.8 \\ -1.2 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 \\ 0.2 \\ -0.4 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.9 \\ 0.2 \\ 0.7 \end{bmatrix}.$$

(57)

Uncertainties are also defined by the following matrices in (2):

$$M_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} -0.1 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0.1 & 0.3 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.1 & 0 & 0 \end{bmatrix},$$

$$N_3 = 0.1. \tag{58}$$

The dynamic characteristic of fault is considered as

$$A_\theta = 0, \qquad F_\theta = 1. \tag{59}$$

The first step to design the fault detection system is to solve the LMIs (26) and (32) in Theorems 3 and 4. The Yalmip LMI toolbox is used to solve the LMIs. To start the iterative optimization method presented in Remark 6, the initial values $\alpha_{int} = 3$ and $\beta_{int} = 0$ are selected. Using this procedure, the following results are obtained:

$$V_1^* = \begin{bmatrix} -0.0994 & -0.0532 & 0.2714 \\ -0.0532 & -5.4751 & 1.3859 \\ 0.2714 & 1.3859 & -0.7651 \end{bmatrix},$$

$$V_2^* = \begin{bmatrix} -0.6315 & -0.8304 & 0.1871 \\ -0.8304 & 0.6711 & 0.5971 \\ 0.1871 & 0.5971 & -0.5845 \end{bmatrix},$$

$$\Phi_1^* = \begin{bmatrix} 662.5 & -250.8 & -2628.5 \\ -250.8 & 3855.7 & -281.2 \\ -2628.5 & -281.2 & 2055.8 \end{bmatrix}, \tag{60}$$

$$\Phi_2^* = \begin{bmatrix} 984.8 & -1453.4 & -842.7 \\ -1453.4 & 2292 & 1253.8 \\ -842.7 & 1253.8 & 733.6 \end{bmatrix},$$

$$\Phi_3^* = \begin{bmatrix} -4.7395 & 13.0831 & 0.7977 \\ 13.0831 & -8.5691 & -14.9116 \\ 0.7977 & -14.9116 & -3.0645 \end{bmatrix},$$

$$\alpha = 2.4, \qquad \beta = 1.$$

Using these values, the LMI (51) is solved and the observer dynamic matrices are obtained as follows:

$$F = \begin{bmatrix} -3.6557 & -3.3806 & 2.9608 \\ -0.3878 & -1.4059 & 0.3008 \\ -2.4404 & -3.2245 & 1.9455 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.0559 & -0.7284 & -0.1686 \\ -0.0153 & -0.1629 & -0.0350 \\ -0.0445 & -0.557 & -0.1273 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.6878 \\ 0.1535 \\ 0.5328 \end{bmatrix},$$

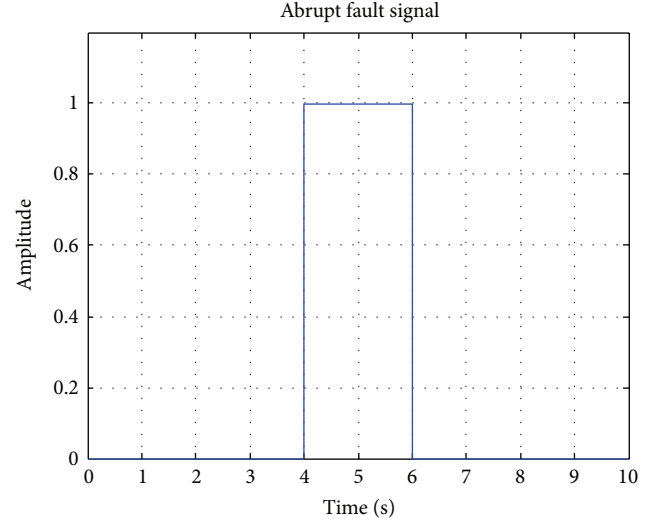FIGURE 1: An abrupt fault occurs at $t = 5$.

$$L_1 = \begin{bmatrix} 0.0372 & -1.1695 & 0.8939 \\ -0.2145 & 0.7355 & 0.1978 \\ -0.7462 & -0.9036 & 1.6937 \end{bmatrix},$$

$$K_{1z} = \begin{bmatrix} -0.7701 & -0.1957 & 0.9507 \\ -0.1730 & -0.0567 & 0.2159 \\ -0.5857 & -0.1278 & 0.7254 \end{bmatrix},$$

$$K_{2z} = \begin{bmatrix} 0.2122 & -0.3395 & -0.7167 \\ 0.0496 & -0.0777 & -0.1591 \\ 0.1603 & -0.2648 & -0.5534 \end{bmatrix}. \tag{61}$$

To verify the sensitivity of designed UIO, an abrupt fault, shown in Figure 1, occurs in the 4 seconds elapsed from running of the system. The step disturbance signal exerted to the system between 3 to 7 seconds. The noise signal is assumed to be white Gaussian noise with power 0.0005, and the uncertainty $\Sigma_i(t)$ is considered sinusoidal signal. The residual signals are shown in Figure 2. It can be seen that the residual signals change when the fault occurs; however, the residual signals do not show any sensitivity to the exerted disturbance. The value of threshold J_{th} is presented in Figure 3. This figure indicates that the fault is detected rapidly and the difference between threshold and norm of faulty residual signal is high enough to detect the occurrence of fault in the system.

The RMS of residual signals (56) has been depicted in Figure 4. It can be seen that the RMS of faulty signals suddenly changes in contrast to RMS of fault-free signals. Therefore, the occurrence of fault can be effectively realized.

4. Conclusions

In this paper, a novel UIO-based residual generator is developed for robust fault detection purposes. The developed method is applicable to a variety of linear uncertain time-delay systems. The proposed approach is able to decouple
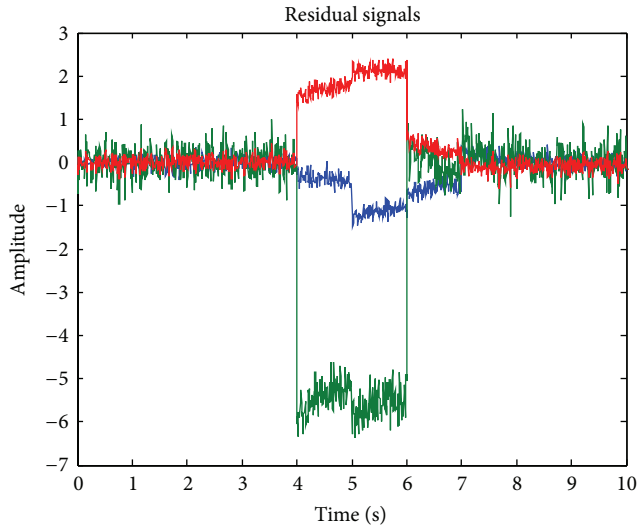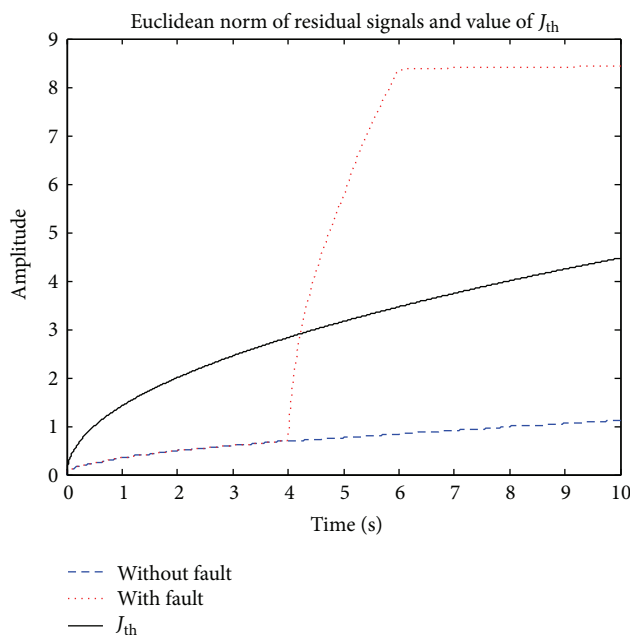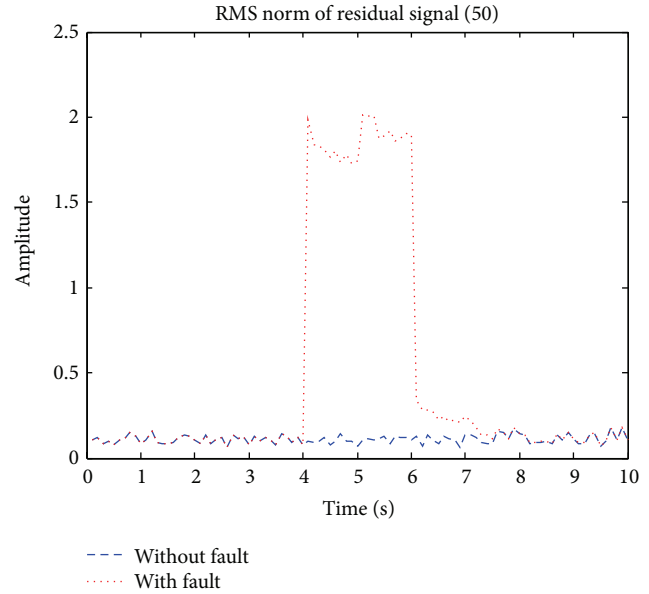
FIGURE 2: Residual signals of UIO fault detection filter.

FIGURE 3: Euclidean norm of residual fault-free and faulty signals and J_{th}.

FIGURE 4: Detecting occurrence of fault using RMS norm of $\|r(t)\|_2^T$.

the data framework for designing unknown input observer for time-delay system is an interesting area.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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thoroughly exogenous disturbances while minimizing uncertainties and noise effects. The fault effect is also maximized at the same time. To this end, first, the optimal fault detection filter is designed for system without considering uncertainties. Then, the fault detection filter is designed so that the H_∞ norm between the fault detection filter and the optimal one is minimized. Superiority of the proposed approach has been verified through a numerical example. Simulation results show that the proposed approach is able to detect dynamic faults. As a future work, one can extend this approach to nonlinear systems and descriptor system. Moreover, developing

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