Research Article

Ulam-Hyers Stability Results for Fixed Point Problems via α-ψ-Contractive Mapping in (b)-Metric Space

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We will investigate some existence, uniqueness, and Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mapping of type-(b) in the framework of b-metric spaces. The presented theorems extend, generalize, and unify several results in the literature, involving the results of Samet et al. (2012).

1. Introduction and Preliminaries

Very recently, Samet et al. [1] introduced the notion of α-ψ-contractive type mapping and proved some fixed point results for such mapping. The authors [1] also reported that several fixed point theorems, including the celebrated Banach contraction mapping principle, can be derived from their main results.

The aim of this manuscript is to investigate the existence and/or uniqueness of a fixed point of α-ψ-contractive type mapping in the context of a b-metric space, a generalization of a usual metric space, which was introduced by Czerwik [2, 3]. In fact, such general settings of metric spaces were considered earlier, for example, by Bourbaki [4], Bakhtin [5], and Heinonen [6]. Following these initial papers, b-metric spaces and related fixed point theorems have been investigated by a number of authors; see for example, Boriceanu et al. [7], Boriceanu [8, 9], Bota [10], and Aydi et al. [11, 12]. As a subsidiary purpose, we consider Ulam-Hyers stability of the observed results. The stability problem of functional equations, originated from a question of Ulam [13], in 1940, concerns the stability of group homomorphisms. The first affirmative partial answer to the question of Ulam for Banach spaces was given by Hyers [14] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability and has attracted attention of several authors. In particular, Ulam-Hyers stability results in fixed point theory have been studied

densely; see for example, Bota-Boriceanu and Petruşel [15], Lazăr [16], Rus [17, 18], and F. A. Tişe and I. C. Tişe [19]. Moreover, there are several remarkable results on the stability of certain classes of functional equations via fixed point approach. Most particularly and recently, Brzdęk et al. [20], Brzdęk and Ciepliński [21, 22], and Cadariu [23] reported some interesting results in this direction.

We recollect some essential definitions and fundamental results. We first recall the definition of a b-metric space.

Definition 1 (Bakhtin [5], Czerwik [2]). Let X be a set, and let $s \geq 1$ be a given real number. A functional $d : X \times X \rightarrow [0, \infty)$ is said to be a b-metric space if the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$,

for all x, y, and $z \in X$. A pair (X, d) is called a b-metric space.

It is clear that b-metric turns into usual metric when we take $s = 1$. Hence, we conclude that the class of b-metric spaces is larger than the class of usual metric spaces. For more details and examples on b-metric spaces, see, for example, [2–6, 24]. We state the following example for the sake of completeness.

Example 2. Let X be a set with the cardinal card$(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that card$(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \rightarrow [0, \infty)$ defined by

$$d(x, y) := \begin{cases} 0, & x = y, \\ 2s, & x, y \in X_1, \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

is a b-metric on X with coefficient $s > 1$.

Let (X, d) be a b-metric space. We consider next the following families of subsets of

$$\mathscr{P}(X) := \{Y \mid Y \subset X \text{ and } Y \neq \emptyset\}. \tag{2}$$

In this case D is a generalized functional on a b-metric space (X, d) defined by

$$D : P(X) \times P(X) \longrightarrow [0, \infty) \cup \{+\infty\},$$

$$D(A, B) = \begin{cases} \inf\{d(a, b) \mid a \in A, b \in B\}, & A \neq \emptyset \neq B; \\ 0, & A = \emptyset = B; \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

The following basic lemmas will be useful in the proof of the main results.

Lemma 3 (Czerwik [2]). *Let (X, d) be a b-metric space. Then, ones has*

$$D(x, A) \leq s[d(x, y) + D(y, A)], \quad \forall x, y \in X \text{ and } A \subset X. \tag{4}$$

Lemma 4 (Czerwik [2]). *Let (X, d) be a b-metric space, and let $\{x_k\}_{k=0}^n \subset X$. Then*

$$d(x_n, x_0) \leq sd(x_0, x_1) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n). \tag{5}$$

A mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \rightarrow 0, n \rightarrow \infty$, for any $t \in [0, \infty)$. We denote by Φ the class of the comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$. For more details and examples, see, for example, [25, 26]. Among them, we recall the following essential result.

Lemma 5 (Berinde [26], Rus [25]). *If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a comparison function, then*

(1) *each iterate φ^k of φ, $k \geq 1$, is also a comparison function;*

(2) *φ is continuous at 0;*

(3) *$\varphi(t) < t$, for any $t > 0$.*

Later, Berinde [26] introduced the concept of (c)-comparison function in the following way.

Definition 6 (Berinde [26]). A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be a (c)-comparison function if

(c$_1$) φ is increasing,

(c$_2$) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

The notion of a (c)-comparison function was improved as a (b)-comparison function by Berinde [27] in order to extend some fixed point results to the class of b-metric space.

Definition 7 (Berinde [27]). Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled:

(1) φ is monotonically increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$, and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

We denote by Ψ_b for the class of (b)-comparison function $\varphi : [0, \infty) \rightarrow [0, \infty)$. It is evident that the concept of (b)-comparison function reduces to that of (c)-comparison function when $s = 1$.

The following lemma has a crucial role in the proof of our main result.

Lemma 8 (Berinde [24]). *If $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a (b)-comparison function, then ones has the following:*

(1) *the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;*

(2) *the function $b_s : [0, \infty) \rightarrow [0, \infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.*

We note that any (b)-comparison function is a comparison function due to Lemma 8.

Next, we will present the definition of α-ψ-contractive and α-admissible mappings introduced by Samet et al. [1].

We denote by Ψ the family of nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$. It is clear that if $\Psi \subset \Phi$ (see, e.g., [28]) and, hence, by Lemma 5 (14), for $\psi \in \Psi$, we have $\psi(t) < t$, for any $t > 0$.

Definition 9 (Samet et al. [1]). Let (X, d) be a metric space and $f : X \rightarrow X$ a given mapping. One says that f is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(f(x), f(y)) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{6}$$

Remark 10. If $f : X \rightarrow X$ satisfies the Banach contraction principle, then f is an α-ψ-contractive mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$.

Definition 11 (Samet et al. [1]). Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. One says that f is α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(f(x), f(y)) \geq 1. \tag{7}$$

Let $\mathscr{F}_f(X)$ be the class of fixed points of a self-mapping f defined on a nonempty set X; that is, $\mathscr{F}_f(X) = \{x \in X : f(x) = x\}$.

Example 12 (Samet et al. [1]). Let $X = (0, +\infty)$. Define $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

    (1) $f(x) = \ln(x)$, for all $x \in X$, and

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x \geq y; \\ 0, & \text{if } x < y. \end{cases} \qquad (8)$$

Then f is α-admissible.

    (2) $f(x) = \sqrt{x}$, for all $x \in X$, and

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y; \\ 0, & \text{if } x < y. \end{cases} \qquad (9)$$

Then f is α-admissible.

Example 13. Let $(X, \preceq)$ be a partially ordered set and d a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$; that is, $x, y \in X$, $x \preceq y \Rightarrow Tx \preceq Ty$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \preceq y \quad \text{or} \quad x \succeq y, \\ 0, & \text{otherwise.} \end{cases} \qquad (10)$$

Then, T is α-admissible. Since there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, we have $\alpha(x_0, Tx_0) \geq 1$. On the other hand, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y$$

$$\text{or} \quad x \preceq y \Longrightarrow Tx \succeq Ty \qquad (11)$$

$$\text{or} \quad Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Thus T is α-admissible.

Theorem 14 (Samet et al. [1]). *Let (X, d) be a complete metric space and $f : X \to X$ an α-ψ-contractive mapping satisfying the following conditions:*

    (i) *f is α-admissible;*

    (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*

    (iii) *f is continuous.*

Then, f has a fixed point; that is, there exists $x^* \in \mathscr{F}_f(X)$.

Theorem 15 (Samet et al. [1]). *Let (X, d) be a complete metric space and $f : X \to X$ an α-ψ-contractive mapping satisfying the following conditions:*

    (i) *f is α-admissible;*

    (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*

    (iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then, f has a fixed point; that is, there exists $x^* \in \mathscr{F}_f(X)$.

2. Main Results

First we give the following definition as a generalization of Definition 9.

Definition 16. Let (X, d) a b-metric space and $f : X \to X$ be a given mapping. We say that f is an α-ψ-contractive mapping of type-(b) if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_b$ such that

$$\alpha(x, y) \, d(f(x), f(y)) \leq \psi(d(x, y)), \quad \forall x, y \in X. \quad (12)$$

Our first main result is the following.

Theorem 17. *Let (X, d) be a complete b-metric space with constant $s > 1$. Let $f : X \to X$ be an α-ψ-contractive mapping of type-(b) satisfying the following conditions:*

    (i) *f is α-admissible;*

    (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*

    (iii) *f is continuous.*

Then the fixed point equation (29) *has a solution; that is, there exists $x^* \in \mathscr{F}_f(X)$.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ (such a point exists from condition (ii)). Define the sequence $\{x_n\}$ in X by

$$x_{n+1} = f(x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \qquad (13)$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point for f, and the proof finishes. Hence we assume that

$$x_n \neq x_{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}. \qquad (14)$$

Since f is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0))$$

$$\geq 1 \Longrightarrow \alpha(f(x_0), f(x_1)) \qquad (15)$$

$$= \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \qquad (16)$$

Applying the inequality (12) with $x = x_{n-1}$ and $y = x_n$ and using (16), we obtain

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n))$$

$$\leq \alpha(x_{n-1}, x_n) \, d(f(x_{n-1}), f(x_n)) \qquad (17)$$

$$\leq \psi(d(x_{n-1}, x_n)).$$

By induction, we get

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \quad \forall n \in \mathbb{N} \cup \{0\}. \qquad (18)$$

From (18) and using the triangular inequality, for all $p \geq 1$, we have

$$d\left(x_n, x_{n+p}\right) \leq sd\left(x_n, x_{n+1}\right) + s^2 d\left(x_{n+1}, x_{n+2}\right)$$
$$+ \cdots + s^{p-2} d\left(x_{n+p-3}, x_{n+p-2}\right)$$
$$+ s^{p-1} d\left(x_{n+p-2}, x_{n+p-1}\right)$$
$$+ s^p d\left(x_{n+p-1}, x_{n+p}\right)$$
$$\leq s\psi^n\left(d\left(x_0, x_1\right)\right) + s^2 \psi^{n+1}\left(d\left(x_0, x_1\right)\right)$$
$$+ \cdots + s^{p-2} \psi^{n+p-3}\left(d\left(x_0, x_1\right)\right)$$
$$+ s^{p-1} \psi^{n+p-2}\left(d\left(x_0, x_1\right)\right) \tag{19}$$
$$+ s^{p-1} \psi^{n+p-1}\left(d\left(x_0, x_1\right)\right)$$
$$= \frac{1}{s^{n-1}}\left[s^n \psi^n\left(d\left(x_0, x_1\right)\right)\right.$$
$$+ s^{n+1} \psi^{n+1}\left(d\left(x_0, x_1\right)\right)$$
$$+ \cdots + s^{n+p-2} \psi^{n+p-2}\left(d\left(x_0, x_1\right)\right)$$
$$\left.+ s^{n+p-1} \psi^{n+p-1}\left(d\left(x_0, x_1\right)\right)\right].$$

Denoting $S_n = \sum_{k=0}^{n} s^k \psi^k(d(x_0, x_1))$, $n \geq 1$, we obtain

$$d\left(x_n, x_{n+p}\right) \leq \frac{1}{s^{n-1}}\left[S_{n+p-1} - S_{n-1}\right], \quad n \geq 1, \ p \geq 1. \tag{20}$$

Due to the assumption (14) and Lemma 8, we conclude that the series $\sum_{k=0}^{n} s^k \psi^k(d(x_0, x_1))$ is convergent. Thus there exists $S = \lim_{n \to \infty} S_n \in [0, \infty)$. Regarding $s \geq 1$ and by (20), we obtain that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the b-metric space (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. From the continuity of f, it follows that $x_{n+1} = f(x_n) \to f(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = f(x^*)$; that is, x^* is a fixed point of f. □

In the following theorem, we are able to omit the continuity hypothesis of f by adding a new condition.

Theorem 18. *Let (X, d) be a complete b-metric space with constant $s > 1$. Let $f : X \to X$ be an α-ψ-contractive mapping of type-(b) satisfying the following conditions:*

   (i) *f is α-admissible;*

   (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$;*

   (iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then the fixed point equation (29) has a solution.

Proof. Following the proof of Theorem 17, we know that $\{x_n\}$ is a Cauchy sequence in the complete b-metric space (X, d).

Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, from (16) and the hypothesis (iii), we have

$$\alpha\left(x_n, x^*\right) \geq 1, \quad \forall n \in \mathbb{N}. \tag{21}$$

Now, using the triangular inequalities, (12) and (21), we get

$$d\left(f\left(x^*\right), x^*\right)$$
$$\leq s\left[d\left(f\left(x^*\right), f\left(x_n\right)\right) + d\left(x_{n+1}, x^*\right)\right]$$
$$\leq s\left[\alpha\left(x_n, x^*\right) d\left(f\left(x^*\right), f\left(x_n\right)\right) + d\left(x_{n+1}, x^*\right)\right] \tag{22}$$
$$\leq s\left[\psi\left(d\left(x_n, x^*\right)\right) + d\left(x_{n+1}, x^*\right)\right].$$

Letting $n \to \infty$, since ψ is continuous at $t = 0$, we obtain $d(f(x^*), x^*) = 0$; that is, $x^* = f(x^*)$. □

To assure the uniqueness of the fixed point, we will consider the following hypothesis:

$$\forall x, y \in X, \text{there exists } z \in X$$
$$\text{such that } \alpha(x, z) \geq 1, \quad \alpha(y, z) \geq 1. \tag{H}$$

Theorem 19. *Adding condition (H) to the hypotheses of Theorem 17 (resp., Theorem 18) one obtains uniqueness of the fixed point of f.*

Proof. Suppose that x^* and y^* are two fixed points of f. From (H), there exists $z \in X$ such that

$$\alpha\left(x^*, z\right) \geq 1, \quad \alpha\left(y^*, z\right) \geq 1. \tag{23}$$

Since f is α-admissible, from (23), we get

$$\alpha\left(x^*, f^n(z)\right) \geq 1, \quad \alpha\left(y^*, f^n(z)\right) \geq 1. \tag{24}$$

Using (24) and (12), we have

$$d\left(x^*, f^n(z)\right) = d\left(f\left(x^*\right), f\left(f^{n-1}(z)\right)\right)$$
$$\leq \alpha\left(x^*, f^{n-1}(z)\right) d\left(f\left(x^*\right), f\left(f^{n-1}(z)\right)\right)$$
$$\leq \psi\left(d\left(x^*, f^{n-1}(z)\right)\right). \tag{25}$$

This implies that

$$d\left(x^*, f^n(z)\right) \leq \psi^n\left(d\left(x^*, z\right)\right), \quad \forall n \in \mathbb{N}. \tag{26}$$

Then, letting $n \to \infty$, we have

$$f^n(z) \longrightarrow x^*. \tag{27}$$

Similarly, using (24) and (12), we get

$$f^n(z) \longrightarrow y^* \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Using (27) and (28), the uniqueness of the limit gives us $x^* = y^*$. This finishes the proof. □

Remark 20. Theorem 14 (resp., Theorem 15) can be derived from Theorem 17 (resp., Theorem 18) by taking $s = 1$. Consequently, all results in [1] can be considered as corollaries of our main results.

3. Ulam-Hyers Stability Results through the Fixed Point Problems

Definition 21. Let (X, d) be a metric space and $f : X \rightarrow X$ an operator. By definition, the fixed point equation

$$x = f(x) \tag{29}$$

is called generalized Ulam-Hyers stability if and only if there exists $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each $w^* \in X$ an ε-solution of the fixed point equation (29), that is, w^*, satisfies the inequality

$$d(w^*, f(w^*)) \leq \varepsilon. \tag{30}$$

There exists a solution $x^* \in X$ of (29) such that

$$d(w^*, x^*) \leq \psi(\varepsilon). \tag{31}$$

If there exists $c > 0$ such that $\psi(t) = c \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point equation (29) is said to be Ulam-Hyers stability.

For Ulam-Hyers stability results in the case of fixed point problems see Bota-Boriceanu, Petruşel [15], Lazăr [16], and Rus [17, 18].

Regarding the Ulam-Hyers stability problem the ideas given in Petru et al. [29] allow us to obtain the following result.

Theorem 22. *Let (X, d) be a complete b-metric space with constant $s > 1$. Suppose that all the hypotheses of Theorem 19 hold and additionally that the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) := r - s\psi(r)$ is strictly increasing and onto. Then the following hold.*

(a) *The fixed point equation (29) is generalized Ulam-Hyers stability.*

(b) *Fix$(f) = \{x^*\}$ and if $x_n \in X$, $n \in \mathbb{N}$ are such that $d(x_n, f(x_n)) \rightarrow 0$, as $n \rightarrow \infty$, then $x_n \rightarrow x^*$, as $n \rightarrow \infty$; that is, the fixed point equation (29) is well posed.*

(c) *If $g : X \rightarrow X$ is such that there exists $\eta \in [0, \infty)$ with*

$$d(f(x), g(x)) \leq \eta, \quad \forall x \in X, \tag{32}$$

then

$$y^* \in \mathrm{Fix}(g) \Longrightarrow d(x^*, y^*) \leq \beta^{-1}(s \cdot \eta). \tag{33}$$

Proof. (a) Since $f : X \rightarrow X$ is a Picard operator, so Fix$(f) = \{x^*\}$. Let $\varepsilon > 0$ and $w^* \in X$ be a solution of (30); that is,

$$d(w^*, f(w^*)) \leq \varepsilon. \tag{34}$$

Since f is α-ψ-contractive mapping of type-(b) and since $x^* \in$ Fix(f), from (H), there exists $w^* \in X$ such that $\alpha(x^*, w^*) \geq 1$; we obtain

$$
\begin{aligned}
d(x^*, w^*) &= d(f(x^*), w^*) \\
&\leq s[d(f(x^*), f(w^*)) + d(f(w^*), w^*)] \\
&\leq s[\alpha(x^*, w^*) d(f(x^*), f(w^*)) + \varepsilon] \\
&\leq s[\psi(d(x^*, w^*)) + \varepsilon].
\end{aligned} \tag{35}
$$

Therefore,

$$
\begin{aligned}
\beta(d(x^*, w^*)) &:= d(x^*, w^*) - s\psi(d(x^*, w^*)) \\
&\leq s \cdot \varepsilon \Longrightarrow d(x^*, w^*) \\
&\leq \beta^{-1}(s \cdot \varepsilon).
\end{aligned} \tag{36}
$$

Consequently, the fixed point equation (29) is generalized Ulam-Hyers stability.

(b) Since f is α-ψ-contractive mapping of type-(b) and since $x^* \in$ Fix(f), from (H), there exists $x_n \in X$ such that $\alpha(x^*, x_n) \geq 1$; we obtain

$$
\begin{aligned}
d(x_n, x^*) &\leq s[d(x_n, f(x_n)) + d(f(x_n), x^*)] \\
&= s[d(x_n, f(x_n)) + d(f(x_n), f(x^*))] \\
&\leq s[d(x_n, f(x_n)) + \alpha(x_n, x^*) d(f(x_n), f(x^*))] \\
&\leq s[d(x_n, f(x_n)) + \psi(d(x_n, x^*))].
\end{aligned} \tag{37}
$$

Therefore

$$
\begin{aligned}
\beta(d(x_n, x^*)) &:= d(x_n, x^*) - s\psi(d(x_n, x^*)) \\
&\leq s d(x_n, f(x_n)) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \\
&\Longrightarrow d(x_n, x^*) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \\
&\Longrightarrow x_n \longrightarrow x^*, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{38}
$$

So, the fixed point equation (29) is well posed.

(c) Since f is α-ψ-contractive mapping of type-(b) and since $x^* \in$ Fix(f), from (H), there exists $x \in X$ such that $\alpha(x^*, x) \geq 1$; we obtain

$$
\begin{aligned}
d(x, x^*) &\leq s[d(x, f(x)) + d(f(x), x^*)] \\
&= s[d(x, f(x)) + d(f(x), f(x^*))] \\
&\leq s[d(x, f(x)) + \alpha(x, x^*) d(f(x), f(x^*))] \\
&\leq s[d(x, f(x)) + \psi(d(x, x^*))].
\end{aligned} \tag{39}
$$

Therefore

$$
\begin{aligned}
\beta(d(x, x^*)) &:= d(x, x^*) - s\psi(d(x, x^*)) \\
&\leq s \cdot d(x, f(x)).
\end{aligned} \tag{40}
$$

So, we have the following estimation:

$$d(x, x^*) \leq \beta^{-1}(s \cdot d(x, f(x))). \tag{41}$$

Writing (41) for $x := y^*$ we get

$$\begin{aligned}
d(x^*, y^*) &\leq \beta^{-1}(s \cdot d(y^*, f(y^*))) \\
&= \beta^{-1}(s \cdot d(s(y^*), f(y^*))) \Longrightarrow d(x^*, y^*) \\
&\leq \beta^{-1}(s \cdot \eta).
\end{aligned} \tag{42}$$

$\square$

Acknowledgments

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