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Research Article

The Existence of Multiple Solutions for Nonhomogeneous Kirchhoff Type Equations in $\mathbb{R}^3$

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We are concerned with the existence of multiple solutions to the nonhomogeneous Kirchhoff type equation $-(a+b\int_{\mathbb{R}^3}|\nabla u|^2)\Delta u+u=|u|^{p-1}u+h(x)$ in $\mathbb{R}^3$, where a, b are positive constants, $p\in(1,5)$, $0\leqslant h(x)=h(|x|)\in C^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, we can find a constant $m_p>0$ such that for all $p\in(1,5)$ the equation has at least two radial solutions provided $|h|_2<m_p$.

1. Introduction and Main Result

In this paper, we consider the existence of multiple solutions to the following nonhomogeneous Kirchhoff type equation:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=|u|^{p-1}u+h(x)\quad \text{in } \mathbb{R}^3, \quad (1)$$

where a, b are positive constants and $p\in(1,5)\cdot h\in C^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ satisfies the following conditions:

(h_1) $0\leqslant h(x)=h(|x|)\in L^2(\mathbb{R}^3)$ and $|h|_2\leqslant m_p$, where

$$m_p=\frac{p-1}{2p\gamma_2}\left(\frac{p+1}{2p\gamma_{p+1}^{p+1}}\right)^{1/(p-1)}, \quad (2)$$

γ_s is the embedding coefficient of $H^1(\mathbb{R}^3)\hookrightarrow L^s(\mathbb{R}^3)$ and $s\in[2,6]$;

(h_2) $(\nabla h(x),x)\in L^2(\mathbb{R}^3)$, where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^3$.

Recently, there have been many references about the existence of nontrivial solutions to the following Kirchhoff type equation by using variational method [1–5]:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V(x)u=f(x,u)\quad \text{in } \mathbb{R}^N, \quad (3)$$

where a, b are positive constants. $V:\mathbb{R}^N\to\mathbb{R}$, $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $N=1,2,3$. A main tool to deal with problem (3) is the mountain pass theorem. For this purpose, one usually assumes that $f(x,t)$ is subcritical, superlinear at the origin, and either 4-superlinear at infinity or satisfies the following global Ambrosetti-Rabinowitz type condition (AR in short):

(AR) there exists $\mu>4$ such that $0<\mu F(x,t)=\int_0^t f(x,s\leqslant tf(x,t)$ for all $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$. Under the above assumptions, the mountain pass geometry structure and the boundedness of Palais-Smale sequence or Cerami sequence can be obtained.

For example, in [5], when f satisfies above assumptions and the potential V satisfies the following conditions:

(V) $V\in C(\mathbb{R}^N,\mathbb{R})$, $\inf_{\mathbb{R}^N}V>0$ and for each $M>0$, $\operatorname{meas}\{x\in\mathbb{R}^N: V(x)\leqslant M\}<\infty$, where meas denotes the Lebesgue measure,

which ensure the compact imbedding of $E=\{u\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N}Vu^2<\infty\}\hookrightarrow L^q(\mathbb{R}^N)$, $q\in[2,2^*)$, the author obtained the existence of a nontrivial solution to problem (3).

The existence of infinitely many solutions was considered in [2, 3] respectively, by the fountain theorem and a variant version of the fountain theorem, where f is odd on $t\in\mathbb{R}$ and is also subcritical, superlinear at the origin, and either 4-superlinear at infinity or satisfies AR condition or some conditions weaker than AR condition. In [2], $N = 2, 3$, $V \equiv 1$ and in [3], $N = 3$, $V \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies the condition (V).

The existence of ground state solutions to problem (3) was also considered in [1, 4]. In [1], the authors studied (3) under the conditions: $N = 3$, a positive potential satisfies $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} > 0 \cdot f(x,t) = f(t) \in C^1(\mathbb{R},\mathbb{R})$ satisfies (AR), $\lim_{t\to 0}(f(t)/|t|^3) = 0$, $\lim_{t\to\infty}(f(t)/|t|^q) = 0$ for some $q \in (3,5)$ and $f(t)/t^3$ increases for all $t > 0$. They obtained a positive ground state solution by using the Nehari manifold.

Under the same condition of V in [1], the authors in [4] discussed the existence of multiple ground state solutions, where $f(x,t) = \lambda f(t) + |t|^4 t$, which contains a critical growth term.

Recently, in [6], the authors studied the existence of a positive solution for the following Kirchhoff equation:

$$\left(a + \lambda \int_{\mathbb{R}^3} \left[|\nabla u|^2 + bu^2\right]\right)(-\Delta u + bu) = f(u) \quad \text{in } \mathbb{R}^3, \tag{4}$$

where $N \geqslant 3$, $a, b > 0$, f is subcritical, superlinear at the origin and infinity. In order to construct the mountain pass geometry structure and obtain the bounded PS sequence, they combined a truncation argument with a monotonicity trick introduced by Jeanjean [7], and obtained that there exists $\lambda_0 > 0$ such that problem (4) has at least one positive solution for $\lambda \in (0, \lambda_0)$.

Motivated by the aforementioned references, we consider the existence of multiple solutions to the nonhomogeneous Kirchhoff equation (1), where $p \in (1,5)$. By using the variational method, we obtain that the problem has at least two positive radial solutions. Under proper assumptions on h, the problem has a local minimum around the origin with negative energy by Ekeland variational principle. Note that the term $|u|^{p-1}u$ is neither 4-superlinear nor satisfies AR condition for $p \in (1,3]$. In order to obtain the bounded PS sequence, we also use the indirect method in [7]. Meanwhile, for $w \in H^1(\mathbb{R}^3)$, we take a transform of $w_t(\cdot) = tw(t^{-2}\cdot)$ to construct the mountain pass geometry structure. Finally, the combination of Pohozaev identity with the method in [7] obtains the bounded PS sequence. Therefore, we obtain the second solution which has positive energy.

Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space equipped with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3} [a\nabla u \cdot \nabla v + uv], \qquad \|u\| = (u,u)^{1/2}. \tag{5}$$

We denote by $|\cdot|_s$ the usual $L^s(\mathbb{R}^3)$ norm. Then, we have that $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ continuously for $s \in [2,6]$. Hence, there exists γ_s such that

$$|u|_s \leqslant \gamma_s \|u\|, \quad u \in H^1(\mathbb{R}^3). \tag{6}$$

Let $H = H^1_r(\mathbb{R}^3)$ be the subspace of $H^1(\mathbb{R}^3)$ containing only the radial functions. Then the imbedding $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s \in (2,6)$ [8, Corollary 1.26, page 18]. Let $\mathscr{D}^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\mathscr{D}^{1,2}} = (\int_{\mathbb{R}^3} |\nabla u|^2)^{1/2}$.

Define the energy functional $J : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\begin{aligned} J(u) = {} & \frac{1}{2}\int_{\mathbb{R}^3}\left[a|\nabla u|^2 + u^2\right] + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 \\ & - \frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1} - \int_{\mathbb{R}^3} hu. \end{aligned} \tag{7}$$

By $p \in (1,5)$, $h \in L^2(\mathbb{R}^3)$, we have $J \in C^1(H^1(\mathbb{R}^3),\mathbb{R})$. And, for any $u, v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} \left(J'(u), v\right) = {} & \int_{\mathbb{R}^3}[a\nabla u\cdot\nabla v + uv] \\ & + b\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)\int_{\mathbb{R}^3}\nabla u\cdot\nabla v \\ & - \int_{\mathbb{R}^3}|u|^{p-1}uv - \int_{\mathbb{R}^3} hv. \end{aligned} \tag{8}$$

Furthermore, by (h_1), $h(x) = h(|x|)$, the functional J is also a C^1 functional defined on H. By standard argument, the weak solution of (1) is corresponding to the critical point of the functional J on H.

Our main result is as follows.

Theorem 1. *Let $p \in (1,5)$ and h satisfy* (h_1)*-*(h_2)*. Then, problem* (1) *has at least two nontrivial radial solutions u_0 and v_0, satisfying $J(u_0) < 0 < J(v_0)$.*

The paper is organized as follows. In Section 2, we give the existence of the negative energy solution u_0. The existence of positive energy solution v_0 and the proof of Theorem 1 are given in Section 3.

2. Existence of Negative Energy Solution

In this section, we give the existence of the negative energy solution. In order to obtain our first solution, we need the following preliminaries.

Lemma 2. *Let $p \in (1,5)$ and h satisfy* (h_1)*. Then, there exists $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, where $B_\rho = \{u \in H : \|u\| < \rho\}$.*

Proof. For $u \in H$, by (7), the Hölder inequality and the Sobolev inequality imply that

$$\begin{aligned} J(u) & \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\gamma_{p+1}^{p+1}\|u\|^{p+1} \\ & \quad - |h|_2 |u|_2 \\ & \geqslant \|u\|\left(\frac{1}{2}\|u\| - \frac{1}{p+1}\gamma_{p+1}^{p+1}\|u\|^p - \gamma_2|h|_2\right). \end{aligned} \tag{9}$$

Set

$$g(t) = \frac{1}{2}t - \frac{1}{p+1}\gamma_{p+1}^{p+1}t^p, \quad t \geqslant 0, \tag{10}$$

since $p > 1$, by calculating directly, we see that $\max_{t\geqslant 0} g(t) = g(\rho) > 0$, where $\rho = ((p+1)/2p\gamma_{p+1}^{p+1})^{1/(p-1)}$, $g(\rho) = ((p-1)/2p)\rho$. Then it follows that, if $|h|_2\gamma_2 < g(\rho)$, that is, $|h|_2 < \gamma_2^{-1} g(\rho) \triangleq m_p$, there exists $\alpha = \rho(g(\rho) - \gamma_2|h|_2) > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, where

$$m_p = \frac{p-1}{2p\gamma_2}\left(\frac{p+1}{2p\gamma_{p+1}^{p+1}}\right)^{1/(p-1)}. \tag{11}$$

□

Lemma 3. *Let $p \in (1,5)$ and h satisfy (h_1). Then $c = \inf_{\overline{B}_\rho} J < 0$, where ρ is given by Lemma 2 and $B_\rho = \{u \in H : \|u\| < \rho\}$.*

Proof. By (h_1), $h \in L^2(\mathbb{R}^3)$, $h \neq 0$, then for $\varepsilon \in (0, |h|_2)$, there exists $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $|h-\psi|_2 < \varepsilon$. Then $\int_{\mathbb{R}^3}(h^2 - h\psi) \leqslant \int_{\mathbb{R}^3}|h^2 - h\psi| \leqslant \varepsilon|h|_2$, so $\int_{\mathbb{R}^3} h\psi \geqslant |h|_2^2 - \varepsilon|h|_2 > 0$. Hence, by (7), for $t > 0$ small enough, we have

$$\begin{aligned} J(t\psi) = {} & \frac{t^2}{2}\|\psi\|^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla\psi|^2\right)^2 \\ & - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3}|\psi|^{p+1} - t\int_{\mathbb{R}^3} h\psi < 0. \end{aligned} \tag{12}$$

Then, by the definition of B_ρ, $c = \inf_{\overline{B}_\rho} J < 0$. □

Lemma 4. *Let $p \in (1,5)$ and h satisfy (h_1). The bounded PS sequence of the functional J possesses a convergent subsequence.*

Proof. Let $\{u_n\}$ be a bounded PS sequence of J, that is $\{u_n\}$ and $\{J(u_n)\}$ are bounded, $J'(u_n) \to 0$ in H', where H' is the dual space of H. We may assume that, up to a subsequence,

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H, \qquad u_n \longrightarrow u \quad \text{in } L^{p+1}(\mathbb{R}^3), \\ u_n \longrightarrow u \quad \text{a.e. on } \mathbb{R}^3. \end{gathered} \tag{13}$$

It follows that

$$\int_{\mathbb{R}^3}\left(|u_n|^{p-1}u_n - |u|^{p-1}u\right)(u_n - u) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{14}$$

By (8), we can obtain that

$$\begin{aligned} & \left(J'(u_n) - J'(u), u_n - u\right) \\ & = \|u_n - u\|^2 + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\int_{\mathbb{R}^3}\nabla u_n\cdot(\nabla u_n - \nabla u) \\ & \quad - b\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)\int_{\mathbb{R}^3}\nabla u\cdot(\nabla u_n - \nabla u) \\ & \quad - \int_{\mathbb{R}^3}\left(|u_n|^{p-1}u_n - |u|^{p-1}u\right)(u_n - u) \\ & = \|u_n - u\|^2 + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\int_{\mathbb{R}^3}|\nabla u_n - \nabla u|^2 \\ & \quad + b\left(\int_{\mathbb{R}^3}\left(|\nabla u_n|^2 - |\nabla u|^2\right)\right)\int_{\mathbb{R}^3}\nabla u\cdot(\nabla u_n - \nabla u) \\ & \quad - \int_{\mathbb{R}^3}\left(|u_n|^{p-1}u_n - |u|^{p-1}u\right)(u_n - u). \end{aligned} \tag{15}$$

Since $\{u_n\}$ is also bounded in $\mathscr{D}^{1,2}(\mathbb{R}^3)$, then

$$b\left(\int_{\mathbb{R}^3}\left(|\nabla u_n|^2 - |\nabla u|^2\right)\right)\int_{\mathbb{R}^3}\nabla u\cdot(\nabla u_n - \nabla u) \longrightarrow 0. \tag{16}$$

Therefore,

$$\begin{aligned} 0 & \leqslant \|u_n - u\|^2 \leqslant \|u_n - u\|^2 \\ & \quad + b\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)\int_{\mathbb{R}^3}|\nabla u_n - \nabla u|^2 \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{17}$$

That is, $u_n \to u$ in H. □

Theorem 5. *Let $p \in (1,5)$ and h satisfy (h_1). Then, there exists $u_0 \in H$ such that*

$$J(u_0) = c = \inf_{\overline{B}_\rho} J(u) < 0, \tag{18}$$

where ρ is given by Lemma 2 and $B_\rho = \{u \in H : \|u\| < \rho\}$.

Proof. By Lemma 3, $c = \inf\{J(u) : u \in \overline{B}_\rho\} < 0$, then by Ekeland variational principle [9], there exists $\{u_n\} \subset \overline{B}_\rho$ such that

$$\begin{gathered} c \leqslant J(u_n) \leqslant c + \frac{1}{n}, \\ J(\omega) \geqslant J(u_n) - \frac{1}{n}\|\omega - u_n\| \qquad \forall \omega \in \overline{B}_\rho. \end{gathered} \tag{19}$$

Then, by Lemma 2 $u_n \in B_\rho$, then $\{u_n\}$ is a bounded PS sequence of J. Therefore, Lemma 4 implies that there exists $u_0 \in H$ such that $u_n \to u_0$, up to a subsequence. So $J(u_0) = c < 0$ and $J'(u_0) = 0$. □

3. Proof of Theorem 1

In this section, we will show the existence of the second solution. Note that $p \in (1,5)$, when $p \in (1,3]$, $|u|^{p-1}u$ neither satisfies (AR) condition nor is 4-superlinear. So, in order to obtain the bounded PS sequence, following the argument in [6], we also use a direct method in [7]. Firstly, we recall the following main result in [7]. The "monotonicity trick" at the core of this theorem was invented by Struwe (see [9]).

Theorem 6 (see [7]). *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ be an interval. Consider the family of C^1 functionals on X*

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in I, \tag{20}$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}, \quad \forall\lambda \in J, \tag{21}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}. \tag{22}$$

Then, for almost every $\lambda \in I$ there is a sequence $\{u_n(\lambda)\} \subset X$ such that

(i) *$\{u_n(\lambda)\}$ is bounded;*

(ii) *$J_\lambda(u_n(\lambda)) \to c_\lambda$;*

(iii) *$J'_\lambda(u_n(\lambda)) \to 0$ in the dual X^{-1} of X.*

In our case, $X = H$, $I = [1/2, 1]$, and define $J_\lambda : X \to \mathbb{R}$ by

$$J_\lambda(u) = A(u) - \lambda B(u), \tag{23}$$

where $\lambda \in I$,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^3} hu, \quad B(u) = \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}. \tag{24}$$

Then $\{J_\lambda\}_{\lambda\in I}$ is a family of C^1 functionals on H. For any $u \in H$, $B(u) \geqslant 0$, and $A(u) \geqslant (1/2)\|u\|^2 - \gamma_2|h|_2\|u\| \to \infty$ as $\|u\| \to \infty$.

In the following, we verify that the functional J_λ satisfies the conditions of Theorem 6.

Lemma 7. *Let $p \in (1,5)$ and h satisfy (h_1)-(h_2). Then, the following claims hold:*

(i) *there exist $r, a > 0$ and $e \in H$ such that for all $\lambda \in I$*

$$J_\lambda|_{\partial B_\rho}(u) \geqslant a > 0, \quad J_\lambda(e) < 0 \quad \text{with } \|e\| > r; \tag{25}$$

(ii) *for any $\lambda \in I$,*

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(0), J_\lambda(e)\}, \tag{26}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}. \tag{27}$$

Proof. (i) Since for all $u \in H$ and $\lambda \in I = [1/2, 1]$, $J_\lambda(u) \geqslant J_1(u)$. By Lemma 2, there exist $r, a > 0$ independent of $\lambda \in I$ such that $J_\lambda(u) \geqslant a > 0$ with $\|u\| = r$.

We choose a function $w \in H$ and $w \neq 0$. Set $w_t(\cdot) = tw(t^{-2}\cdot)$ for $t > 0$. Then, for all $\lambda \in I$, by (7) and (h_1), we have

$$J_\lambda(w_t) \leqslant \frac{at^4}{2}\int_{\mathbb{R}^3} |\nabla w|^2 + \frac{t^8}{2}\int_{\mathbb{R}^3} |w|^2 + \frac{bt^8}{4}\left(\int_{\mathbb{R}^3} |\nabla w|^2\right)^2 - \frac{t^{p+7}}{2(p+1)}\int_{\mathbb{R}^3} |w|^{p+1}. \tag{28}$$

Noting $p > 1$, then there exists t_0 large enough satisfying $\|w_{t_0}\| > r$, which is independent of $\lambda \in I$, such that for all $\lambda \in I$, $J_\lambda(e) < 0$ with $e = w_{t_0}$.

(ii) Since c_λ is nonincreasing on $\lambda \in I$, then by the definition of c_λ and (i), for all $\lambda \in I$, we have $c_{1/2} \geqslant c_\lambda \geqslant c_1 \geqslant a > 0$. □

By Lemma 7 and Theorem 6, for almost every $\lambda \in I$, there exists a bounded sequence $\{u_n^\lambda\} \subset H$ such that $J_\lambda(u_n^\lambda) \to c_\lambda$, $(J_\lambda)'(u_n^\lambda) \to 0$. By Lemma 4, there exists $u^\lambda \in H$ such that $u_n^\lambda \to u^\lambda$ in H. Therefore, $(J_\lambda)'(u^\lambda) = 0$ and $J_\lambda(u^\lambda) = c_\lambda$. It follows from (ii) of Lemma 7 that $u^\lambda \neq 0$.

Therefore, there exists $\{\lambda_n\} \subset I$ with $\lambda_n \to 1^-$ and a nonnegative sequence $\{u^{\lambda_n}\}$ (denoted by $\{u_n\}$ for simplicity) satisfying

$$J_{\lambda_n}(u_n) = c_{\lambda_n}, \qquad (J_{\lambda_n})'(u_n) = 0. \tag{29}$$

In order to obtain the boundedness of $\{u_n\}$, we need the following Pohozaev identity. The proof is similar to the argument in [10].

Lemma 8. *Under the conditions of (h_1) and (h_2), if $u \in H$ is a weak solution of* (1), *the following Pohozaev identity holds:*

$$\frac{1}{2}\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}\int_{\mathbb{R}^3} u^2 = \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} + \int_{\mathbb{R}^3} (3h + (\nabla h(x), x))\, u. \tag{30}$$

Lemma 9. *Consider $\{u_n\}$ in* (29) *is bounded in H.*

Proof. Firstly, since $(J_{\lambda_n})'(u_n) = 0$, then by Lemma 8, u_n satisfies the following Pohozaev identity:

$$\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \frac{3}{2}\int_{\mathbb{R}^3} u_n^2 - \lambda_n\frac{3}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} - 3\int_{\mathbb{R}^3} h(x)\, u_n - \int_{\mathbb{R}^3} (\nabla h(x), x)\, u_n = 0. \tag{31}$$

On the other hand, by $((J_{\lambda_n})'(u_n), u_n) = 0$ and $J_{\lambda_n}(u_n) = c_{\lambda_n}$, we have that

$$a\int_{\mathbb{R}^3} |\nabla u_n|^2 + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \int_{\mathbb{R}^3} u_n^2 - \lambda_n\int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} hu_n = 0, \tag{32}$$

$$\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 - \frac{\lambda_n}{p+1}\int_{\mathbb{R}^3} |u_n|^{p+1} - \int_{\mathbb{R}^3} hu_n = c_{\lambda_n}. \tag{33}$$

Then, combining (33) with (31), we can obtain

$$a\int_{\mathbb{R}^3}|\nabla u_n|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\right)^2=3c_{\lambda_n}-\int_{\mathbb{R}^3}(\nabla h(x),x)\,u_n. \tag{34}$$

Since $c_{\lambda_n}\leqslant c_{1/2}$ by Lemma 7, and $(\nabla h(x),x)\in L^2(\mathbb{R}^3)$, so in order to prove the boundedness of $\{u_n\}$ in H, we only need to prove that $|u_n|_2$ is bounded. By contradiction, we assume that $|u_n|_2\to\infty$, up to a subsequence. Let $v_n=u_n/|u_n|_2$, $X_n=a\int_{\mathbb{R}^3}|\nabla v_n|^2$, $Y_n=b|u_n|_2^2(\int_{\mathbb{R}^3}|\nabla v_n|^2)^2$, $Z_n=\lambda_n|u_n|_2^{p-1}\int_{\mathbb{R}^3}|v_n|^{p+1}$.

Note that c_{λ_n} is bounded and h, $(\nabla h(x),x)\in L^2(\mathbb{R}^3)$. Multiplying (31)–(33) by $1/|u_n|_2^2$, we have that

$$\begin{gathered}\frac{X_n}{2}+\frac{Y_n}{2}-\frac{3}{p+1}Z_n=-\frac{3}{2}+o(1),\\ X_n+Y_n-Z_n=-1+o(1),\\ \frac{X_n}{2}+\frac{Y_n}{4}-\frac{Z_n}{p+1}=-\frac{1}{2}+o(1),\end{gathered}\tag{35}$$

where $o(1)$ denotes the quantity tends to zero as $n\to\infty$. By calculating, we obtain that

$$X_n=\frac{1-p}{5-p}+o(1). \tag{36}$$

Since $p\in(1,5)$ and $X_n\geqslant 0$ for $n\in\mathbb{N}$, so this is a contradiction for n large enough. Therefore, $\{u_n\}$ is bounded in H. □

Proof of Theorem 1. Since

$$\begin{gathered}J(u_n)=J_{\lambda_n}(u_n)+(\lambda_n-1)\frac{1}{p+1}\int_{\mathbb{R}^3}|u_n|^{p+1},\\ \left(J'(u_n),v\right)=\left(\left(J_{\lambda_n}\right)'(u_n),v\right)\\ +(\lambda_n-1)\int_{\mathbb{R}^3}|u_n|^{p-1}u_nv,\quad v\in H.\end{gathered}\tag{37}$$

By Lemma 9 and $H\hookrightarrow L^{p+1}(\mathbb{R}^3)$, $|u_n|_{p+1}$ is bounded and $|\int_{\mathbb{R}^3}|u_n|^{p-1}u_nv|\leqslant\gamma_{p+1}^{p+1}|u_n|_{p+1}^p\|v\|$. Thus, when $\lambda_n\to 1^-$, we have that $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$. Therefore $\{u_n\}$ is a bounded PS sequence of J. By Lemma 4, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n\to v_0$, up to a subsequence. Consequently, $J'(v_0)=0$. According to Lemma 7, we have $J(v_0)=\lim_{n\to\infty}J(u_n)=\lim_{n\to\infty}J_{\lambda_n}(u_n)\geqslant a>0$. Thus v_0 is a positive energy solution to problem (1). Hence, by Theorem 5 problem (1) has two solutions u_0 and v_0 satisfying $J(u_0)<0<J(v_0)$. □

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