

Research Article

Borel Directions and Uniqueness of Meromorphic Functions

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We investigate the relationship between Borel directions and uniqueness of meromorphic functions and obtain some results of meromorphic functions sharing four distinct values IM and one set in an angular domain containing a Borel line. Our result is an improvement of a recent theorem given by Long and Wu (2012).

1. Introduction and Main Results

We use \mathbb{C} to denote the open complex plane, $\widehat{\mathbb{C}} (= \mathbb{C} \cup \{\infty\})$ to denote the extended complex plane, and $\Omega (\subset \mathbb{C})$ to denote an angular domain. The fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used in [1, 2]. In addition, the order of meromorphic function f is defined by

$$\rho = \rho(f) = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}. \quad (1)$$

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega := \{z : \alpha \leq \arg z \leq \beta\} \subset \mathbb{C}$. Define

$$\begin{aligned} E(S, \Omega, f) &= \bigcup_{a \in S} \{z \in \Omega \mid f_a(z) = 0, \text{ counting multiplicities}\}, \\ \overline{E}(S, \Omega, f) &= \bigcup_{a \in S} \{z \in \Omega \mid f_a(z) = 0, \text{ ignoring multiplicities}\}, \end{aligned} \quad (2)$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = 1/f(z)$.

Let f and g be two nonconstant meromorphic functions in \mathbb{C} . If $E(S, \Omega, f) = E(S, \Omega, g)$, then we say f and g share

the set S CM (counting multiplicities) in Ω . If $\overline{E}(S, \Omega, f) = \overline{E}(S, \Omega, g)$, then we say f and g share the set S IM (ignoring multiplicities) in Ω . In particular, when $S = \{a\}$, $a \in \widehat{\mathbb{C}}$, we say f and g share the value a CM in Ω if $E(S, \Omega, f) = E(S, \Omega, g)$, and we say f and g share the value a IM in Ω if $\overline{E}(S, \Omega, f) = \overline{E}(S, \Omega, g)$. When $\Omega = \mathbb{C}$, we give the simple notation as before, $E(S, f)$, $\overline{E}(S, f)$, and so on (see [3]).

Nevanlinna (see [4]) proved the following well-known theorems.

Theorem 1 (see [4]). *If f and g are two nonconstant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in \mathbb{C} , then $f(z) \equiv g(z)$.*

Theorem 2 (see [4]). *If f and g are two distinct nonconstant meromorphic functions that share four distinct values a_1, a_2, a_3, a_4 CM in \mathbb{C} , then f is a Möbius transformation of g , two of the shared values, say a_1 and a_2 , are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

After their very work, many investigations studied the uniqueness of meromorphic functions with shared values in the whole complex plane (see [5]). Zheng studied the uniqueness problem under the condition that five values and four values are shared in some angular domain in \mathbb{C} around 2003 (see [6, 7]). It is an interesting topic to investigate the uniqueness with shared values in the angular domain;

see [3, 6–12]. The basic notations and definitions of meromorphic functions in an angular domain will be introduced as follows (see [1, 6, 7]).

Let f be a meromorphic function on the angular domain $\Omega(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$ and $0 < \beta - \alpha \leq 2\pi$. Define

$$\begin{aligned}
 A_{\alpha, \beta}(r, f) &= \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \\
 &\quad \times \left\{ \log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})| \right\} \frac{dt}{t}, \\
 B_{\alpha, \beta}(r, f) &= \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta, \\
 C_{\alpha, \beta}(r, f) &= 2 \sum_{1 < |b_\mu| < r} \left(\frac{1}{|b_\mu|^\omega} - \frac{|b_\mu|^\omega}{r^{2\omega}} \right) \sin \omega(\theta_\mu - \alpha), \\
 S_{\alpha, \beta}(r, f) &= A_{\alpha, \beta}(r, f) + B_{\alpha, \beta}(r, f) + C_{\alpha, \beta}(r, f), \tag{3}
 \end{aligned}$$

where $\omega = \pi/(\beta - \alpha)$ and $b_\mu = |b_\mu|e^{i\theta_\mu}$ ($\mu = 1, 2, \dots$) are the poles of f on $\Omega(\alpha, \beta)$ counted according to their multiplicities. $S_{\alpha, \beta}(r, f)$ is called the Nevanlinna’s angular characteristic, and $C_{\alpha, \beta}(r, f)$ is called the angular counting function of the poles of f on $\Omega(\alpha, \beta)$, and $\bar{C}_{\alpha, \beta}(r, f)$ is the reduced function of $C_{\alpha, \beta}(r, f)$. Similarly, when $a \neq \infty$, we will use the notations $A_{\alpha, \beta}(r, 1/(f - a))$, $B_{\alpha, \beta}(r, 1/(f - a))$, $C_{\alpha, \beta}(r, 1/(f - a))$, $S_{\alpha, \beta}(r, 1/(f - a))$, and so on.

It is well known that angular distribution is an interesting topic of value distribution of meromorphic function in complex analysis, and Borel directions played an important role in the topic of angular distribution (see [13–24]). Valiron [16] proved that every meromorphic function of finite order $\rho > 0$ has at least one Borel direction of order ρ . Chuang [25] investigated the existence of Borel direction of meromorphic function of infinite order. To state the Chuang’s results, we will introduce the definition as follows.

Definition 3 (see [25]). Let f be a meromorphic function of infinite order, $\rho(r)$ is a real function satisfying the following conditions:

- (i) $\rho(r)$ is continuous, nondecreasing for $r \geq r_0$ and $\rho(r) \rightarrow \infty$ as $r \rightarrow \infty$;
- (ii)

$$\lim_{r \rightarrow \infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}, \tag{4}$$

where $U(r) = r^{\rho(r)}$ ($r \geq r_0$);

- (iii)

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log U(r)} = 1. \tag{5}$$

Then $\rho(r)$ is called infinite order of meromorphic function f . This definition is given by Chuang [25].

Let $\rho(r)$ be infinite order of meromorphic function f ; we will denote by $M(\rho(r))$ the set of meromorphic function g satisfying $\limsup_{r \rightarrow \infty} (\log T(r, g)/\rho(r) \log r) \leq 1$; that is,

$$M(\rho(r)) := \left\{ g : \limsup_{r \rightarrow \infty} \frac{\log T(r, g)}{\rho(r) \log r} \leq 1 \right\}. \tag{6}$$

Let $\alpha < \beta$, $\beta - \alpha < 2\pi$, $r > 0$, and $\Omega(\alpha, \beta, r) := \{z : \alpha \leq \arg z \leq \beta, 0 < |z| \leq r\}$. The definition of Borel direction of meromorphic functions f of infinite order $\rho(r)$ is given as follows.

Definition 4 (see [25]). Let f be meromorphic functions of infinite order $\rho(r)$; if for any ε ($0 < \varepsilon < \pi$), the equality

$$\limsup_{r \rightarrow \infty} \frac{\log n(\Omega(\theta - \varepsilon, \theta + \varepsilon, r), f = a)}{\rho(r) \log r} = 1 \tag{7}$$

holds for any complex number $a \in \widehat{\mathbb{C}}$, at most except two exceptions, where $n(\Omega(\theta - \varepsilon, \theta + \varepsilon, r), f = a)$ is the counting function of zeros of the function $f - a$ in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, counting multiplicities, then the ray $\arg z = \theta$ is called a Borel direction of $\rho(r)$ order of meromorphic function f .

Remark 5. Chuang [25] proved that every meromorphic function f of infinite order $\rho(r)$ has at least one Borel direction of infinite order $\rho(r)$.

In 2012, Long and Wu [26] investigated the problem concerning Borel direction and shared value of meromorphic functions and obtained the following theorems.

Theorem 6 (see [26, Theorem 1.1]). *Let f be a meromorphic function of infinite order $\rho(r)$, and let $g \in M(\rho(r))$, $\arg z = \theta$ ($0 \leq \theta < 2\pi$) be one Borel direction of $\rho(r)$ order of meromorphic function f ; let $a_i \in \widehat{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) be five distinct complex numbers. If f and g share a_i ($i = 1, 2, 3, 4, 5$) IM in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any ε ($0 < \varepsilon < \pi$), then $f \equiv g$.*

Theorem 7 (see [26, Theorem 1.2]). *Let f be a meromorphic function of infinite order $\rho(r)$, and let $g \in M(\rho(r))$, $\arg z = \theta$ ($0 \leq \theta < 2\pi$) be one Borel direction of $\rho(r)$ order of meromorphic function f ; let $a_i \in \widehat{\mathbb{C}}$ ($i = 1, 2, 3, 4$) be four distinct complex numbers. If f and g share a_i ($i = 1, 2, 3, 4$) CM in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any ε ($0 < \varepsilon < \pi$), then f is a Möbius transformation of g .*

Thus, a question arises naturally: could the nature of sharing the values α_i be further relaxed in Theorems 6 and 7?

In this paper, we will deal with the above question and obtain the following result which is an improvement of Theorem 6.

Theorem 8. *Let f be a meromorphic function of infinite order $\rho(r)$, and let $g \in M(\rho(r))$, $\arg z = \theta$ ($0 \leq \theta < 2\pi$) be one Borel direction of $\rho(r)$ order of meromorphic function f ; we assume that f and g share four distinct values a_j ($j = 1, 2, 3, 4$) IM in*

$\Omega(\theta - \varepsilon, \theta + \varepsilon)$, and $\bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subset \bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), g)$, for any ε ($0 < \varepsilon < \pi$), where $S = \{b_1, \dots, b_m\}$, $m \geq 1$ and $b_1, \dots, b_m \in \widehat{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}$. Then f and g share all values CM; thus, it follows that either $f \equiv g$ or f is a Möbius transformation of g . Furthermore, if the number of the values in S is odd, then $f \equiv g$.

Remark 9. The special case $m = 1$ of this theorem immediately yields Theorem 6. In fact, when $m = 1$, set $S = \{a_5\}$. If f, g share a_5 IM, which implies $\bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subset \bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), g)$, then by Theorem 8, we can get $f \equiv g$.

2. Some Lemmas

To prove our result, we require the following lemmas.

Lemma 10 (see [27]). *Let f be a nonconstant meromorphic function on $\Omega(\alpha, \beta)$. Then for arbitrary complex number a , we have*

$$S_{\alpha, \beta} \left(r, \frac{1}{f - a} \right) = S_{\alpha, \beta}(r, f) + \varepsilon(r, a), \tag{8}$$

where $\varepsilon(r, a) = O(1)$ as $r \rightarrow \infty$.

Lemma 11 (see [7, 28]). *Suppose that f is a nonconstant meromorphic function in one angular domain $\Omega(\alpha, \beta)$ with $0 < \beta - \alpha \leq 2\pi$; then, for arbitrary q distinct $a_j \in \widehat{\mathbb{C}}$ ($1 \leq j \leq q$), we have*

$$(q - 2) S_{\alpha, \beta}(r, f) \leq \sum_{j=1}^q \bar{C}_{\alpha, \beta} \left(r, \frac{1}{f - a_j} \right) + R_{\alpha, \beta}(r, f), \tag{9}$$

where the term $\bar{C}_{\alpha, \beta}(r, 1/(f - a_j))$ will be replaced by $\bar{C}_{\alpha, \beta}(r, f)$ when some $a_j = \infty$ and

$$\begin{aligned} R_{\alpha, \beta}(r, f) &= A_{\alpha, \beta} \left(r, \frac{f'}{f} \right) + B_{\alpha, \beta} \left(r, \frac{f'}{f} \right) \\ &+ \sum_{j=1}^q \left\{ A_{\alpha, \beta} \left(r, \frac{f'}{f - a_j} \right) + B_{\alpha, \beta} \left(r, \frac{f'}{f - a_j} \right) \right\} \\ &+ O(1). \end{aligned} \tag{10}$$

Lemma 12 (see [27, Page 138]). *Let f be a nonconstant meromorphic function in the whole complex plane \mathbb{C} . Given one angular domain on $\Omega(\alpha, \beta)$. Then, for any $1 \leq r < R$, we have*

$$\begin{aligned} A_{\alpha, \beta} \left(r, \frac{f'}{f} \right) &\leq K \left\{ \left(\frac{R}{r} \right)^\omega \int_1^R \frac{\log^+ T(r, f)}{t^{1+\omega}} dt \right. \\ &\left. + \log^+ \frac{r}{R-r} + \log \frac{R}{r} + 1 \right\}, \end{aligned} \tag{11}$$

$$B_{\alpha, \beta} \left(r, \frac{f'}{f} \right) \leq \frac{4\omega}{r^\omega} m \left(r, \frac{f'}{f} \right), \tag{12}$$

where $\omega = \pi/(\beta - \alpha)$ and K is a positive constant not depending on r and R .

Remark 13. Nevanlinna conjectured that

$$A_{\alpha, \beta} \left(r, \frac{f'}{f} \right) + B_{\alpha, \beta} \left(r, \frac{f'}{f} \right) = o \left(S_{\alpha, \beta}(r, f) \right) \tag{13}$$

when r tends to $+\infty$ outside an exceptional set of finite linear measure, and he proved that $A_{\alpha, \beta}(r, f'/f) + B_{\alpha, \beta}(r, f'/f) = O(1)$ when the function f is meromorphic in \mathbb{C} and has finite order. In 1975, Goldberg [28] constructed a counter example to show that (13) is not valid.

Remark 14. From Lemmas 11 and 12, we can get the following conclusion:

$$R_{\alpha, \beta}(r, f) = \begin{cases} O(1), & f \text{ is of finite order,} \\ O(\log U(r)), r \notin E, & f \text{ is of infinite order,} \end{cases} \tag{14}$$

where $R_{\alpha, \beta}(r, f)$ is stated as in (10), $U(r) = r^{\rho(r)}$, $\rho(r)$ is infinite order of meromorphic function f , and E is a set of finite linear measure.

Remark 15. From the definition of $A_{\alpha, \beta}(r, f)$, $B_{\alpha, \beta}(r, f)$, $C_{\alpha, \beta}(r, f)$, $S_{\alpha, \beta}(r, f)$, and Lemmas 10–12, we can see that the properties of $C_{\alpha, \beta}(r, f)$, $(A+B)_{\alpha, \beta}(r, f)$, and $S_{\alpha, \beta}(r, f)$ are the same as for the more familiar quantities $N(r, f)$, $m(r, f)$, and $T(r, f)$, respectively.

Lemma 16 (see [29]). *Let f be meromorphic function of infinite order $\rho(r)$. Then the ray $\arg z = \theta$ is one Borel direction of $\rho(r)$ order of meromorphic function f if and only if f satisfies the equality*

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta - \varepsilon, \theta + \varepsilon}(r, f)}{\rho(r) \log r} = 1, \tag{15}$$

for any ε ($0 < \varepsilon < \pi/2$).

By using the same argument as in [8, Lemma 1] and [5], we can get the lemma below easily.

Lemma 17. *Suppose that f is a nonconstant meromorphic function with infinite order $\rho(r)$, and the ray $\arg z = \theta$ is one Borel direction of $\rho(r)$ order of meromorphic function f . Let $P(f) = a_0 f^p + a_1 f^{p-1} + \dots + a_p$ ($a_0 \neq 0$) be a polynomial of f with degree p , where the coefficients a_j ($j = 0, 1, \dots, p$) are constants, and let b_j ($j = 1, 2, \dots, q$) be q ($q \geq p + 1$) distinct finite complex numbers. Then for any ε ($0 < \varepsilon < \pi/2$),*

$$\begin{aligned} (A + B)_{\theta - \varepsilon, \theta + \varepsilon} \left(r, \frac{P(f) \cdot f'}{(f - b_1)(f - b_2) \dots (f - b_q)} \right) \\ = R_{\theta - \varepsilon, \theta + \varepsilon}(r, f). \end{aligned} \tag{16}$$

Lemma 18. *Suppose that f is a nonconstant meromorphic function with infinite order $\rho(r)$, and the ray $\arg z = \theta$ is one Borel direction of $\rho(r)$ order of meromorphic function f . We assume that for any for any ε ($0 < \varepsilon < \pi/2$), f and g share*

four distinct values a_j ($j = 1, 2, 3, 4$) IM in angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ and $f \neq g$. Let

$$\begin{aligned} \Psi &= f' g' (f - g)^2 \times ((f - a_1)(f - a_2)(f - a_3)(f - a_4) \\ &\quad \times (g - a_1)(g - a_2) \\ &\quad \times (g - a_3)(g - a_4))^{-1}. \end{aligned} \tag{17}$$

Then, $S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi) = R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, g)$.

Proof. Since the ray $\arg z = \theta$ is one Borel direction of meromorphic function f of $\rho(r)$ order, thus, for any ε ($0 < \varepsilon < \pi/2$), we can assume that $z_0 \in \Omega(\theta - \varepsilon, \theta + \varepsilon)$ and $f(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity p and $g(z_0) = a_1$ (or a_2, a_3, a_4) with multiplicity q . From (17), we can get

$$\Psi(z) = O\left((z - z_0)^{2\min(p,q)-2}\right). \tag{18}$$

Hence, Ψ is analytic in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. By Lemmas 12 and 17, we have

$$\begin{aligned} S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi) &= (A + B)_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi) \\ &\leq (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{f'}{(f - a_2)(f - a_3)(f - a_4)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{f'}{(f - a_1)(f - a_2)(f - a_3)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{f'}{(f - a_1)(f - a_2)(f - a_3)(f - a_4)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{f' P_1(f)}{(f - a_1)(f - a_2)(f - a_3)(f - a_4)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{g'}{(g - a_2)(g - a_3)(g - a_4)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{g'}{(g - a_1)(g - a_2)(g - a_3)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \\ &\quad \times \left(r, \frac{g'}{(g - a_1)(g - a_2)(g - a_3)(g - a_4)}\right) \\ &\quad + (A + B)_{\theta-\varepsilon, \theta+\varepsilon} \end{aligned}$$

$$\begin{aligned} &\times \left(r, \frac{g' P_2(g)}{(g - a_1)(g - a_2)(g - a_3)(g - a_4)}\right) \\ &\quad + O(1) \\ &= R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, g), \end{aligned} \tag{19}$$

where $P_1(f)$ is a polynomial of degree no more than 2 in f and $P_2(g)$ is a polynomial of degree no more than 2 in g . Thus, we complete the proof of this lemma. \square

3. Proof of Theorem 8

Proof. Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z = \theta$ ($0 \leq \theta < 2\pi$) is one Borel direction of $\rho(r)$ order of the meromorphic function f , by Lemma 16, we can get for any ε ($0 < \varepsilon < \pi$)

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, f)}{\rho(r) \log r} = 1. \tag{20}$$

And since $g \in M(\rho(r))$, we have

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, g)}{\rho(r) \log r} \leq 1. \tag{21}$$

Set $R(r) = O(\rho(r) \log r)$ as $r \rightarrow \infty$, $r \notin E$ where E is a set of finite linear measure; then $R(r) = o(S_{\theta-\varepsilon, \theta+\varepsilon}(r, f))$ as $r \rightarrow \infty$, $r \notin E$.

Since f and g share four distinct values a_j ($j = 1, 2, 3, 4$) IM in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, suppose that $f \neq g$ and none of the a_j ($j = 1, 2, 3, 4$) is ∞ . By Lemma 11, we have

$$\begin{aligned} 2S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) &\leq \sum_{j=1}^4 \overline{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f - a_j}\right) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) \\ &\leq C_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f - g}\right) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) \\ &\leq S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) \\ &\quad + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f), \end{aligned} \tag{22}$$

and by interchanging f and g , we can get that

$$\begin{aligned} S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) &= S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f), \\ S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) &= S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, g). \end{aligned} \tag{23}$$

Thus, it follows $R_{\theta-\varepsilon, \theta+\varepsilon}(r, g) = R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) = R(r)$ and

$$\sum_{j=1}^4 \overline{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f - a_j}\right) = 2S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R(r). \tag{24}$$

Let Ψ be the function expressed in Lemma 18. Then, $\Psi \neq 0$. By Lemma 11 and (24), for any $b_j \in S$ ($j = 1, 2, \dots, m$), we have

$$\begin{aligned} 3S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) &\leq \sum_{j=1}^4 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-a_j} \right) \\ &\quad + \bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-b_j} \right) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f) \\ &\leq 2S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + \bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-b_j} \right) \\ &\quad + R(r); \end{aligned} \tag{25}$$

that is,

$$\bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-b_j} \right) = S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R(r). \tag{26}$$

Similarly, we have

$$\bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{g-b_j} \right) = S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + R(r). \tag{27}$$

From (26) and (27), it follows

$$\begin{aligned} (A+B)_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-b_j} \right) &= R(r), \\ (A+B)_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{g-b_j} \right) &= R(r), \end{aligned} \tag{28}$$

for any $b_j \in S$ ($j = 1, 2, \dots, m$).

Set

$$\begin{aligned} \Psi_1 &:= \frac{(g-b_1) \cdots (g-b_m)}{(f-b_1) \cdots (f-b_m)} \cdot \left(\frac{g'(f-g)}{(g-a_1) \cdots (g-a_4)} \right)^m, \\ \Psi_2 &:= \frac{(f-b_1) \cdots (f-b_m)}{(g-b_1) \cdots (g-b_m)} \cdot \left(\frac{f'(f-g)}{(f-a_1) \cdots (f-a_4)} \right)^m. \end{aligned} \tag{29}$$

By Lemma 17 and (28), we can get that

$$\begin{aligned} (A+B)_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-b_j} \cdot \frac{g'(f-g)(g-b_j)}{(g-a_1) \cdots (g-a_4)} \right) &= R(r), \\ (A+B)_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{g-b_j} \cdot \frac{f'(f-g)(f-b_j)}{(f-a_1) \cdots (f-a_4)} \right) &= R(r). \end{aligned} \tag{30}$$

Thus, it follows that $(A+B)_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi_j) = R(r)$, $j = 1, 2$. From (28), we see that ‘‘almost all’’ of poles and b_j -points of f and g in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ are simple. Since f, g share the four distinct values a_j , $j = 1, 2, 3, 4$ in the

angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ and $\bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subset \bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), g)$, we can easily get that $C_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi_1) = R(r)$. Therefore, we have

$$S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi_1) = R(r). \tag{31}$$

Since $\Psi_1 \Psi_2 \equiv \Psi^m$, we can have

$$S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi_2) = R(r). \tag{32}$$

Let $\Xi_{\Omega}^{pq}(a_j)$ be the set of those a_j -points of f and g in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ in which the multiplicities of f and g at these points are p and q , respectively. For any $z_0 \in \Xi_{\Omega}^{pq}(a_1)$, by simple computation, we have

$$\begin{aligned} \Psi_1(z_0) &= \left(q \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} \right)^m, \\ \Psi_2(z_0) &= \left(p \cdot \frac{f'(z_0) - g'(z_0)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)} \right)^m. \end{aligned} \tag{33}$$

Hence,

$$\frac{1}{q^m} \Psi_1(z_0) - \frac{1}{p^m} \Psi_2(z_0) = 0. \tag{34}$$

Similarly, we can see that (34) holds for any $z_0 \in \Xi_{\Omega}^{pq}(a_j)$, $j = 2, 3, 4$.

Now, two cases will be considered below.

Case 1. Suppose that $\Psi^{pq} := (1/q^m)\Psi_1 - (1/p^m)\Psi_2 \neq 0$, for all positive integers p, q .

First, we use $C_{\theta-\varepsilon, \theta+\varepsilon}^{pq}(r, 1/(f-a_j))$ to denote the counting function of f in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with respect to the set $\Xi_{\Omega}^{pq}(a_j)$, and we also use $\bar{C}_{\theta-\varepsilon, \theta+\varepsilon}^{pq}(r, 1/(f-a_j))$ to denote the corresponding reduced counting function. Thus, we have

$$\begin{aligned} C_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-a_j} \right) &= \sum_{p, q=1}^{\infty} C_{\theta-\varepsilon, \theta+\varepsilon}^{pq} \left(r, \frac{1}{f-a_j} \right), \\ \bar{C}_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{f-a_j} \right) &= \sum_{p, q=1}^{\infty} \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}^{pq} \left(r, \frac{1}{f-a_j} \right). \end{aligned} \tag{35}$$

From the above two equations and (28)–(32), we can see that $S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi^{pq}) = R(r)$. And by (34) each zero of $f - a_j$ is a zero of Ψ^{pq} , so with the help of Lemma 10 and $\Psi^{pq} \neq 0$, we can get

$$\begin{aligned} \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}^{pq} \left(r, \frac{1}{f-a_j} \right) &\leq \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}^{pq} \left(r, \frac{1}{\Psi^{pq}} \right) \leq S_{\theta-\varepsilon, \theta+\varepsilon} \left(r, \frac{1}{\Psi^{pq}} \right) \\ &\leq S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Psi^{pq}) + O(1) = R(r), \end{aligned} \tag{36}$$

for some p, q . Since $S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) = S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + R(r)$, it follows

$$\begin{aligned} \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{f-a_j}\right) &= \sum_{\max(p, q) \geq 5} \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}^{pq}\left(r, \frac{1}{f-a_j}\right) \\ &\quad + R(r, f) \\ &\leq \frac{1}{5} \sum_{\max(p, q) \geq 5} C_{\theta-\varepsilon, \theta+\varepsilon}^{pq}\left(r, \frac{1}{f-a_j}\right) \\ &\quad + \frac{1}{5} \sum_{\max(p, q) \geq 5} C_{\theta-\varepsilon, \theta+\varepsilon}^{pq}\left(r, \frac{1}{g-a_j}\right) \\ &\quad + R(r) \\ &\leq \frac{1}{5} \left(C_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{f-a_j}\right) \right. \\ &\quad \left. + C_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) \right) + R(r) \\ &\leq \frac{2}{5} S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R(r), \\ &\quad j = 1, 2, 3, 4. \end{aligned} \tag{37}$$

From the above inequality and (24), we can get

$$2S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) \leq \frac{8}{5} S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + R(r). \tag{38}$$

Since f is of infinite order $\rho(r)$ and $R(r) = O(\rho(r) \log r)$, we can get a contradiction to (38).

Case 2. Suppose that $\Psi^{pq} := (1/q^m)\Psi_1 - (1/p^m)\Psi_2 \equiv 0$, for some positive integers p, q . From the definitions of Ψ_1 and Ψ_2 , we have

$$\begin{aligned} \left(\frac{p}{q}\right)^m \cdot \frac{(g-b_1)^2 \cdots (g-b_m)^2}{(f-b_1)^2 \cdots (f-b_m)^2} \\ \equiv \left(\frac{f'(g-a_1) \cdots (g-a_4)}{g'(f-a_1) \cdots (f-a_4)}\right)^m. \end{aligned} \tag{39}$$

Next, we take the following two subcases into consideration.

Subcase 2.1. Suppose that $p \neq q$. Without loss of generality, we may assume that $p < q$. For some two positive integers p_1 and q_1 , if $z_1 \in \Xi_{\Omega}^{p_1 q_1}(a_j)$ for some $j \in \{1, 2, 3, 4\}$, then (39) implies that $p/q = p_1/q_1$. Hence, $q_1 > p_1 \geq 1$, and $q_1 \geq 2$ which

means that any a_j -points ($j = 1, 2, 3, 4$) of g in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ are multiple. By Lemma 11 and (24), we can get

$$\begin{aligned} 2S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) &= \sum_{j=1}^4 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) + R(r) \\ &\leq \frac{1}{2} \sum_{j=1}^4 C_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) + R(r) \\ &\leq 2S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + R(r); \end{aligned} \tag{40}$$

it follows that

$$S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) = C_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) + R(r), \tag{41}$$

$$C_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) = 2\bar{C}_{\theta-\varepsilon, \theta+\varepsilon}\left(r, \frac{1}{g-a_j}\right) + R(r). \tag{42}$$

From (41) and (42), we can see that ‘‘almost all’’ of a_j -points of g have multiplicity 2, and ‘‘almost all’’ of a_j -points of f are simple in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. Without loss of generality, we may assume that f and g attain the values a_3 and a_4 in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. Set

$$\begin{aligned} \Phi_1 &:= \frac{2f'(f-a_4)}{(f-a_1)(f-a_2)(f-a_3)} \\ &\quad - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)(g-a_3)}, \\ \Phi_2 &:= \frac{2f'(f-a_3)}{(f-a_1)(f-a_2)(f-a_4)} \\ &\quad - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)(g-a_4)}. \end{aligned} \tag{43}$$

Since Φ_i ($i = 1, 2$) is analytic at the poles of f and of g and also at those common a_j -points of f and g which have multiplicity 1 with respect to f and multiplicity 2 with respect to g , by Lemma 17, we have $S_{\theta-\varepsilon, \theta+\varepsilon}(r, \Phi_i) = R(r)$, $i = 1, 2$. If $\Phi_i \not\equiv 0$, then $C_{\theta-\varepsilon, \theta+\varepsilon}(r, 1/(g-a_4)) \leq C_{\theta-\varepsilon, \theta+\varepsilon}(r, 1/\phi_1) = R(r)$, which contradicts to (41). Then, $\Phi_1 \equiv 0$. Similarly, we have $\Phi_2 \equiv 0$. Therefore, from the definitions of Φ_1 and Φ_2 , we have

$$\left(\frac{f-a_4}{f-a_3}\right)^2 \equiv \left(\frac{g-a_4}{g-a_3}\right)^2. \tag{44}$$

Since $f \neq g$, from (44), we have

$$\frac{f-a_4}{f-a_3} \equiv -\frac{g-a_4}{g-a_3}, \tag{45}$$

which implies that f and g share a_3, a_4 CM in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. Since f and g assume the value a_3 , there exist positive integers

p_1, q_1 such that $\Xi_{\Omega}^{p_1 q_1}(a_3) \neq \emptyset$. From the considerations above we get $q_1 > p_1$, contradicting the fact that f and g share a_3 CM.

Subcase 2.2. Suppose that $p = q$.
In this subcase, (39) becomes

$$\frac{(g - b_1)^2 \cdots (g - b_m)^2}{(f - b_1)^2 \cdots (f - b_m)^2} \equiv \left(\frac{f'(g - a_1) \cdots (g - a_4)}{g'(f - a_1) \cdots (f - a_4)} \right)^m. \tag{46}$$

which implies that f and g share the four values a_j ($j = 1, 2, 3, 4$) CM in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. Then by applying Theorem 7, g is a Möbius transformation of f . Furthermore, two of the four values, say a_1, a_2 , are Picard exceptional values of f and g in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. Set

$$\begin{aligned} \Lambda_1 &:= \frac{f'(f - a_4)}{(f - a_1)(f - a_2)(f - a_3)} \\ &\quad - \frac{g'(g - a_4)}{(g - a_1)(g - a_2)(g - a_3)}, \\ \Lambda_2 &:= \frac{f'(f - a_3)}{(f - a_1)(f - a_2)(f - a_4)} \\ &\quad - \frac{g'(g - a_3)}{(g - a_1)(g - a_2)(g - a_4)}. \end{aligned} \tag{47}$$

Similar to the discussion in Subcase 2.1 for Λ_1, Λ_2 , we can get

$$\frac{f - a_3}{f - a_4} \equiv -\frac{g - a_3}{g - a_4}. \tag{48}$$

We define the Möbius transformations $T, H,$ and J by

$$T(w) := \frac{w - a_3}{w - a_4}, \quad H(w) := -w, \quad J := T^{-1} \circ H \circ T. \tag{49}$$

Then, we have

$$T \circ f = -T \circ g; \quad \text{hence, } g = J \circ f. \tag{50}$$

Obviously, a_3 and a_4 are the fixed points of J . Therefore, there exist no fixed points of J in the set S . Let some $b \in S$ be given. Then, in view of $b \neq a_1, a_2$, there exists a $z_0 \in \mathbb{C}$ such that $b = f(z_0)$, and from $\bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subseteq \bar{E}(S, \Omega(\theta - \varepsilon, \theta + \varepsilon), g)$ we obtain

$$J(b) = J(f(z_0)) = g(z_0) \in S. \tag{51}$$

So S is invariant under J . Furthermore, we have $J \circ J = I$ where I denotes the identical transformation. Hence, S must contain an even number of values. Thus, the proof of Theorem 8 is completed. \square

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