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Research Article

Linear Sequences and Weighted Ergodic Theorems

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We present a simple way to produce good weights for several types of ergodic theorem including the Wiener-Wintner type multiple return time theorem and the multiple polynomial ergodic theorem. These weights are deterministic and come from orbits of certain bounded linear operators on Banach spaces. This extends the known results for nilsequences and return time sequences of the form $(g(S^n y))$ for a measure preserving system (Y, S) and $g \in L^\infty(Y)$, avoiding in the latter case the problem of finding the full measure set of appropriate points y.

1. Introduction

The classical mean and pointwise ergodic theorems due to von Neumann and Birkhoff, respectively, take their origin in questions from statistical physics and found applications in quite different areas of mathematics such as number theory, stochastics, and harmonic analysis. Over the years, they were extended and generalised in many ways. For example, to multiple ergodic theorems, see Furstenberg [1], Bergelson et al. [2], Host and Kra [3], Ziegler [4], and Tao [5], to the Wiener-Wintner theorem, see Assani [6], Lesigne [7], Frantzikinakis [8], Host and Kra [9], and Eisner and Zorin-Kranich [10], to the return time theorem and its generalisations, see Bourgain et al. [11], Demeter et al. [12], Rudolph [13], Assani and Presser [14, 15], and Zorin-Kranich [16], and to further weighted, modulated, and subsequential ergodic theorems, see Berend et al. [17], Below and Losert [18], Bourgain [19, 20], and Wierdl [21].

The return time theorem due to Bourgain, solving a quite long standing open problem, is a classical example of a weighted pointwise ergodic theorem. It states that for every measure preserving system (Y, μ, S) and $g \in L^\infty(Y, \mu)$, the sequence $(g(S^n y))$ is for μ-almost every y a good weight for the pointwise ergodic theorem. This means that for every other system (Y_1, μ_1, S_1) and every $g_1 \in L^\infty(Y_1, \mu_1)$, the weighted ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} g(S^n y)\, g_1(S_1^n y_1) \tag{1}$$

converge almost everywhere in y_1. The proof due to Bourgain et al. [11], see also Lesigne et al. [22] and Zorin-Kranich [23], is descriptive and gives conditions on y to produce a good weight. However, these conditions can be quite difficult to check in a concrete situation. Later, Rudolph [13], see also Assani and Presser [14] and Zorin-Kranich [16], gave a generalisation of the return time theorem and showed that (in the previous notation) the sequence $(g(S^n y))$ is for almost every y a universally good weight for multiple ergodic averages; see Definition 4 later. However, the conditions on the point y did not become easier to check.

The most general class of systems for which the convergence in the multiple return time theorem is known to hold *everywhere*, hence, leading to good weights which are easy to construct, are nilsystems, that is, systems of the form $Y = G/\Gamma$ for a nilpotent Lie group G, a discrete cocompact subgroup Γ, the Haar measure μ on G/Γ, and the rotation S by some element of G. For such system (Y, μ, S), $g \in C(Y)$ and $y \in Y$, the sequence $(g(S^n y))$ is called a basic nilsequence. A *nilsequence* is a uniform limit of basic nilsequences of the same step, or, equivalently, a sequence of the form $(g(S^n y))$ for an inverse limit Y of nilsystems of the same step, $y \in Y$, a rotation S on Y and $g \in C(Y)$; see Host and Maass [24]. Indeed, recently Zorin-Kranich [16] proved the Wiener-Wintner type return time theorem for nilsequences showing universal convergence of averages

$$\frac{1}{N}\sum_{n=1}^{N} a_n g_1(S_1^n y_1)\cdots g_k(S_k^n y_k) \tag{2}$$

for every $k \in \mathbb{N}$ and every nilsequence (a_n), where the universal sets of convergence do not depend on (a_n). This generalised an earlier result by Assani et al. [25] for sequences of the form (λ^n), $\lambda \in \mathbb{T}$, and $k = 2$.

In this paper, we search for good weights for ergodic theorems using a functional analytic perspective and produce deterministic good weights. We first introduce sequences of the form $(\langle T^n x, x' \rangle)$, which we call *linear sequences* if x is in a Banach space X, $x' \in X'$ and T is a linear operator on X with relatively weakly compact orbits; see Section 2 later. Using a structure result for linear sequences, we show that they are good weights for the multiple polynomial ergodic theorem (Section 4) and for the Wiener-Wintner type multiple return time theorem discussed (Section 3). In the last section, we present a counterexample showing that the assumption on the operators cannot be dropped even for positive isometries on Banach lattices and the mean ergodic theorem.

We finally remark that all results in this paper hold if we replace linear sequences by a larger class of "asymptotic nilsequences," that is, for sequences (a_n) of the form $a_n = b_n + c_n$, where (b_n) is a nilsequence and (c_n) is a bounded sequence satisfying $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|c_n| = 0$ (cf. Theorem 3). Examples of asymptotic nilsequences (of step ≥2 in general) are *multiple polynomial correlation sequences* (a_n) of the form

$$a_n = \int_Y S^{p_1(n)} g_1 \cdots S^{p_k(n)} g_k \, d\mu \tag{3}$$

for an ergodic invertible measure preserving system (Y, μ, S), $k \in \mathbb{N}$, $g_j \in L^\infty(Y, \mu)$, and polynomials p_j with integer coefficients, $j = 1, \ldots, k$. This follows from Leibman [26, Theorem 3.1] and, in the case of linear polynomials, is due to Bergelson et al. [27, Theorem 1.9]. Thus, multiple polynomial correlation sequences provide another class of deterministic examples of good weights for the Wiener-Wintner type multiple return time theorem and the multiple polynomial ergodic theorem discussed in Sections 3 and 4.

2. Linear Sequences and Their Structure

A linear operator T on a Banach space X has *relatively weakly compact orbits* if for every $x \in X$, the orbit $\{T^n x, n \in \mathbb{N}_0\}$ is relatively weakly compact in X.

Definition 1. We call a sequence $(a_n) \subset \mathbb{C}$ a linear sequence if there exists an operator T on a Banach space X with relatively weakly compact orbits and $x \in X$, $x' \in X'$, such that $a_n = \langle T^n x, x' \rangle$ holds for every $n \in \mathbb{N}$.

A large class of operators with relatively weakly compact orbits, leading to a large class of linear sequences, are power bounded operators on reflexive Banach spaces. Recall that an operator T is called *power bounded* if it satisfies $\sup_{n\in\mathbb{N}}\|T^n\| < \infty$. Another class of operators with relatively weakly compact orbits are power bounded positive operators on a Banach lattice $L^1(\mu)$ preserving the order interval generated by a strictly positive function; see, for example, Schaefer [28, Theorem II.5.10(f) and Proposition II.8.3]. See [29, Section I.1] and [30, Section 16.1] for further discussion.

Remark 2. By restricting to the closed linear invariant subspace $Y := \overline{\text{lin}}\{T^n x, n \in \mathbb{N}_0\}$ induced by the orbit and using the decomposition $X' = Y' \oplus Y_0'$ for $Y_0' := \{x' : x'|_Y = 0\}$, it suffices to assume that only the relevant orbit $\{T^n x, n \in \mathbb{N}_0\}$ is relatively weakly compact in the definition of a linear sequence $(\langle T^n x, x' \rangle)$. Note that in this case T has relatively weakly compact orbits on Y by a limiting argument; see, for example, [29, Lemma I.1.6].

We obtain the following structure result for linear sequences as a direct consequence of an extended Jacobs-Glicksberg-deLeeuw decomposition for operators with relatively weakly compact orbits.

Theorem 3. *Every linear sequence is a sum of an almost periodic sequence and a (bounded) sequence (c_n) satisfying $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|c_n| = 0$.*

Proof. Let T be an operator on a Banach space X with relatively weakly compact orbits. By the Jacobs-Glicksberg-deLeeuw decomposition, see, for example, [29, Theorem II.4.8], $X = X_r \oplus X_s$, where

$$X_r = \overline{\text{lin}}\{x : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}, \tag{4}$$

while every $x \in X_s$ satisfies $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|\langle T^n x, x' \rangle| = 0$ for every $x' \in X'$. (Recall that by the Koopman-von Neumann lemma, see, for example, Petersen [31, p. 65], for bounded sequences the condition $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|c_n| = 0$ is equivalent to $\lim_{j\to\infty} c_{n_j} = 0$ for some subsequence $\{n_j\} \subset \mathbb{N}$ with density 1.)

Let $x \in X$, $x' \in X'$ and define the sequence (a_n) by $a_n := \langle T^n x, x' \rangle$. For $x \in X_s$ we have $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|a_n| = 0$ by the aforementioned. If now x is an eigenvector corresponding to an eigenvalue $\lambda \in \mathbb{T}$, then $a_n = \lambda^n \langle x, x' \rangle$. Therefore, for every $x \in X_r$, the sequence (a_n) is a uniform limit of finite linear combinations of sequences (λ^n), $\lambda \in \mathbb{T}$, and is therefore almost periodic. The assertion follows. □

3. A Wiener-Wintner Type Result for the Multiple Return Time Theorem

In this section, we show that one can take linear sequences as weights in the multiple Wiener-Wintner type generalisation of the return time theorem due to Zorin-Kranich [16] and Assani et al. [25] discussed in the introduction.

First we recall the definition of a property satisfied universally.

Definition 4. Let $k \in \mathbb{N}$ and P be a pointwise property for k measure preserving dynamical systems. We say that a property P is satisfied universally almost everywhere if for every system (Y_1, μ_1, S_1) and every $g_1 \in L^\infty(Y_1, \mu_1)$ there is a set $Y_1' \subset Y_1$ of full measure such that for every $y_1 \in Y_1'$ and every system $(Y_2, \mu_2, S_2) \ldots$ for every system (Y_k, μ_k, S_k) and $g_k \in L^\infty(Y_k, \mu_k)$ there is a set $Y_k' \subset Y_k$ of full measure such that for every $y_k \in Y_k'$ the property P holds.

We show the following linear version of the Wiener-Wintner type multiple return time theorem.

Theorem 5. *For every $k \in \mathbb{N}$, the weighted averages (2) converge universally almost everywhere for every linear sequence (a_n), where the universal sets Y_j', $j = 1, \dots, k$, of full measure are independent of (a_n).*

Proof. By Theorem 3, we can show the assertion for almost periodic sequences and for (a_n) satisfying $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|a_n| = 0$ separately. For sequences from the second class, the assertion follows from the estimate

$$\left|\frac{1}{N}\sum_{n=1}^{N} a_n g_1(S_1^n y_1)\cdots g_k(S_k^n y_k)\right| \le \|g_1\|_\infty \cdots \|g_k\|_\infty \frac{1}{N}\sum_{n=1}^{N}|a_n| \tag{5}$$

with a clear choice of $Y_1', \dots, Y_k'$.

Universal convergence for almost periodic sequences is a consequence of Zorin-Kranich's result [16, Theorem 1.3] which shows the assertion for the larger class of nilsequences. □

4. Weighted Multiple Polynomial Ergodic Theorem

Using the Host-Kra Wiener-Wintner type result for nilsequences and extending their result for linear polynomials from [9], Chu [32] showed the following (see also [10] for a slightly different proof). Let (Y, μ, S) be a system and $g \in L^\infty(Y, \mu)$. Then, for almost every $y \in Y$, the sequence $(g(S^n y))$ is a *good weight for the multiple polynomial ergodic theorem*; that is, for the sequence of weights (a_n) given by $a_n := g(S^n y)$ and for every $k \in \mathbb{N}$, the weighted multiple polynomial averages

$$\frac{1}{N}\sum_{n=1}^{N} a_n S_1^{p_1(n)} g_1 \cdots S_1^{p_k(n)} g_k \tag{6}$$

converge in L^2 for every system (Y_1, μ_1, S_1) with invertible S_1, every $g_1, \dots, g_k \in L^\infty(Y_1, \mu_1)$, and every polynomial $p_1, \dots, p_k$ with integer coefficients.

The following result is a consequence of Chu [32, Theorem 1.3], with the fact that the product of two nilsequences is again a nilsequence and equidistribution theory for nilsystems; see, for example, Parry [33] and Leibman [34].

Theorem 6. *Every nilsequence is a good weight for the multiple polynomial ergodic theorem.*

This remains true when replacing a nilsequence by a linear sequence.

Theorem 7. *Every linear sequence is a good weight for the multiple polynomial ergodic theorem.*

Proof. For an almost periodic sequence (a_n), the averages (6) converge in L^2 by Theorem 6. It is also clear that the averages (6) converge to 0 in L^∞ for every sequence (a_n) satisfying $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N}|a_n| = 0$. The assertion follows now from Theorem 3. □

5. A Counter Example

The following example shows that if one does not assume relative weak compactness in the definition of linear sequences, each of the previous results can fail dramatically even for positive isometries on Banach lattices.

Example 8. Let $X := l^1$ and T be the right shift operator; that is,

$$T(t_1, t_2, \dots) := (0, t_1, t_2, \dots). \tag{7}$$

We first show that for every $\lambda \in \mathbb{T}$, $x = (t_j) \in X$, and $x' = (s_j) \in X'$, we have

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\lambda^n\left\langle T^n x, x'\right\rangle - \frac{1}{N}\sum_{n=1}^{N}\lambda^n s_n \sum_{j=1}^{\infty}\overline{\lambda}^j t_j\right| = 0. \tag{8}$$

Indeed, take $\varepsilon > 0$ and $J \in \mathbb{N}$ such that $\sum_{j=J+1}^{\infty}|t_j| < \varepsilon$. Then, for $N \in \mathbb{N}$ we have

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{n=1}^{N}\lambda^n\left\langle T^n x, x'\right\rangle - \frac{1}{N}\sum_{n=1}^{N}\lambda^n s_n\sum_{j=1}^{\infty}\overline{\lambda}^j t_j\right| \\
&= \left|\frac{1}{N}\sum_{n=1}^{N}\lambda^n\sum_{j=1}^{\infty}t_j s_{n+j} - \frac{1}{N}\sum_{n=1}^{N}\lambda^n s_n\sum_{j=1}^{\infty}\overline{\lambda}^j t_j\right| \\
&\le \left|\frac{1}{N}\sum_{n=1}^{N}\lambda^n\sum_{j=1}^{J}t_j s_{n+j} - \frac{1}{N}\sum_{n=1}^{N}\lambda^n s_n\sum_{j=1}^{J}\overline{\lambda}^j t_j\right| \\
&\quad + 2\left\|x'\right\|_\infty \varepsilon \\
&= \left|\sum_{j=1}^{J}\overline{\lambda}^j t_j\frac{1}{N}\sum_{n=1+j}^{N+j}\lambda^n s_n - \sum_{j=1}^{J}\overline{\lambda}^j t_j\frac{1}{N}\sum_{n=1}^{N}\lambda^n s_n\right| \\
&\quad + 2\left\|x'\right\|_\infty \varepsilon \\
&\le \frac{2J\|x\|_1\left\|x'\right\|_\infty}{N} + 2\left\|x'\right\|_\infty \varepsilon.
\end{aligned} \tag{9}$$

Choosing, for example, $N > J\|x\|_1/\varepsilon$ finishes the proof of (8).

In particular, for $\lambda = 1$, we see that the sequence $(\langle T^n x, x'\rangle)$ is Cesàro divergent for every $x = (t_j) \in l^1$ with $\sum_{j=1}^{\infty} t_j \neq 0$ and for every $x' \in l^\infty$ which is Cesàro divergent. Note that the sets of such x and x' are open and dense in l^1 and l^∞, respectively. (The assertion for l^1 is clear as well as the openness of the set of Cesàro divergent sequences in l^∞, and density follows from the fact that one can construct Cèsaro divergent sequences of arbitrarily small supremum norm.) Thus, for topologically very big sets of x and x' (with complements being nowhere dense), the sequence $(\langle T^n x, x'\rangle)$ is not a good weight for the mean ergodic theorem.

We further show that in fact for every $0 \neq x \in l^1$, there is $\lambda \in \mathbb{T}$ so that for every $x' \in l^\infty$ from a dense open set, the sequence $(\lambda^n\langle T^n x, x'\rangle)$ is Cesàro divergent, implying that the sequence $(\langle T^n x, x'\rangle)$ is not a good weight for the mean ergodic theorem.

Take $0 \neq x = (t_j) \in l^1$ and define the function f on the unit disc $\mathbb{D}$ by $f(z) := \sum_{j=1}^{\infty} t_j z^j$. Then, f is a nonzero holomorphic function belonging to the Hardy space $H^1(\mathbb{D})$. By Hardy space theory, see, for example, Rosenblum and Rovnyak [35, Theorem 4.25], there is a set $M \subset \mathbb{T}$ of positive Lebesgue measure such that for every $\lambda \in M$, we have

$$\lim_{r \to 1-} f\left(r\overline{\lambda}\right) = \sum_{j=1}^{\infty} \overline{\lambda}^j t_j \neq 0. \quad (10)$$

For every such λ, by (8), we see that the sequence $(\lambda^n \langle T^n x, x' \rangle)$ is Cesàro divergent for every $x' = (s_j) \in l^\infty$ such that $(\lambda^j s_j)$ is Cesàro divergent. The set of such x' is open and dense in l^∞ since it is the case for $\lambda = 1$, and the multiplication operator $(s_j) \mapsto (\lambda^j s_j)$ is an invertible isometry. Thus, for every $0 \neq x \in l^1$, there is an open dense set of $x' \in l^\infty$ such that the sequence $(\langle T^n x, x' \rangle)$ fails to be a good weight for the mean ergodic theorem.

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