Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 827826, 4 pages
http://dx.doi.org/10.1155/2013/827826

Research Article

Comparison Theorems for Single and Double Splittings of Matrices

Cui-Xia Li, Qun-Fa Cui, and Shi-Liang Wu

School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China

Correspondence should be addressed to Cui-Xia Li; lixiatk@126.com

Received 5 February 2013; Accepted 14 March 2013

Academic Editor: Giuseppe Marino

Some comparison theorems for the spectral radius of double splittings of different matrices under suitable conditions are presented, which are superior to the corresponding results in the recent paper by Miao and Zheng (2009). Some comparison theorems between the spectral radius of single and double splittings of matrices are established and are applied to the Jacobi and Gauss-Seidel double SOR method.

1. Introduction

Consider the linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, $b \in \mathbb{R}^{n\times 1}$ is given, and $x \in \mathbb{R}^{n\times 1}$ is unknown. The splitting of the coefficient matrix

$$A = M - N, \tag{2}$$

where M is nonsingular, is called a single splitting of A in [1]; the basic iterative method for solving (1) is

$$x^{k+1} = M^{-1}Nx^k + M^{-1}b \equiv Tx^k + M^{-1}b, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where matrix $T = M^{-1}N$ is the iteration matrix in (3). Obviously, the iterative method (3) converges to the unique solution of the linear system (1) if and only if the spectral radius $\rho(M^{-1}N)$ of the iteration matrix is smaller than 1. The spectral radius of the iteration matrix is decisive for the convergence and stability, and the smaller it is, the faster the iterative method converges when the spectral radius is smaller than 1. So far, many comparison theorems of single splitting of matrices have been arisen in some papers and books [2–8].

The double splitting of A was introduced by Woźnicki [1] and can be described as follows. Splitting the matrix A in the form

$$A = P - R - S \tag{4}$$

is called the double splitting of A, where P is a nonsingular matrix; the corresponding iterative scheme is spanned by three successive iterations:

$$x^{k+1} = P^{-1}Rx^k + P^{-1}Sx^{k-1} + P^{-1}b, \quad k = 0, 1, 2, \ldots. \tag{5}$$

Following the idea of Golub and Varga [9], Woźnicki wrote (5) in the following equivalent form:

$$\begin{bmatrix} x^{k+1} \\ x^k \end{bmatrix} = \begin{bmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{bmatrix} \begin{bmatrix} x^k \\ x^{k-1} \end{bmatrix} + \begin{bmatrix} P^{-1}b \\ 0 \end{bmatrix}, \tag{6}$$

where I is the identity matrix. Then, the iterative method (6) converges to the unique solution of (1) for all initial vectors x^0, x^1 if and only if the spectral radius of the iteration matrix

$$W = \begin{bmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{bmatrix} \tag{7}$$

is less than one, that is, $\rho(W) < 1$.

Recently, some comparison theorems for double splittings of monotone matrices and Hermitian positive definite matrices were presented in [8, 10–13]. Elsner et al. [14]

presented some comparison theorems of single splittings of different monotone matrices, that is, matrices with nonnegative inverses. Our basic purpose here is to derive some new comparison theorems for the spectral radius of double splittings of different matrices. Under suitable conditions, new comparison theorems are superior to the corresponding results in the recent paper [12]. Some comparison theorems between the spectral radius of single and double splittings of matrices are also established and are applied to the Jacobi and Gauss-Seidel double SOR method.

2. Preliminaries

For convenience, we give some of the notations, definitions, and lemmas which will be used in the sequel.

The matrix A is called nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$. We write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for $i, j = 1, 2, \ldots, n$. The matrix A is called a monotone matrix if $A^{-1} \geq 0$. Matrix A is an L-matrix if $a_{ii} > 0 \quad (i = 1, \ldots, n)$ and $a_{ij} < 0$ for all $i, j = 1, \ldots, n; i \neq j$.

Definition 1. Let A be a nonsingular matrix. Then, the double splitting $A = P - R - S$ is

(i) convergent if and only if $\rho(W) < 1$;

(ii) a regular double splitting if $P^{-1} \geq 0$, $R \geq 0$ and $S \geq 0$;

(iii) a weak regular double splitting if $P^{-1} \geq 0$, $P^{-1}R \geq 0$, and $P^{-1}S \geq 0$;

(iv) a nonnegative splitting if $P^{-1}R \geq 0$ and $P^{-1}S \geq 0$.

Lemma 2 (see [3]). *Let $A \geq 0$. Then, $\alpha x \leq Ax$, $x \geq 0$, implies $\alpha \leq \rho(A)$ and $Ax \leq \beta x$, $x > 0$, implies $\rho(A) \leq \beta$.*

Lemma 3 (see [10]). *Let $A^{-1} \geq 0$ and $A = P - R - S$ be a weak regular double splitting. Then, $\rho(W) < 1$.*

3. Comparison Theorem

In [12], Miao and Zheng gave a comparison theorem for the spectral radius of double splittings of different monotone matrices. That is, [12, Theorem 3.1] is a major result and is described as follows.

Let A_1 and A_2 be two monotone matrices, and let $A_1 = P_1 - R_1 - S_1$, and let $A_2 = P_2 - R_2 - S_2$ be double splittings of A_1 and A_2, respectively. Consequently,

$$W_1 = \begin{bmatrix} P_1^{-1}R_1 & P_1^{-1}S_1 \\ I & 0 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} P_2^{-1}R_2 & P_2^{-1}S_2 \\ I & 0 \end{bmatrix}. \quad (8)$$

Theorem 4 (see [12]). *Let A_1 and A_2 be two nonsingular matrices with $A_1^{-1} \geq 0$ and $A_2^{-1} \geq 0$, $A_1 = P_1 - R_1 - S_1$, and let $A_2 = P_2 - R_2 - S_2$ be weak regular double splittings. If $P_1^{-1}A_1 \geq P_2^{-1}A_2$ and $P_1^{-1}R_1 \geq P_2^{-1}R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$.*

Based on the forms of W_1 and W_2, we have the following theorem.

Theorem 5. *Let A_1 and A_2 be two nonsingular matrices, and let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be nonnegative splittings. If $P_1^{-1}R_1 \leq P_2^{-1}R_2$ and $P_1^{-1}S_1 \leq P_2^{-1}S_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $\rho(W_2) < 1$.*

Proof. Obviously, if $P_1^{-1}R_1 \leq P_2^{-1}R_2$ and $P_1^{-1}S_1 \leq P_2^{-1}S_2$, then $0 \leq W_1 \leq W_2$. Therefore, we obtain that $\rho(W_1) \leq \rho(W_2) < 1$ for $\rho(W_2) < 1$. □

Based on Definition 1, we obtain the following Theorem 6, which is superior to Theorem 4 [12].

Theorem 6. *Let A_1 and A_2 be two nonsingular matrices, and let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be nonnegative splittings. If $P_1^{-1}A_1 \geq P_2^{-1}A_2$ and $P_1^{-1}R_1 \geq P_2^{-1}R_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

Proof. Obviously, $W_1 \geq 0$ and $W_2 \geq 0$. By the Perron-Frobenius theorem [3], there exists a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0, \quad x \neq 0 \quad (9)$$

such that $W_2 x = \rho(W_2) x$; that is,

$$\begin{gathered} P_2^{-1}R_2 x_1 + P_2^{-1}S_2 x_2 = \rho(W_2) x_1, \\ x_1 = \rho(W_2) x_2. \end{gathered} \quad (10)$$

Then, we have

$$\begin{aligned}
& W_1 x - \rho(W_2) x \\
& = \begin{bmatrix} P_1^{-1}R_1 x_1 + P_1^{-1}S_1 x_2 - \rho(W_2) x_1 \\ x_1 - \rho(W_2) x_2 \end{bmatrix} \\
& = \begin{bmatrix} \left(P_1^{-1}R_1 - P_2^{-1}R_2\right) x_1 + \dfrac{1}{\rho(W_2)} \left(P_1^{-1}S_1 - P_2^{-1}S_2\right) x_1 \\ x_1 - \rho(W_1) x_2 \end{bmatrix} \\
& \leq \frac{1}{\rho(W_2)} \begin{bmatrix} \left(P_1^{-1}R_1 - P_2^{-1}R_2\right) x_1 + \left(P_1^{-1}S_1 - P_2^{-1}S_2\right) x_1 \\ 0 \end{bmatrix} \\
& = \frac{1}{\rho(W_2)} \begin{bmatrix} \left(P_1^{-1}R_1 + P_1^{-1}S_1\right) x_1 - \left(P_2^{-1}R_2 + P_2^{-1}S_2\right) x_1 \\ 0 \end{bmatrix} \\
& = \frac{1}{\rho(W_2)} \begin{bmatrix} P_1^{-1}(R_1 + S_1) x_1 - P_2^{-1}(R_2 + S_2) x_1 \\ 0 \end{bmatrix} \\
& = \frac{1}{\rho(W_2)} \begin{bmatrix} P_1^{-1}(P_1 - A_1) x_1 - P_2^{-1}(P_2 - A_2) x_1 \\ 0 \end{bmatrix} \\
& = \frac{1}{\rho(W_2)} \begin{bmatrix} \left(P_2^{-1}A_2 - P_1^{-1}A_1\right) x_1 \\ 0 \end{bmatrix} \\
& \leq 0.
\end{aligned} \quad (11)$$

From Lemma 2, we obtain that $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$. □

By investigating Theorem 6, it is easy to see that the conditioners, Theorem 6 are weaker than those of Theorem 4 [12]. That is, the result of Theorem 6 holds without $A_1^{-1} \geq 0$ and $A_2^{-1} \geq 0$.

Similarly, we have the following result.

Theorem 7. *Let A_1 and A_2 be two nonsingular matrices, and let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be nonnegative splittings. If $P_1^{-1}A_1 \geq P_2^{-1}A_2$ and $P_1^{-1}S_1 \leq P_2^{-1}S_2$, then $\rho(W_1) \leq \rho(W_2) < 1$ for $0 < \rho(W_2) < 1$.*

4. Convergence for the Jacobi and Gauss-Seidel Double SOR Method

To establish some comparison theorems between the spectral radius of single and double splittings of matrices, based on (3) and (5), we obtain that $M = P$ and $N = R + S$. Here and now, $T = P^{-1}(R + S)$.

The result for comparing $\rho(W)$ with $\rho(T)$ is stated as in the following theorem.

Theorem 8. *Let $A = P - R - S$ be a nonnegative splitting. Then,*

(1) $\rho(T) \leq \rho(W) < 1$ *for* $\rho(W) < 1$;

(2) $\rho(T) \geq \rho(W) > 1$ *for* $\rho(W) > 1$.

Proof. By Definition 1, obviously, matrix $W \geq 0$. Based on the Perron-Frobenius theorem [3], there exists a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0, \quad x \neq 0 \tag{12}$$

such that $Wx = \rho(W)x$, that is,

$$\begin{bmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \rho(W) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{13}$$

The above equation is equivalent to

$$P^{-1}Rx_1 + P^{-1}Sx_2 = \rho(W)x_1, \tag{14}$$

$$x_1 = \rho(W)x_2. \tag{15}$$

From (15), we get that $x_2 = (1/\rho(W))x_1$. Substituting it into (14) yields

$$\rho(W)x_1 = P^{-1}Rx_1 + \frac{1}{\rho(W)}P^{-1}Sx_1. \tag{16}$$

If $\rho(W) < 1$, then

$$\rho(W)x_1 \geq P^{-1}Rx_1 + P^{-1}Sx_1. \tag{17}$$

That is, $\rho(W)x_1 \geq Tx_1$. By Lemma 2, it is easy to obtain that $\rho(T) \leq \rho(W) < 1$.

Obviously, we also obtain that $\rho(T) \geq \rho(W) > 1$ for $\rho(W) > 1$. □

Example 9. Let

$$\begin{aligned} A &= \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \\ &\equiv P - R - S. \end{aligned} \tag{18}$$

Then,

$$\begin{aligned} P^{-1} &= \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \leq 0, \\ P^{-1}R &= \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \geq 0, \\ P^{-1}S &= \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \geq 0. \end{aligned} \tag{19}$$

By the simple computations, we have $\rho(W) = 0.63$ and $\rho(T) = 0.5$. Clearly, $\rho(T) \leq \rho(W) < 1$ holds.

Let the matrix A be split as

$$A = D - L - U, \tag{20}$$

where $D = \text{diag}(A)$, and L, U are strictly lower and upper triangular matrices, respectively, for $\omega \neq 0$ and $\sigma \neq 0$. Let

$$\begin{aligned} P_J &= \frac{1}{\omega\sigma}D, \\ R_J &= \frac{1}{\omega\sigma}\left[\omega\sigma(L+U) - (\omega-1)D - (\sigma-1)D\right], \\ S_J &= \frac{(\omega-1)(\sigma-1)}{\omega\sigma}D. \end{aligned} \tag{21}$$

Then, the iterative method (5) corresponding to the double splitting

$$A = P_J - R_J - S_J \tag{22}$$

is called the Jacobi double SOR method [1, 15].

Based on (21), we have the following lemma.

Lemma 10. *Let A be an L-matrix, and let the double splittings be defined by* (21) *and* (22). *Suppose*

$$2 - \sigma - \omega \geq 0, \qquad (\omega-1)(\sigma-1) \geq 0. \tag{23}$$

Then, the double splitting defined by (22) *is regular.*

Let

$$W_J = \begin{bmatrix} P_J^{-1}R_J & P_J^{-1}S_J \\ I & 0 \end{bmatrix}, \qquad T_J = P_J^{-1}(R_J + S_J). \tag{24}$$

Then, we have the following result.

Theorem 11. *Under the conditions of Lemma 10, then*

(1) $\rho(T_J) \leq \rho(W_J) < 1$ *for* $\rho(W_J) < 1$;

(2) $\rho(T_J) \geq \rho(W_J) > 1$ *for* $\rho(W_J) > 1$.

Proof. From Theorem 8, it is easy to see that Theorem 11 holds. □

Let

$$P_G = \frac{1}{\omega\sigma} D\left(I - \sigma D^{-1}U\right),$$
$$S_G = \frac{(\omega - 1)(\sigma - 1)}{\omega\sigma} D,$$
$$R_G = \frac{1}{\omega\sigma}\left[\omega\sigma L - (\omega - 1) D\left(I - \sigma D^{-1}U\right) - (\sigma - 1) D\right]. \tag{25}$$

Then, the iterative method (5) corresponding to the double splitting

$$A = P_G - R_G - S_G \tag{26}$$

is called the Gauss-Seidel double SOR method [1, 15].

Let

$$W_G = \begin{bmatrix} P_G^{-1}R_G & P_G^{-1}S_G \\ I & 0 \end{bmatrix}, \qquad T_G = P_G^{-1}\left(R_G + S_G\right). \tag{27}$$

Similarly, we have the following result.

Theorem 12. *Let A be an L-matrix, and let the double splittings be defined by* (25) *and* (26)*, for* $0 < \sigma \leq 1$ *and* $0 < \omega \leq 1$*; then*

(1) $\rho(T_G) \leq \rho(W_G) < 1$ *for* $\rho(W_G) < 1$;

(2) $\rho(T_G) \geq \rho(W_G) > 1$ *for* $\rho(W_G) > 1$.

From Theorems 8, 11, and 12, it is easy to see that the spectral radius of single splitting method is less than the spectral radius of double splitting method under suitable conditions. That is, the efficiency of the single splitting method maybe be superior to that of the double splitting method under suitable conditions.

Acknowledgment

This research was supported by NSFC Tianyuan Mathematics Youth Fund (11026040, 11226337), Science & Technology Development Plan of Henan Province (no. 122300410316) and by Natural Science Foundations of Henan Province (no. 13A110022).

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