Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 803540, 7 pages
http://dx.doi.org/10.1155/2013/803540

Research Article

Post-Lie Algebra Structures on the Lie Algebra $gl(2, C)$

Yuqiu Sheng and Xiaomin Tang

Department of Mathematics, Heilongjiang University, Harbin 150080, China

Correspondence should be addressed to Xiaomin Tang; x.m.tang@163.com

Received 1 August 2013; Accepted 15 September 2013

Academic Editor: Teoman Özer

The post-Lie algebra is an enriched structure of the Lie algebra. We give a complete classification of post-Lie algebra structures on the Lie algebra $gl(2, C)$ up to isomorphism.

1. Introduction

Post-Lie algebras were introduced around 2007 by Vallette [1], who found the structure in a purely operadic manner as the Koszul dual of a commutative trialgebra. Moreover, Vallette [1] proves that post-Lie algebras have the important algebraic property of being Koszul. This property is shared by many other important algebras, such as Lie algebras, associative algebras, commutative algebras, pre-Lie algebras, LR-algebras, and dendriform algebras, see [2, 3]. Recently, many authors study some post-Lie algebras and post-Lie algebra structures [4–8]. We recall the definition of the post-Lie algebra (structure) as follows, see [1, 8].

Definition 1. A (left) post-Lie C-algebra is a C-vector space L with two binary operations $\circ$ and $[,]$ which satisfy the following relations:

$$[x, y] = -[y, x], \tag{1}$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0, \tag{2}$$

$$\begin{aligned} z \circ (y \circ x) - y \circ (z \circ x) + (y \circ z) \circ x \\ - (z \circ y) \circ x + [y, z] \circ x = 0, \end{aligned} \tag{3}$$

$$z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0. \tag{4}$$

Let $(\sigma(L), [,])$ denote the Lie algebra defined by (1) and (2). Call $(L, [,], \circ)$ a post-Lie algebra on $(\sigma(L), [,])$.

Remark 2. Suppose $(g, [,])$ is a Lie algebra. Two post-Lie algebra $(g, [,], \circ)$ and $(g, [,], \star)$ on the Lie algebra g are called isomorphic on the Lie algebra $(g, [,])$ if there is an automorphism φ of the Lie algebra $(g, [,])$ such that

$$\varphi(x \circ y) = \varphi(x) \star \varphi(y), \quad \forall x, y \in g. \tag{5}$$

One of the key problems in the study of post-Lie algebras is to find the post-Lie algebra structures on the (given) Lie algebras. In [8], the authors determined all isomorphic classes of post-Lie algebra structures on $(sl(2, C), [,])$, the special linear Lie algebra of order 2. They use an important fact that the derivation of a semisimple Lie algebra is inner. But for the nonsemisimple Lie algebra, this fact dos not hold. So that we must find another way to study such problem for nonsemisimple Lie algebra. The purpose of this paper is to give a complete classification of post-Lie algebra structures on nonsemisimple Lie algebra $gl(2, C)$, the general linear Lie algebra of order 2, up to isomorphism. Now, we recall the above two Lie algebras.

Denote

$$\begin{aligned} u_1 &:= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad u_2 := \frac{1}{2\sqrt{-1}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ u_3 &:= \frac{1}{2\sqrt{-1}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad u_4 := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{6}$$

It is obvious that the previous four matrices form a C-linear basis of $gl(2, C)$ and determine the Lie algebra $(gl(2, C), [,])$ through the Lie product

$$\begin{aligned} [u_1, u_2] = u_3, \qquad [u_2, u_3] = u_1, \qquad [u_3, u_1] = u_2, \\ [u_i, u_4] = 0, \quad i = 1, 2, 3. \end{aligned} \tag{7}$$

It is also well known that u_1, u_2, u_3 form a C-linear basis of $sl(2,C)$ and determine the Lie algebra $(sl(2,C),[,])$ through the relations (7). The authors in [8] got the following classification theorem, which will be used in our proof.

Theorem 3 (see [8]). *The following is a complete set of representatives for the isomorphic classes of post-Lie algebra* $(sl(2,C),[,],\circ)$ *on the Lie algebra* $(sl(2,C),[,])$.

(1) $u_i \circ u_j = 0,\ i,j = 1,2,3$;

(2) $u_i \circ u_j = [u_i, u_j],\ i,j = 1,2,3$;

(3) $u_1 \circ u_i = [-u_1, u_i], u_2 \circ u_i = [-((1+\sqrt{-1})/2)u_2 + ((\sqrt{-1}-1)/2)u_3, u_i], u_3 \circ u_i = [-((1+\sqrt{-1})/2)u_2 + ((\sqrt{-1}-1)/2)u_3, u_i], i = 1,2,3$;

(4) $u_1 \circ u_i = [(\sqrt{-1}-(1/2))u_1 + (1-(\sqrt{-1}/2))u_2, u_i]$, $u_2 \circ u_i = [(1+(\sqrt{-1}/2))u_1 - (\sqrt{-1}+(1/2))u_2, u_i]$, $u_3 \circ u_i = 0$, $i = 1,2,3$;

(5) $u_1 \circ u_i = [ku_1, u_i], u_2 \circ u_i = [-(1/2)u_2 + (\sqrt{-1}/2)u_3, u_i]$, $u_3 \circ u_i = [-(\sqrt{-1}/2)u_2 - (1/2)u_3, u_i]$, $i = 1,2,3, k \in C^*$.

2. Equations from Post-Lie Algebras

From (4), we obtain $u_1 \circ u_2 = u_1 \circ [u_3, u_1] = [u_1 \circ u_3, u_1] + [u_3, u_1 \circ u_1] \in sl(2,C)$ and $u_4 \circ u_1 = u_4 \circ [u_2, u_3] = [u_4 \circ u_2, u_3] + [u_2, u_4 \circ u_3] \in sl(2,C)$. Similarly, we have $u_i \circ u_j \in sl(2,C)$ for any $i \in \{1,2,3,4\}$ and $j \in \{1,2,3\}$. Thus, we get the following.

Lemma 4. $(sl(2,C),[,],\circ)$ *is a post-Lie subalgebra of* $(gl(2,C),[,],\circ)$.

Proposition 5. *Let* $(gl(2,C),[,],\circ)$ *and* $(gl(2,C),[,],\star)$ *be post-Lie algebras on the Lie algebra* $gl(2,C)$*, they are isomorphic through automorphism* φ *of the Lie algebra* $gl(2,C)$*. Then,* $sl(2,C)$ *is a* φ*-subspace of* $gl(2,C)$*, and* $\varphi|_{sl(2,C)}$ *is an isomorphism from* $(sl(2,C),[,],\circ)$ *to* $(sl(2,C),[,],\star)$.

Proof. Suppose $Q = (q_{ij})_{4\times 4}$ is the matrix of φ with respect to the basis $\{u_1, u_2, u_3, u_4\}$; that is,

$$\varphi(u_i) = \sum_{j=1}^{3} q_{ij} u_j, \quad i = 1,2,3,4. \tag{8}$$

From $\varphi([u_i, u_4]) = \varphi(0) = 0, i = 1,2,3,4$, we obtain $\varphi(u_4) = q_{44}u_4$. Moreover, $\varphi(u_1) = \varphi([u_2,u_3]) = [\varphi(u_2), \varphi(u_3)] \in sl(2,C)$ means $q_{41} = 0$. Similarly, we have $q_{42} = q_{43} = 0$. Now, we see that Q has the form

$$Q = \begin{bmatrix} q_{11} & q_{21} & q_{31} & 0 \\ q_{12} & q_{22} & q_{32} & 0 \\ q_{13} & q_{23} & q_{33} & 0 \\ 0 & 0 & 0 & q_{44} \end{bmatrix}. \tag{9}$$

From this we can easily get the conclusion. □

Proposition 6. *Let* $(gl(2,C),[,],\circ)$ *be post-Lie algebra on the Lie algebra* $gl(2,C)$.

(1) *There exists a linear map* $f : sl(2,C) \to sl(2,C)$ *such that*

$$x \circ y = [f(x), y], \quad \forall x, y \in sl(2,C). \tag{10}$$

(2) *There exist* $b_1, b_2, b_3, b_4 \in C$ *such that* $u_i \circ u_4 = b_i u_4$, $i = 1,2,3,4$.

(3) *There exist* $l_{12}, l_{13}, l_{23} \in C$ *such that*

$$(u_4 \circ u_1, u_4 \circ u_2, u_4 \circ u_3) = (u_1, u_2, u_3) \begin{bmatrix} 0 & l_{12} & l_{13} \\ -l_{12} & 0 & l_{23} \\ -l_{13} & -l_{23} & 0 \end{bmatrix}. \tag{11}$$

Proof. (1) The conclusion is given by [8].

(2) For any $x, z \in gl(2,C)$, we have $[x, z \circ u_4] = z \circ [x, u_4] - [z \circ x, u_4] = 0$ from (4) and (7). Thus, $z \circ u_4$ is in the center of $gl(2,C)$, and so $z \circ u_4 = h(z)u_4$, where h is a linear map from $gl(2,C)$ to C. Let $b_i = h(u_i)$. The conclusion of (2) is proved.

(3) Let $z = u_4$, $\{x, y\} = \{u_1, u_2\}, \{u_2, u_3\}$, and $\{u_3, u_1\}$ in (4), one can get the conclusion (3) by a simple computation. □

Definition 7. Suppose A is the matrix of f (from Proposition 6) with respect to the basis $\{u_1, u_2, u_3\}$; that is, $f(u_i) = \sum_{j=1}^{3} a_{ij} u_j, i = 1,2,3$. Denote

$$L = \begin{bmatrix} 0 & l_{12} & l_{13} \\ -l_{12} & 0 & l_{23} \\ -l_{13} & -l_{23} & 0 \end{bmatrix}, \quad \beta = [b_1, b_2, b_3, b_4]. \tag{12}$$

The matrix set $\{A, L, \beta\}$ is unique for a given $\circ$. On the other hand, $\circ$ is defined uniquely by the matrix set $\{A, L, \beta\}$. Because of their uniqueness, the matrix set $\{A, L, \beta\}$ is called the matrix set of the post-Lie algebra $(gl(2,C),[,],\circ)$ and is also denoted by $\{A_\circ, L_\circ, \beta_\circ\}$.

Proposition 8. *Suppose that* $\{A_\circ, L_\circ, \beta_\circ\}$ *and* $\{A_\star, L_\star, \beta_\star\}$ *are the matrix sets of post-Lie algebras* $(gl(2,C),[,],\circ)$ *and* $(gl(2,C),[,],\star)$*, respectively.* φ *is an isomorphic map from* $(gl(2,C),[,],\circ)$ *to* $(gl(2,C),[,],\star)$*. Then,*

(1)

$$\varphi(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4) Q, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & q_{44} \end{bmatrix}, \tag{13}$$

where $Q_1 \in SO(3,C)$*, the group of all* 3×3 *complex orthogonal matrices whose determinants are* (1);

(2) $Q_1 A_\circ Q_1^{-1} = A_\star$;

(3) $\beta_\circ = \beta_\star Q$;

(4) $Q_1 L_\circ Q_1^{-1} = q_{44} L_\star$.

Proof. (1) The conclusion is given by Proposition 5 and [8].

(2) It is given by [8].

(3) Let $\beta_\circ = (b_1, b_2, b_3, b_4)$ and $\beta_\star = (b'_1, b'_2, b'_3, b'_4)$. Note that

$$\begin{aligned}
&\varphi(u_1 \circ u_4, u_2 \circ u_4, u_3 \circ u_4, u_4 \circ u_4) \\
&\quad = \varphi(b_1u_4, b_2u_4, b_3u_4, b_4u_4) \\
&\quad = q_{44}(b_1u_4, b_2u_4, b_3u_4, b_4u_4), \\
&(\varphi(u_1) \star \varphi(u_4), \varphi(u_2) \star \varphi(u_4), \\
&\quad \varphi(u_3) \star \varphi(u_4), \varphi(u_4) \star \varphi(u_4)) \\
&\quad = q_{44}(\varphi(u_1) \star u_4, \varphi(u_2) \star u_4, \varphi(u_3) \star u_4, \varphi(u_4) \star u_4) \\
&\quad = q_{44}(u_1 \star u_4, u_2 \star u_4, u_3 \star u_4, u_4 \star u_4)Q \\
&\quad = q_{44}(b'_1u_4, b'_2u_4, b'_3u_4, b'_4u_4)Q.
\end{aligned} \tag{14}$$

So, we have by $\varphi(u_i \circ u_j) = \varphi(u_i) \star \varphi(u_j)$ that $q_{44}(b'_1, b'_2, b'_3, b'_4)Q = q_{44}(b_1, b_2, b_3, b_4)$, that is, $\beta_\circ = \beta_\star Q$.

(4) Since

$$\begin{aligned}
&\varphi(u_4 \circ u_1, u_4 \circ u_2, u_4 \circ u_3) \\
&\quad = \varphi(u_1, u_2, u_3) L_\circ = (u_1, u_2, u_3) Q_1 L_\circ,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&(\varphi(u_4) \star \varphi(u_1), \varphi(u_4) \star \varphi(u_2), \varphi(u_4) \star \varphi(u_3)) \\
&\quad = q_{44}(u_4 \star \varphi(u_1), u_4 \star \varphi(u_2), u_4 \star \varphi(u_3)) \\
&\quad = q_{44}(u_4 \star u_1, u_4 \star u_2, u_4 \star u_3) Q_1 \\
&\quad = q_{44}(u_1, u_2, u_3) L_\star Q_1 \\
&\quad = (u_1, u_2, u_3) Q_1 L_\circ,
\end{aligned} \tag{16}$$

we deduce by $\varphi(u_i \circ u_j) = \varphi(u_i) \star \varphi(u_j)$ that $Q_1 L_\circ Q_1^{-1} = q_{44} L_\star$, the proof is completed. □

Proposition 9. *Suppose that $\{A, L, \beta\}$ is the matrix set of post-Lie algebra $(gl(2, C), [,], \circ)$, then we have the following equations:*

$$\begin{aligned}
&l_{12}b_2 + l_{13}b_3 + b_1b_4 = 0, \\
&l_{12}b_1 - l_{23}b_3 - b_2b_4 = 0, \\
&l_{13}b_1 + l_{23}b_2 - b_3b_4 = 0, \\
&i.e., (b_4I_3 + L)\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0,
\end{aligned} \tag{17}$$

$$\begin{aligned}
&(a_{11} + a_{22} + 1)b_3 - a_{13}b_1 - a_{23}b_2 = 0, \\
&(a_{22} + a_{33} + 1)b_1 - a_{21}b_2 - a_{31}b_3 = 0, \\
&(a_{11} + a_{33} + 1)b_2 - a_{12}b_1 - a_{32}b_3 = 0, \\
&i.e., ((trA + 1)I_3 - A)\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&(b_1 - a_{23})l_{12} + (a_{11} - a_{33})l_{13} + a_{12}l_{23} = 0, \\
&(a_{12} + a_{21})l_{12} + (a_{13} + a_{31})l_{13} - b_1l_{23} = 0, \\
&(a_{12} + a_{21})l_{12} - b_2l_{13} - (a_{23} + a_{32})l_{23} = 0, \\
&(a_{11} - a_{22})l_{12} - a_{23}l_{13} + (b_2 - a_{31})l_{23} = 0, \\
&(a_{22} - a_{11})l_{12} + (a_{32} + b_1)l_{13} + a_{13}l_{23} = 0, \\
&(b_2 + a_{13})l_{12} + a_{21}l_{13} + (a_{22} - a_{33})l_{23} = 0, \\
&-a_{31}l_{12} + (b_3 - a_{12})l_{13} + (a_{33} - a_{22})l_{23} = 0, \\
&-a_{32}l_{12} + (a_{11} - a_{33})l_{13} + (b_3 + a_{21})l_{23} = 0, \\
&b_3l_{12} + (a_{13} + a_{31})l_{13} + (a_{23} + a_{32})l_{23} = 0.
\end{aligned} \tag{19}$$

Proof. We consider (3) and (10). Let $y = x = u_4$ and $z = u_1, u_2$, and u_3, respectively, we get (17); Let $x = u_4$, $\{z, y\} = \{u_2, u_1\}, \{u_3, u_2\}$ and $\{u_3, u_1\}$, respectively, we get (18); let $z = u_4$ and $\{y, x\} = \{u_1, u_2\}, \{u_2, u_3\}, \{u_1, u_1\}, \{u_2, u_1\}, \{u_3, u_3\}$, and $\{u_3, u_1\}$, respectively, we get (19). □

3. Classification of Post-Lie Algebra $(gl(2,C),[,], \circ)$

Lemma 10 (see [9]). *Suppose that A is a complex skewsymmetric 3×3 matrix, there exists $T \in O(3, C)$ such that*

$$\begin{aligned}
&TAT^{-1} \\
&= \begin{bmatrix} 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2} \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{20}$$

Based on Definition 7, we get the main result in this paper as follows.

Theorem 11. *The following is a complete set of matrix sets of representatives for the isomorphic classes of post-Lie algebra $(gl(2, C), [,], \circ)$ on the Lie algebra $(gl(2, C), [,])$*

(1) $A = 0$, $\quad L = \begin{bmatrix} 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2} \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 \end{bmatrix}$,

$$\beta = [0, 0, 0, b_4], \quad b_4 = 0 \text{ or } 1;$$

(2) $A = 0, \quad L = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$\beta = [0, 0, 0, b_4], \quad a \in C, \; b_4 = 0 \text{ or } 1;$

(3) $A = -I_3, \quad L = \begin{bmatrix} 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2} \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 \end{bmatrix},$

$\beta = [0, 0, 0, b_4], \quad b_4 = 0 \text{ or } 1;$

(4) $A = -I_3, \quad L = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$\beta = [0, 0, 0, b_4], \quad a \in C, \; b_4 = 0 \text{ or } 1;$

(5) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{-1-\sqrt{-1}}{2} & \frac{-1-\sqrt{-1}}{2} \\ 0 & \frac{-1+\sqrt{-1}}{2} & \frac{-1+\sqrt{-1}}{2} \end{bmatrix}, \quad L = 0,$

$\beta = [b_1, \sqrt{-1}b_3, b_3, 0], \quad b_1, b_3 \in C;$

(6) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{-1-\sqrt{-1}}{2} & \frac{-1-\sqrt{-1}}{2} \\ 0 & \frac{-1+\sqrt{-1}}{2} & \frac{-1+\sqrt{-1}}{2} \end{bmatrix},$

$L = 0, \quad \beta = [0, 0, 0, 1];$

(7) $A = \begin{bmatrix} k & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix}, \quad L = 0,$

$\beta = [b_1, 0, 0, 0], \quad k \neq -1, \; b_1 \in C^*;$

(8) $A = \begin{bmatrix} k & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix},$

$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}, \quad \beta = [0, 0, 0, b_4],$

$k \neq -1, \; a \in C, \; b_4 = 0 \text{ or } 1;$

(9) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix},$

$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}, \quad \beta = [0, 0, 0, 1], \quad a \in C;$

(10) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix}, \quad L = 0,$

$\beta = [b_1, \sqrt{-1}b_3, b_3, 0], \quad b_1, b_3 \in C;$

(11) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix},$

$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \beta = 0;$

(12) $A = \begin{bmatrix} \sqrt{-1}-\frac{1}{2} & 1+\frac{\sqrt{-1}}{2} & 0 \\ 1-\frac{\sqrt{-1}}{2} & -\sqrt{-1}-\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$L = 0, \quad \beta = [0, 0, 0, 1];$

(13) $A = \begin{bmatrix} \sqrt{-1}-\frac{1}{2} & 1+\frac{\sqrt{-1}}{2} & 0 \\ 1-\frac{\sqrt{-1}}{2} & -\sqrt{-1}-\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$

$L = 0, \quad \beta = [0, 0, b_3, 0], \quad b_3 \in C;$

(14) $A = \begin{bmatrix} \sqrt{-1}-\frac{1}{2} & 1+\frac{\sqrt{-1}}{2} & 0 \\ 1-\frac{\sqrt{-1}}{2} & -\sqrt{-1}-\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = 0,$

$$\beta = [\sqrt{-1}b_2, b_2, b_3, 0], \quad b_2 \in C^*, \; b_3 \in C. \tag{21}$$

Proof. Case 1. $A = 0$. By (18), we obtain $b_1 = b_2 = b_3 = 0$; that is, $\beta = [0, 0, 0, b_4]$. By Lemma 10 we have

$$L = \begin{bmatrix} 0 & \frac{1+\sqrt{-1}}{2} & 0 \\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2} \\ 0 & \frac{1-\sqrt{-1}}{2} & 0 \end{bmatrix} \tag{22}$$

$$\text{or} \quad \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Case 1.1. Consider

$$L=\begin{bmatrix}0 & \frac{1+\sqrt{-1}}{2} & 0\\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2}\\ 0 & \frac{1-\sqrt{-1}}{2} & 0\end{bmatrix}. \tag{23}$$

If $b_4 \neq 0$, let

$$Q_1=\begin{bmatrix}\frac{b_4^2+1}{2b_4} & 0 & \frac{\sqrt{-1}\left(b_4^2-1\right)}{2b_4}\\ 0 & 1 & 0\\ \frac{\sqrt{-1}\left(1-b_4^2\right)}{2b_4} & 0 & \frac{b_4^2+1}{2b_4}\end{bmatrix}, \tag{24}$$

$$Q=\begin{bmatrix}Q_1 & 0\\ 0 & b_4\end{bmatrix}; \tag{25}$$

then

$$Q_1AQ_1^{-1}=A,\qquad Q_1LQ_1^{-1}=b_4L,\qquad \beta=[0,0,0,1]\,Q. \tag{26}$$

In view of Proposition 8, we can suppose that

$$A=0,\qquad L=\begin{bmatrix}0 & \frac{1+\sqrt{-1}}{2} & 0\\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2}\\ 0 & \frac{1-\sqrt{-1}}{2} & 0\end{bmatrix},$$
$$\beta=[0,0,0,b_4],\qquad b_4=0\quad\text{or}\quad 1. \tag{27}$$

Case 1.2. Consider $L=\begin{bmatrix}0 & a & 0\\ -a & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. If $b_4 \neq 0$, let

$$Q_1=I_3,\qquad Q=\begin{bmatrix}I_3 & 0\\ 0 & b_4\end{bmatrix}; \tag{28}$$

then

$$Q_1AQ_1^{-1}=A,$$
$$Q_1\begin{bmatrix}0 & a & 0\\ -a & 0 & 0\\ 0 & 0 & 0\end{bmatrix}Q_1^{-1}=b_4\begin{bmatrix}0 & ab_4^{-1} & 0\\ -ab_4^{-1} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$
$$\beta=[0,0,0,1]\,Q. \tag{29}$$

Thus, we can suppose that $A=0, L=\begin{bmatrix}0 & a & 0\\ -a & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, and $\beta=[0,0,0,b_4]$, $a\in C$, $b_4=0$ or 1.

Case 2. Consider $A=-I_3$. By Lemma 10 we have

$$L=\begin{bmatrix}0 & \frac{1+\sqrt{-1}}{2} & 0\\ \frac{-1-\sqrt{-1}}{2} & 0 & \frac{-1+\sqrt{-1}}{2}\\ 0 & \frac{1-\sqrt{-1}}{2} & 0\end{bmatrix}$$
$$\text{or}\quad\begin{bmatrix}0 & a & 0\\ -a & 0 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{30}$$

By (18), we obtain $\beta=[0,0,0,b_4]$. In a similar way with the proof of Case 1, we can suppose that $\beta=[0,0,0,b_4]$, $b_4=0$ or 1.

Case 3. Consider

$$A=\begin{bmatrix}-1 & 0 & 0\\ 0 & \frac{-1-\sqrt{-1}}{2} & \frac{-1-\sqrt{-1}}{2}\\ 0 & \frac{-1+\sqrt{-1}}{2} & \frac{-1+\sqrt{-1}}{2}\end{bmatrix}. \tag{31}$$

By (18), we obtain $b_2=\sqrt{-1}b_3$. Then, by (19), we get

$$l_{23}=0,$$
$$l_{12}=l_{13}, \tag{32}$$
$$b_2l_{12}=b_3l_{12}=\left(b_1-i\right)l_{12}=0.$$

Therefore, by (17), we can have

$$l_{12}=l_{13}=0,\qquad b_1b_4=b_2b_4=b_3b_4=0. \tag{33}$$

Hence, $L=0$. If $b_4\neq 0$, then $b_1=b_2=b_3=0$. In a similar way with the proof of Case 1, we can suppose that $\beta=[0,0,0,1]$. Else, if $b_4=0$, then $\beta=[b_1,\sqrt{-1}b_3,b_3,0]$, $b_1,b_3\in C$.

Case 4. Consider

$$A=\begin{bmatrix}k & 0 & 0\\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2}\\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2}\end{bmatrix}. \tag{34}$$

By (18), we obtain

$$b_2=b_3=0,\quad k\neq -1,$$
$$b_2=\sqrt{-1}b_3,\quad k=-1. \tag{35}$$

Case 4.1. When $k\neq -1$, $b_2=b_3=0$. By (17), we obtain $b_1b_4=b_1l_{12}=b_1l_{13}=0$. Then, by (19), we get $l_{12}=l_{13}=b_1l_{23}=0$.

If $b_1\neq 0$, then $b_4=l_{23}=0$.

Therefore,

$$L=0,\qquad \beta=\left[b_1,0,0,0\right],\quad b_1\in C^*\ \text{or}$$
$$L=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & a\\ 0 & -a & 0\end{bmatrix},\qquad \beta=\left[0,0,0,b_4\right],\quad a,b_4\in C. \tag{36}$$

In a similar way with the proof of Case 1, we can suppose that $b_4 = 0$ or 1.

Case 4.2. When $k = -1$, $b_2 = \sqrt{-1}b_3$. By (19), we obtain

$$\begin{gathered} b_i l_{12} = b_i l_{13} = b_i l_{23} = 0 \quad i = 2, 3, \\ b_1 l_{23} = 0, \\ l_{13} = \sqrt{-1} l_{12}, \\ \left(b_1 - \sqrt{-1}\right) l_{13} = 0. \end{gathered} \tag{37}$$

This, together with (17), implies that $l_{12} = l_{13} = 0, b_i b_4 = 0$, and $i = 1, 2, 3$.

If $b_4 \neq 0$, then $b_i = 0$, and $i = 1, 2, 3$. Hence, in a similar way with the proof of Case 1, we can suppose that

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}, \qquad \beta = [0, 0, 0, 1]. \tag{38}$$

If $b_4 = 0$ and $l_{23} = 0$, then

$$L = 0, \qquad \beta = \left[a, \sqrt{-1}c, c, 0\right], \quad a, c \in C. \tag{39}$$

If $b_4 = 0$ and $l_{23} \neq 0$, then $b_i = 0, i = 1, 2, 3$. Hence,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}, \qquad \beta = 0, \quad a = l_{23} \in C^*. \tag{40}$$

Let

$$Q_1 = I_3, \qquad Q = \begin{bmatrix} I_3 & 0 \\ 0 & a \end{bmatrix}; \tag{41}$$

then

$$\begin{gathered} Q_1 A Q_1^{-1} = A, \\ Q_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} Q_1^{-1} = a \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \\ 0 = \beta = \beta Q. \end{gathered} \tag{42}$$

Thus, we can suppose that $L = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right]$, $\beta = 0$.

Case 5. Consider

$$A = \begin{bmatrix} \sqrt{-1} - \dfrac{1}{2} & 1 + \dfrac{\sqrt{-1}}{2} & 0 \\ 1 - \dfrac{\sqrt{-1}}{2} & -\sqrt{-1} - \dfrac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{43}$$

By (18), we obtain $b_1 = \sqrt{-1}b_2$. Then, by (19), we get

$$\sqrt{-1} b_2 l_{12} + \left(\sqrt{-1} - \frac{1}{2}\right) l_{13} + \left(1 - \frac{\sqrt{-1}}{2}\right) l_{23} = 0, \tag{44}$$

$$2 l_{12} - \sqrt{-1} b_2 l_{23} = 0, \tag{45}$$

$$2 l_{12} - b_2 l_{13} = 0, \tag{46}$$

$$\left(b_3 - 1 + \frac{\sqrt{-1}}{2}\right) l_{13} + \left(\frac{1}{2} + \sqrt{-1}\right) l_{23} = 0, \tag{47}$$

$$\left(-\frac{1}{2} + \sqrt{-1}\right) l_{13} + \left(b_3 + 1 + \frac{\sqrt{-1}}{2}\right) l_{23} = 0. \tag{48}$$

From (45) and (46), we have $b_2(l_{13} - \sqrt{-1} l_{23}) = 0$.

Case 5.1. When $b_2 = 0$, $b_1 = 0$. From (45), we have $l_{12} = 0$. Then by (44), (47), and (48) we obtain $l_{13} = l_{23} = 0$. Therefore, by (17), we get $b_3 b_4 = 0$. So, $L = 0, \beta = [0, 0, b_3, b_4], b_3 b_4 = 0$. In a similar way with the proof of Case 1, we can suppose that

$$L = 0, \qquad \beta = [0, 0, 0, 1] \text{ or } \beta = \left[0, 0, b_3, 0\right], \quad b_3 \in C. \tag{49}$$

Case 5.2. When $b_2 \neq 0$, $l_{13} = \sqrt{-1} l_{23}$. From (44), we have $b_2 l_{12} = \sqrt{-1} l_{23} = l_{13}$. Then, by (45) and (46), we get $2\sqrt{-1} l_{23} = 2 b_2 l_{12} = \sqrt{-1} b_2^2 l_{23}$, $2 l_{13} = 2 b_2 l_{12} = b_2^2 l_{13}$ and $2 l_{12} = b_2 l_{13} = b_2^2 l_{12}$; that is, $(b_2^2 - 2) l_{23} = (b_2^2 - 2) l_{13} = (b_2^2 - 2) l_{12} = 0$.

If $b_2^2 \neq 2$, then $l_{23} = l_{13} = l_{12} = 0$.

Else, if $b_2^2 = 2$, from (44) we obtain that $\sqrt{-1} b_2^2 l_{12} + (\sqrt{-1} - (1/2)) b_2 l_{13} + (1 - (\sqrt{-1}/2)) b_2 l_{23} = 0$; that is, $2\sqrt{-1} l_{12} + (\sqrt{-1} - (1/2)) 2 l_{12} - \sqrt{-1}(1 - (\sqrt{-1}/2)) \sqrt{-1} b_2 l_{23} = 2\sqrt{-1} l_{12} + (\sqrt{-1} - (1/2)) 2 l_{12} - \sqrt{-1}(1 - (\sqrt{-1}/2)) 2 l_{12} = 0$; thus, $l_{12} = 0$. Therefore, by (45) and (46), we conclude that $l_{13} = l_{23} = 0$.

Therefore, we can get $b_4 = 0$ by (17). So,

$$L = 0, \qquad \beta = \left[\sqrt{-1} b_2, b_2, b_3, 0\right], \quad b_2 \in C^*, \ b_3 \in C. \tag{50}$$

The sufficiency of Theorem 11 is obvious from the proof of the necessity. □

Acknowledgments

This work is supported in part by NSFC (11171294), The Natural Science Foundation of Heilongjiang Province of China (A201013), and the fund of Heilongjiang Education Committee (12531483).

References

[1] B. Vallette, "Homology of generalized partition posets," *Journal of Pure and Applied Algebra*, vol. 208, no. 2, pp. 699–725, 2007.

[2] D. Burde, "Left-symmetric algebras, or pre-Lie algebras in geometry and physics," *Central European Journal of Mathematics*, vol. 4, no. 3, pp. 323–357, 2006.

[3] D. Burde, K. Dekimpe, and S. Deschamps, "LR-algebras," in *New Developments in Lie Theory and Geometry*, vol. 491 of *Contemporary Mathematics*, pp. 125–140, American Mathematical Society, Providence, RI, USA, 2009.

[4] D. Burde and K. Dekimpe, "Post-Lie algebra structures and generalized derivations of semisimple Lie algebras," *Moscow Mathematical Journal*, vol. 13, no. 1, pp. 1–18, 2013.

[5] D. Burde, K. Dekimpe, and K. Vercammen, "Affine actions on Lie groups and post-Lie algebra structures," *Linear Algebra and its Applications*, vol. 437, no. 5, pp. 1250–1263, 2012.

[6] J.-L. Loday, "Generalized bialgebras and triples of operads," *Astérisque*, vol. 320, 2008.

[7] H. Z. Munthe-Kaas and A. Lundervold, "On post-lie algebras, lie—butcher series and moving frames," *Foundations of Computational Mathematics*, vol. 13, no. 4, pp. 583–613, 2013.

[8] Y. Pan, Q. Liu, C. Bai, and L. Guo, "PostLie algebra structures on the Lie algebra $SL(2, C)$," *Electronic Journal of Linear Algebra*, vol. 23, pp. 180–197, 2012.

[9] F. R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea Publishing, New York, NY, USA, 1959.