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Research Article

Numerical Solution of a Kind of Fractional Parabolic Equations via Two Difference Schemes

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A kind of parabolic equation was extended to the concept of fractional calculus. The resulting equation is, however, difficult to handle analytically. Therefore, we presented the numerical solution via the explicit and the implicit schemes. We presented together the stability and convergence of this time-fractional parabolic equation with two difference schemes. The explicit and the implicit schemes in this case are stable under some conditions.

1. Introduction

A parabolic partial differential equation is a type of second-order partial differential equations (PDEs), describing a wide family of problems in science including heat diffusion and ocean acoustic propagation, in physical or mathematical systems with a time variable, which behave essentially like heat diffusing through a solid [1–4]. This mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximate solution helps us to understand the means of these mathematical models. In most cases, it is difficult, or infeasible, to find the analytical solution or good numerical solution of the problems. Numerical solutions or approximate analytical solutions become necessary. Numerical methods typically yield approximate solutions to the governing equation through the discretization of space and time and can relax the rigid idealized conditions of analytical models or lumped-parameter models. They can, therefore, be more realistic and flexible for simulating field conditions. Within the discredited problem domain, the variable internal properties, boundaries, and stresses of the system are approximated. One of the most important aspects of this numerical method is the study of the stability and convergence of the numerical method [5, 6].

The purpose of this work is to study the stability and the convergence of the numerical scheme of the parabolic equation of the following form:

$$\begin{aligned} &\partial_t v(x,t) + {}_0^C D_t^\alpha v(x,t) - a(x,t)\,\partial_{xx}^2 v(x,t) + \delta v(x,t) \\ &\quad = f(x,t), \qquad v(0,x) = 0, \\ &\qquad\qquad\qquad\qquad 0 \le x \le 1, \quad 0 < \alpha \le 1, \\ &\quad v(x,0) = v(0,t), \qquad v_x(t,0) = v_t(x,0), \end{aligned} \tag{1}$$

where $f(x,t)$ is sufficiently given smooth function and $a(x,t) \ge 0$. Here, δ is a sufficiently large positive constant.

2. Useful Tools for the Fractional Calculus

Definition 1 (see [7–17]). A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0,\infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2 (see [7–16]). The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad \alpha > 0,\ x > 0,$$
$$J^0 f(x) = f(x). \tag{2}$$

Properties of the operator can be found in [7–15]; we only mention the following:

$$\text{for } f \in C_\mu, \quad \mu \geq -1, \quad \alpha, \beta \geq 0, \quad \gamma > -1,$$
$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \qquad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \tag{3}$$
$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 3. The Caputo fractional-order derivative is given as follows [7–10]:

$$ {}_0^C D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, \quad n-1 \leq \alpha \leq n. \tag{4}$$

Definition 4. The Riemann-Liouville fractional-order derivative is given as follows [8–16]:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt, \quad n-1 \leq \alpha \leq n. \tag{5}$$

Definition 5. The Jumarie fractional-order derivative is given as follows [16]:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\}\, dt, \quad n-1 \leq \alpha \leq n. \tag{6}$$

Lemma 6. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{7}$$

Definition 7 (partial derivatives of fractional order [7, 8, 11, 18]). Assume now that $f(\mathbf{x})$ is a function of n variables x_i, $i = 1, \ldots, n$, also of class C on $D \in \mathbb{R}_n$. We define partial derivative of order α for f in respect to the x_i the function as follows:

$$a\partial_{\underline{\mathbf{x}}}^\alpha f = \frac{1}{\Gamma(m-\alpha)} \int_a^{x_i} (x_i - t)^{m-\alpha-1} \partial_{x_i}^m f(x_j)\Big|_{x_j=t} dt, \tag{8}$$

where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

3. Examination of the Numerical Solution via Difference Schemes

This section is devoted to the discussion underpinning the numerical simulation of the solution above (1) via the explicit scheme [19–23] and the implicit scheme [23–30]. However, before we present the numerical schemes, we must assume that (1) has a unique and sufficiently smooth solution [23]. In addition, to present the numerical schemes, we let $x_l = lh$, $0 \leq l \leq M$, $Mh = L$, $t_k = k\tau$, $0 \leq k \leq N$, and $N\tau = T$; h is the space step size, and M and N are grid points. We will start with the implicit scheme.

3.1. Implicit Scheme for the Main Problem. It is important to recall that the finite difference approximation for the second-order spatial derivative is known as follows [23]:

$$\frac{\partial^2 v(x_l, t_{k+1})}{\partial x^2} = \frac{v(x_{l+1}, t_{k+1}) - 2v(x_l, t_{k+1}) + v(x_{l-1}, t_{k+1})}{h^2} + O(h^2). \tag{9}$$

The discretization of the Caputo-type time-fractional-order derivative can be presented as follows:

$$\begin{aligned} \frac{\partial^\alpha v(x_l, t_{k+1})}{\partial t^\alpha} &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \Big(v(x_l, t_{k+1}) - v(x_l, t_k) \\ &\quad + \sum_{j=1}^{k} \left[v(x_l, t_{k+1-j}) - v(x_l, t_{k-j}) \right] \\ &\quad \times \left[(j+1)^{1-\alpha} - (j)^{1-\alpha} \right] \Big), \\ \frac{\partial v(x_l, t_{k+1})}{\partial t} &= \frac{v(x_l, t_{k+1}) - v(x_l, t_k)}{\Delta t} + O(\Delta t). \end{aligned} \tag{10}$$

Now, substituting (9) and (10) into (1), we obtained the following expression:

$$\begin{aligned} &\frac{v(x_l, t_{k+1}) - v(x_l, t_k)}{\Delta t} + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \\ &\quad \times \Big(v(x_l, t_{k+1}) - v(x_l, t_k) \\ &\quad + \sum_{j=1}^{k} \left[v(x_l, t_{k+1-j}) - v(x_l, t_{k-j}) \right] \end{aligned}$$

$$\times\left[(j+1)^{1-\alpha}-(j)^{1-\alpha}\right]\Big)$$
$$-a_l^k\left(\frac{v(x_{l+1},t_{k+1})-2v(x_l,t_{k+1})+v(x_{l-1},t_{k+1})}{h^2}\right)$$
$$+\delta v(x_l,t_k)=f(x_l,t_k). \tag{11}$$

For ease, let

$$v_l^k=v(x_l,t_k),\qquad f_l^k=f(x_l,t_k),$$
$$a(x_l,t_k)=a_l^k,\qquad (j+1)^{1-\alpha}-(j)^{1-\alpha}=b_j. \tag{12}$$

Then, (11) can be rephrased as follows:

$$\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_l^k}{h^2}\right)v_l^{k+1}$$
$$=\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+\delta\right)v_l^k+a_l^k v_{l+1}^{k+1} \tag{13}$$
$$+a_l^k v_{l-1}^{k+1}-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}\left[v_l^{k-j+1}-v_l^{k-j}\right]b_j+f_l^k.$$

It is important to inform that if $k=0$, then the term of the sum of the right-hand side automatically vanished. Then, (13) can be divided as follows:

$$\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_l^1}{h^2}\right)v_l^1-a_l^1v_{l+1}^1-a_l^1v_{l-1}^1$$
$$=\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+\delta\right)v_l^0+f_l^0,\quad k=0, \tag{14}$$

$$\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_l^k}{h^2}\right)v_l^{k+1}-a_l^kv_{l+1}^{k+1}-a_l^kv_{l-1}^{k+1}$$
$$=\left(\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+\delta\right)v_l^k \tag{15}$$
$$-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}\left[v_l^{k-j}\right]d_j+b_{k+1}v_j^0+f_l^k,$$

where $d_j=b_{j+1}-b_j$.

The above equation can be written in matrix form as follows:

$$\begin{pmatrix}
\Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_l^1}{h^2} & -a_1^k & 0\ 0\ 0\ \cdots & & 0\\
-a_2^k & \Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_2^1}{h^2} & \ddots\quad 0 & 0 & \\
 & \ddots & \ddots\quad 0 & 0 & 0\\
\vdots & & -a_{m-2}^k\quad \Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_{m-2}^1}{h^2} & -a_{m-2}^k & \\
0\ 0 & & 0\ 0\ -a_{m-1}^k & & \Delta t+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}+2\frac{a_l^1}{h^2}
\end{pmatrix}
\begin{pmatrix} v_1^{k+1}\\ v_2^{k+1}\\ \vdots\\ v_{m-1}^{k+1}\\ v_{m-1}^{k+1}\end{pmatrix} \tag{16}$$

$$=\begin{pmatrix} f_1^k-\frac{a_1^k}{h^2}v(0,t_k)\\ f_2^k\\ \vdots\\ f_{m-2}^k\\ f_{m-1}^k-\frac{a_{m-1}^k}{h^2}v(0,t_k)\end{pmatrix}.$$

3.1.1. Stability of the Implicit Difference Scheme. In this subsection, we present the stability analysis of the implicit difference scheme for solving the time-fractional parabolic equation (1). To accomplish this, we let $\zeta_l^k=v_l^k-V_l^k$, with V_l^k being the approximate solution of the main problem at the point (x_l,t_k), $k=1,\ldots,N$; $l=1,\ldots,M$; in addition, ζ^k is the transpose of the matrix $[\zeta_1^k,\zeta_2^k,\ldots,\zeta_M^k]$.

To analyze the stability, we exploit the Fourier method [22], and the expression of ζ^k can be defined as follows:

$$\zeta^k(x)=\begin{cases}\zeta_l^k, & \text{if } x_l-\frac{h}{2}<x\le x_l+\frac{h}{2},\\ & l=1,2,\ldots,M-1,\\ 0, & \text{if } L-\frac{h}{2}<x\le L.\end{cases} \tag{17}$$

Then, the function $\zeta^k(x)$ can be expressed in Fourier series as follows:

$$\zeta^k(x) = \sum_{m=-\infty}^{m=\infty} \delta_m(m) \exp\left[\frac{2i\pi mk}{L}\right],$$
$$\delta_k(x) = \frac{1}{L}\int_0^L \rho^k(x) \exp\left[\frac{2i\pi mx}{L}\right] dx. \tag{18}$$

It was proven in [22, 29, 30] that

$$\left\|\rho^2\right\|_2^2 = \sum_{m=-\infty}^{m=\infty} \left\|\delta_k(m)\right\|^2. \tag{19}$$

Let us now examine the stability of the implicit scheme of the main problem

$$\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^k}{h^2}\right)\zeta_l^{k+1}$$
$$= \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + \delta\right)\zeta_l^k + a_l^k \zeta_{l+1}^{k+1} + a_l^k \zeta_{l-1}^{k+1}$$
$$- \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}\left[\zeta_l^{k-j}\right] d_j + b_{k+1}\zeta_j^0 + f_l^k, \tag{20}$$
$$k = 1, \ldots, M.$$

We next assume that

$$\zeta_l^k = \delta_k \exp[i\rho lk] \tag{21}$$

with ρ being the real space wave number and $i = \sqrt{-1}$. Then, by replacing (21) into (20), we obtain the following:

$$\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^k}{h^2}\right)\delta_{k+1}$$
$$= \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right)\delta_k$$
$$- \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left[\delta_{k-j}\right] d_j + b_{k+1}\delta_0 + f_l^k, \tag{22}$$
$$k = 0, \ldots, M-1.$$

Lemma 8. *Assume that $\delta_k (k = 0, \ldots, M-1)$ verified* (14) *and that for all (l, k), $\delta \le a_l^k/h^2 + d_1 (l = 1, \ldots, N; k = 1, \ldots, M)$; then, the following inequality is satisfied:*

$$\left|\delta_k\right| \le \left|\delta_0\right|, \quad k = 1, 2, \ldots, M. \tag{23}$$

Proof. To prove this lemma, we make use of the recursive method on the natural number n from (3) and (15), we have that, for all $l = 1, \ldots, N$ and for $k = 0$,

$$\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^0}{h^2}\right)\delta_1$$
$$= \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right)\delta_0. \tag{24}$$

And for $k \ge 1$, we have

$$\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^k}{h^2}\right)\delta_{k+1}$$
$$= \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right)\delta_k \tag{25}$$
$$- \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left[\delta_{k-j}\right] d_j + b_{k+1}\delta_0.$$

Observe that $\delta \le a_l^0/h^2 + d_1$ then, the following is obtained:

$$\delta_1 = \frac{\left(\Delta t + \tau^{-\alpha}/\Gamma(2-\alpha) + 2\sin^2(\rho h/2)\,\delta - d_1\right)}{\left(\Delta t + \tau^{-\alpha}/\Gamma(2-\alpha) + 2\sin^2(\rho h/v)\left(a_l^0/h^2\right)\right)}\delta_0. \tag{26}$$

Applying the absolute value on both sides of (20), we have

$$\left|\delta_1\right| \le \left|\delta_0\right|. \tag{27}$$

Now, let us assume that (18) is true for all $2 \le m \le k$; then,

$$\delta_{k+1} = \left(\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right)\delta_k\right.$$
$$\left.-\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left[\delta_{k-j}\right] d_j + b_{k+1}\delta_0\right) \tag{28}$$
$$\times\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^k}{h^2}\right)^{-1}.$$

Now, applying the absolute value on both sides of (28), and making further use of the inequality theorem, we arrived at the following:

$$\left|\delta_{k+1}\right| \le \left(\left|\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right|\left|\delta_k\right|\right.$$
$$\left.+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left|\delta_{k-j}\right| d_j + b_{k+1}\left|\delta_0\right|\right) \tag{29}$$
$$\times\left(\left|\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^k}{h^2}\right|\right)^{-1}.$$

Making use of the induction hypothesis and factorizing $|\delta_0|$, we obtain

$$\left|\delta_{k+1}\right| \le \left[\left(\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\delta - d_1\right.\right.\right.$$
$$\left.+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1} d_j + b_{k+1}\right)$$
$$\left.\times\left(\left|\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\sin^2\left(\frac{\rho h}{2}\right)\frac{a_l^k}{h^2}\right|\right)^{-1}\right]\left|\delta_0\right|. \tag{30}$$

But we have that

$$\sum_{j=1}^{k-1} d_j = 1 - b_{k+1}, \quad 0 \le d_j \le 1. \tag{31}$$

Therefore,

$$|\delta_{k+1}| \le \left[\frac{\Delta t + \tau^{-\alpha}/\Gamma(2-\alpha) + 2\sin^2(\rho h/2)\,\delta - d_1}{\left|\Delta t + \tau^{-\alpha}/\Gamma(2-\alpha) + 2\sin^2(\rho h/2)\left(a_l^k/h^2\right)\right|} \right] |\delta_0|. \tag{32}$$

Thus,

$$|\delta_{k+1}| \le |\delta_0|. \tag{33}$$

□

Theorem 9. *The implicit difference scheme for the time-fractional parabolic equation* (1) *is stable providing that, for all* $(l,k), \delta \le a_l^k/h^2 + d_1 (l = 1,\ldots,N;\ k = 1,\ldots,M)$.

Proof. From (19) and Lemma 8, we obtain

$$\left\|\zeta^2\right\|_2 \le \left\|\zeta^0\right\|_2, \tag{34}$$

and this proves that the implicit difference scheme for the time-fractional parabolic equation (1) is stable. □

Remark 10. It is observed that, from Theorem 9, the statement of stability of implicit difference scheme for time-fractional parabolic equation (1) depends on the evolution of the function $a(x,t)$. It follows that the stability condition can change in time advancement and space position.

3.1.2. Convergence Analysis of the Implicit Difference Scheme. Assuming that $v(x_l, t_k)$ $(l = 0,\ldots,M-1;\ k = 0,\ldots,N-1)$ is the exact solution of (1) at the point (x_l, t_k), then, by defining, $\beta_l^k = v(x_l,t_k) - v_l^k$, and β^k is the transpose of the matrix $(\beta_1^k, \beta_2^k, \ldots, \beta_N^k)$; here, β^0 is neglected because of being equal to zero. Therefore, we have the following relation for the implicit difference scheme for the time-fractional parabolic equation (1) and from (13):

$$\begin{aligned}
&\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^0}{h^2}\right)\beta_l^1 - a_l^0\beta_{l+1}^1 \\
&\quad - a_l^0\beta_{l-1}^1 - f_l^0 = T_l^1, \quad \text{for } k = 0, \\
&\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^k}{h^2}\right)\beta_l^{k+1} - \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + \delta\right)\beta_l^k \\
&\quad - a_l^k\beta_{l+1}^{k+1} - a_l^k\beta_{l-1}^{k+1} - f_l^k \\
&\quad = -\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\beta_l^{k-j}b_j + T_l^k, \quad \text{for } k \ge 1.
\end{aligned} \tag{35}$$

Here, T_l^{k+1} is the truncate or the remainder term of the approximation and has the following expression:

$$\begin{aligned}
T_l^{k+1} &= v(x_l, t_{k+1}) + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\beta_l^{k-j}b_j \\
&\quad + a(x_l,t_k)\left[v(x_{l+1},t_{k+1})\right. \\
&\qquad \left. - 2v(x_l,t_{k+1}) + v(x_{l-1},t_{k+1})\right] \\
&\quad + \left(v(x_l,t_{k+1}) - v(x_l,t_k)\right) - f(x_l,t_k) + \delta v(x_l,t_k).
\end{aligned} \tag{36}$$

It follows from (3), (4), and (5) that,

$$\begin{aligned}
&\frac{\partial^\alpha v(x_l,t_{k+1})}{\partial t^\alpha} + D_1\tau \\
&\quad = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \\
&\qquad \times \left(v(x_l,t_{k+1}) - v(x_l,t_k) \right. \\
&\qquad\quad \left. + \sum_{j=1}^{k-1} v(x_l,t_{k-j})\,d_j + d_{k+1}v(x_l,t_0) \right),
\end{aligned} \tag{37}$$

$$\frac{\partial^2 v(x_l,t_{k+1})}{\partial x^2} + h^2 D_2 = \frac{v(x_{l+1},t_{k+1}) - 2v(x_l,t_{k+1}) + v(x_{l-1},t_{k+1})}{h^2},$$

$$\frac{\partial v(x_l,t_{k+1})}{\partial t} + \Delta t D_3 = \frac{v(x_l,t_{k+1}) - v(x_l,t_k)}{\Delta t}.$$

Thus, from (36) and (37), we have the following:

$$T_l^{k+1} \le D\left(\tau^{1+\alpha} + h^2\tau^\alpha + \Delta t\tau^\alpha\right), \tag{38}$$

where D_1, D_2, D_3, and D are constant. The interested reader can find the error analysis of the chosen fractional derivative (the Caputo fractional derivative) in [29, 30].

Lemma 11. *One has that* $\|\beta^k\|_\infty \le D(b^{k+1})^{-1}[\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t\tau^\alpha]$ *is true for all* $k = 0, 1, \ldots, M-1$.

Proof. Again we employ the induction method to achieve this, so that, for $k = 0$,

$$\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^0}{h^2}\right)\left|\beta_l^1\right| - a_l^0\left|\beta_{l+1}^1\right| - a_l^0\left|\beta_{l-1}^1\right| \ge \left|\beta_l^1\right|, \tag{39}$$

$$\begin{aligned}
\left|T_l^1\right| &= \left|\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^0}{h^2}\right)\beta_l^1 - a_l^0\beta_{l+1}^1 - a_l^0\beta_{l-1}^1\right| \\
&\ge \left|\beta_l^1\right|, \\
&D\left(b^1\right)^{-1}\left(\tau^{1+\alpha} + h^2\tau^\alpha + \Delta t\tau^\alpha\right) \ge \left|T_l^1\right|.
\end{aligned} \tag{40}$$

Now, assuming that, for all $j = 0, \ldots, M-2$, $\|\beta^j\|_\infty \leq D(b^{j+1})^{-1}[\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t \tau^\alpha]$, then,

$$\begin{gathered}
\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^k}{h^2}\right)\left|\beta_l^{k+1}\right| \\
- \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + \delta\right)\left|\beta_l^k\right| - a_l^k\left|\beta_{l+1}^{k+1}\right| - a_l^k\left|\beta_{l-1}^{k+1}\right| \\
\geq \left|\beta_l^{k+1}\right|, \\
\left|\left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + 2\frac{a_l^k}{h^2}\right)\beta_l^{k+1}\right. \\
\left. - \left(\Delta t + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} + \delta\right)\beta_l^k - a_l^k\beta_{l+1}^{k+1} - a_l^k\beta_{l-1}^{k+1}\right| \\
\geq \left|\beta_l^{k+1}\right|, \\
\left|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\beta_l^{k-j}b_j + T_l^{k+1}\right| \geq \left|\beta_l^{k+1}\right|, \\
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left|\beta_l^{k-j}\right|b_j + T_l^{k+1} \geq \left|\beta_l^{k+1}\right|.
\end{gathered} \tag{41}$$

Making use of the induction hypothesis, we obtain the following:

$$\begin{gathered}
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k-1}\left\|\beta_l^{k-j}\right\|_\infty b_j + \left[\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t \tau^\alpha\right] \geq \left|\beta_l^{k+1}\right|, \\
\left(b^0 + b^{k+1} - b^{k+1}\right)\left(b^{k+1}\right)^{-1} \times D\left[\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t \tau^\alpha\right] \\
\geq \left|\beta_l^{k+1}\right|, \\
\left(b^{k+1}\right)^{-1} \times D\left[\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t \tau^\alpha\right] \geq \left|\beta_l^{k+1}\right|.
\end{gathered} \tag{42}$$

This completes the proof. □

Theorem 12. *The implicit difference scheme of the time-fractional parabolic equation* (1) *is convergent, and there exists a constant D such that,*

$$\begin{gathered}
\left|v(x_l, t_k) - v_l^k\right| \leq D\left(\tau^{1+\alpha} + \tau^\alpha h^2 + \Delta t \tau^\alpha\right), \\
\text{for } (l = 0, \ldots, N-1;\ k = 0, \ldots, M-1).
\end{gathered} \tag{43}$$

3.2. Explicit Difference Scheme for Time-Fractional Parabolic Equation. It is important to recall that the finite-difference approximation for the second-order spatial derivative is known as follows:

$$\frac{\partial^2 v(x_l, t_k)}{\partial x^2} = \frac{v(x_{l+1}, t_k) - 2v(x_l, t_k) + v(x_{l-1}, t_k)}{h^2} + O\left(h^2\right). \tag{44}$$

The discretization of the Caputo-type time-fractional-order derivative can be presented as follows:

$$\begin{aligned}
\frac{\partial^\alpha v(x_l, t_{k+1})}{\partial t^\alpha} &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(v(x_l, t_{k+1}) - v(x_l, t_k)\right. \\
&\quad + \sum_{j=1}^{k}\left[v\left(x_l, t_{k+1-j}\right) - v\left(x_l, t_{k-j}\right)\right] \\
&\quad \left.\times\left[(j+1)^{1-\alpha} - (j)^{1-\alpha}\right]\right), \\
\frac{\partial v(x_l, t_k)}{\partial t} &= \frac{v(x_{l+1}, t_k) - v(x_l, t_k)}{\Delta t} + O(\Delta t).
\end{aligned} \tag{45}$$

Now, substituting (44) and (45) into (1), we obtained the following expression:

$$\begin{aligned}
&\frac{v(x_{l+1}, t_k) - v(x_l, t_k)}{\Delta t} \\
&+ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(v(x_l, t_{k+1}) - v(x_l, t_k)\right. \\
&\quad + \sum_{j=1}^{k}\left[v\left(x_l, t_{k+1-j}\right) - v\left(x_l, t_{k-j}\right)\right] \\
&\quad \left.\times\left[(j+1)^{1-\alpha} - (j)^{1-\alpha}\right]\right) \\
&- a_l^k\left(\frac{v(x_{l+1}, t_k) - 2v(x_l, t_k) + v(x_{l-1}, t_k)}{h^2}\right) \\
&+ \delta v(x_l, t_k) = f(x_l, t_k).
\end{aligned} \tag{46}$$

For simplicity, let

$$\begin{gathered}
v_l^k = v(x_l, t_k), \qquad f_l^k = f(x_l, t_k), \\
a(x_l, t_k) = a_l^k, \qquad (j+1)^{1-\alpha} - (j)^{1-\alpha} = b_j.
\end{gathered} \tag{47}$$

Then, (37) can be rewritten as follows

$$\begin{aligned}
v_l^{k+1} &= v_l^k\left[B_1 + 1 - r_l^k - \delta\right] + v_{l+1}^k\left[r_l^k - B_1\right] \\
&\quad + r_l^k v_{l-1}^k + B_1 f_l^k - \sum_{j=1}^{k}\left[v_l^{k+1-j} - v_l^{k-j}\right]b_j,
\end{aligned} \tag{48}$$

where $B_1 = \Gamma(2-\alpha)\tau^\alpha$ and $r_l^k = (\Gamma(2-\alpha)\tau^\alpha/h^2)a_l^k$.

3.2.1. Stability of the Explicit Difference Scheme of the Time-Fractional Parabolic Equation. Following the discussion presented earlier for the analysis of the implicit scheme, we obtain the following roundoff error equation from explicit scheme equation (1):

$$\beta_l^{k+1} = \beta_l^k \left[B_1 + 1 - r_l^k - \delta\right] + \beta_{l+1}^k \left[r_l^k - B_1\right] + r_l^k \beta_{l-1}^k + B_1 f_l^k - \sum_{j=1}^{k} \left[\beta_l^{k-j}\right] d_j - d_{k+1}\beta^0, \quad \text{for } k = 1, \ldots, M-1, \tag{49}$$

where

$$d_j = b_{j+1} - b_j. \tag{50}$$

Again, we suppose that β_l^k in (39) can be given in exponential form as follows:

$$\beta_l^k = \delta_k e^{(-i\sigma lk)}. \tag{51}$$

Then, replacing this form into (49), we arrive at the following expression:

$$\delta_{k+1} = \delta_k \left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^k - \delta\right] - \sum_{j=1}^{k-1} \delta_{k-j} d_j + \delta_0 d_{k+1}, \quad \text{for } k = 0, \ldots, M-1. \tag{52}$$

Lemma 13. *Suppose that δ_k is the solution of* (40) *for all* ($l = 0, \ldots, N-1;\ k = 0, \ldots, M-1$), $r_l^k \le 1 - \delta + B_1$, *and* $r_l^k \ge B_1 - \delta$, *then, the following inequality holds:*

$$\left|\delta_k\right| \le \left|\delta_0\right|, \quad \text{for } k \ge 1. \tag{53}$$

Proof. To prove this, we make use of the recurrence technique on the natural number k, and we will examine first the case for $k = 1$. Therefore, if $k = 1$, we have that

$$\delta_1 = \delta_0 \left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^0 - \delta\right]. \tag{54}$$

Since $r_l^0 \le 1 - \delta + B_1$ and $r_l^0 \ge B_1 - \delta$, then, we have

$$\left|\delta_1\right| \le \left|\delta_0\right|. \tag{55}$$

We next assume that, for all $j = 1, \ldots, k$, $|\delta_k| \le |\delta_0|$; then

$$\left|\delta_{k+1}\right| = \left|\delta_k \left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^k - \delta\right] - \sum_{j=1}^{k-1} \delta_{k-j} d_j + \delta_0 d_{k+1}\right|. \tag{56}$$

Now, making use of the triangular inequality, we arrive at the following inequality:

$$\left|\delta_{k+1}\right| \le \left|\delta_k \left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^k - \delta\right]\right| + \sum_{j=1}^{k-1} \left|\delta_{k-j}\right| d_j + \left|\delta_0\right| d_{k+1}. \tag{57}$$

Now, making use of the induction hypothesis, we arrive at the following:

$$\left|\delta_{k+1}\right| \le \left|\left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^k - \delta\right] + \sum_{j=1}^{k-1} d_j + d_{k+1}\right| \left|\delta_0\right|. \tag{58}$$

And this produces, since $\sum_{j=1}^{k-1} d_j + d_{k+1} = 1$,

$$\left|\delta_{k+1}\right| \le \left|\left[B_1 + 1 - \sin^2\left(\frac{\sigma h}{2}\right) r_l^k - \delta\right]\right| \left|\delta_0\right| \le \left|\delta_0\right|. \tag{59}$$

This completes the proof. □

Theorem 14. *The explicit difference scheme of time-fractional parabolic equation* (1) *is stable under the condition that, for all* ($l = 0, 1, \ldots, N-1;\ k = 0, \ldots, M-1$), $r_l^k \le 1 - \delta + B_1$ *and* $r_l^k \ge B_1 - \delta$.

Proof. It is straightforward from (11) and Lemma 13, that,

$$\left\|\zeta^2\right\|_2 \le \left\|\zeta^0\right\|_2 \tag{60}$$

and this proves that the explicit difference scheme for the time-fractional parabolic equation (3) is stable. □

3.2.2. Convergence Analysis of the Explicit Scheme

Theorem 15. *The explicit scheme is convergent if, for all* ($l = 0, 1, \ldots, N-1; k = 0, \ldots, M-1$), $r_l^k \le 1-\delta+B_1$ *and* $r_l^k \ge B_1-\delta$ *and there exists a constant D such that*

$$\left|v_l^k - v\left(x_l, t_k\right)\right| \le D\left(\tau + h^2 + \Delta t\right). \tag{61}$$

Using the same method in implicit difference, the above theorem can be proven.

4. Conclusions

Partial differential equations are sometimes very difficult to be solved analytically. It is, therefore, sometimes easy to solve them via numerical techniques. In this paper, we solve the time-fractional parabolic equation (1) via the implicit and explicit difference schemes. We study the stability and the convergence of implicit and explicit difference schemes.

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