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Research Article

Deterministic and Stochastic Bifurcations of the Catalytic CO Oxidation on Ir(111) Surfaces with Multiple Delays

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The main purpose is to investigate both deterministic and stochastic bifurcations of the catalytic CO oxidation. Firstly, super- and subcritical bifurcations are determined by the signs of the Poincaré-Lyapunov coefficients of the center manifold scalar bifurcation equations. Secondly, we explore the stochastic bifurcation of the catalytic CO oxidation on Ir(111) surfaces with multiple delays according to the qualitative changes in the invariant measure, the Lyapunov exponent, and the stationary probability density of system response. Some new criteria ensuring stability and stochastic bifurcation are obtained.

1. Introduction

For most of the practical cases, the dynamical systems will be disturbed by some stochastic perturbation. There are real benefits to be gained in using stochastic rather than deterministic models. It could be intuitively thought that the external noise would smear out the finer details of the nonlinear system, such as bifurcations between well-defined states. However, it could also happen that noise forces the system far away from its deterministic attractors to explore regions of phase space that are otherwise not visited or interact with the nonlinearities giving rise to new phenomena [1–10].

Now, a great care should be taken when reducing the internal/external sources of noise and uncertainties such as thermal noise, observation errors, and model misspecification. The CO oxidation on platinum group crystals offers an experimental setup, where the nonlinear mechanisms have been captured with high accuracy, and sources of internal noise are controlled. The CO oxidation on crystals shows, in the absence of noise, a wide variety of phenomena, for example, oscillations, bistability, bifurcation, excitability, spatiotemporal patterns, and turbulence. These reactions have been receiving an increasing attention as a laboratory analogy of catalysis used at the industrial level to manufacture chemical products and process harmful species. Understanding the effects of external noise on this oxidation reaction can be considered as an effort to bridge the quality gap. Besides material and pressure gap, which have been discussed in the literature before, the quality gap separates small experiments in perfectly controlled settings and the industrial processes that are always under the influence of uncertainties.

In this paper, we investigate the deterministic and stochastic bifurcations of the catalytic CO oxidation on Ir(111) surfaces with multiple delays. Mechanisms and parameters are known with a high degree of accuracy, and the relevant parameters can be controlled. These ideal properties make it possible to go back and forth between theory and experiments. Our paper focuses on homogeneous states and builds on the work of Hayase et al. [11–18] who studied experimentally and numerically the effect of noise on CO oxidation on Ir(111). They verified that the mathematical model was correct and observed probability distributions that are consistent with the presence of multiplicative noise.

The structure of the paper is as follows. In Section 2, we analyzed the single and double Hopf bifurcation of deterministic system. In Section 3, we analyzed the stochastic stability and bifurcation of stochastic system.

2. Deterministic System

In this section, we will investigate the delayed CO oxidation on Ir(111) surfaces as follows:

$$\dot{u}(t) = \alpha\kappa(1-u-v) - \beta u - \gamma u(t-\tau_1)v(t-\tau_2),$$
$$\dot{v}(t) = \delta(1-\kappa)(1-u-v)^3 - \gamma uv, \quad (1)$$

where $\gamma = \gamma_c + \tilde{\gamma}$ is the bifurcation parameter. It is obvious that there is a unique equilibrium point $E(0,1)$ in system (1).

Let $x_1 = u - 0$, $x_2 = v - 1$, and $X = [x_1, x_2]^T$, then by substituting the corresponding variables in (1), we have

$$\dot{x}_1(t) = -\alpha\kappa(x_1+x_2) - \beta x_1 - \gamma x_1(t-\tau_1)[x_2(t-\tau_2)+1],$$
$$\dot{x}_2(t) = -\delta(1-\kappa)(x_1+x_2)^3 - \gamma x_1(x_2+1). \quad (2)$$

Linearized System. As the first step, we analyze the stability of the trivial solution of the linearized system of (1), which is given by

$$\dot{x}_1(t) = -(\alpha\kappa+\beta)x_1 - \alpha\kappa x_2 - \gamma x_1(t-\tau_1),$$
$$\dot{x}_2(t) = -\gamma x_1, \quad (3)$$

in $\mathscr{C} := \mathscr{C}([-\tau,0],\mathfrak{R}^n)$, and its adjoint form

$$\dot{x}_1(\hat{t}) = (\alpha\kappa+\beta)x_1(\hat{t}) + \alpha\kappa x_2(\hat{t}) + \gamma x_1(\hat{t}+\tau_1),$$
$$\dot{x}_2(\hat{t}) = \gamma x_1(\hat{t}), \quad (4)$$

in $\widehat{\mathscr{C}} := \widehat{\mathscr{C}}([-\tau,0],\mathfrak{R}^n)$ with respect to the associated bilinear relation

$$\left(\psi_j(\vartheta), \phi_k(\theta)\right) = \left(\psi_j(0), \phi_k(0)\right) - \gamma\int_{-\tau_1}^{0}\psi_j(\zeta+\tau_1)\phi_k(\zeta)\,d\zeta \quad (5)$$
$$j,k = 1,2,\ldots,$$

has eigenvalues λ satisfying the transcendental characteristic equation

$$\Delta(\lambda,N) := \lambda^2 + \lambda\left[(\alpha\kappa+\beta) + \gamma e^{-\lambda\tau_1}\right] - \gamma\alpha\kappa = 0. \quad (6)$$

By the Hopf bifurcation, we let $\lambda_{1,2} = \pm i\omega(\gamma)$, with, $\omega(\gamma) \neq 0$ and $\mathfrak{R}e\{d\Delta(\lambda,\gamma)/d\gamma\} \neq 0$, be the eigenvalue of (6). All the remaining eigenvalues of $\Delta(\lambda,\gamma) = 0$ have negative real parts. Then, by substituting $\lambda_1 = i\omega(\gamma)$ into $\Delta(\lambda,\gamma) = 0$ and setting the real and imaginary parts of the resulting algebraic equations to zero, we have

$$-\omega^2 - \gamma\alpha\kappa + \omega\gamma\sin\omega\tau_1 + \omega\left[(\alpha\kappa+\beta) + \omega\gamma\cos\omega\tau_1\right]i = 0. \quad (7)$$

Separating the real and imaginary parts of (7) gives the following equations:

$$-\omega^2 - \gamma\alpha\kappa + \omega\gamma\sin\omega\tau_1 = 0,$$
$$\omega\left[(\alpha\kappa+\beta) + \omega\gamma\cos\omega\tau_1\right] = 0, \quad (8)$$

or equivalently

$$\sin\omega\tau_1 = \frac{\omega^2+\gamma\alpha\kappa}{\gamma\omega} > 0,$$
$$\cos\omega\tau_1 = \frac{\alpha\kappa+\beta}{\gamma\omega} > 0. \quad (9)$$

Possible values of $\omega\tau_1$ are of the form $\omega\tau_1 \in (2k\pi, 2k\pi+\pi/2)$, $k = 0,1,2,\ldots$. Thus, we obtain the bifurcation parameter γ as

$$\gamma = \Big(-2\alpha\kappa\omega^2 - \sqrt{4\alpha^2\kappa^2\omega^4 - 4(\alpha^2\kappa^2-\omega^2)(\beta^2+2\alpha\beta\kappa+\alpha^2\kappa^2+\omega^4)}\Big) \times \left(2\left(\alpha^2\kappa^2-\omega^2\right)\right)^{-1} \quad (\alpha\kappa < \omega). \quad (10)$$

Now, we compute Hopf's transversality condition by differentiating implicitly (6) with respect to γ, obtaining

$$\left\{\frac{d\lambda}{d\gamma}\right\}_{\lambda=i\omega} = -\left\{\left(\frac{d\Delta(\lambda,\gamma)}{d\gamma}\right)\left(\frac{d\Delta(\lambda,\gamma)}{d\lambda}\right)^{-1}\right\}_{\lambda=i\omega}$$
$$= -\left\{\frac{\lambda e^{-\lambda\tau_1} - \alpha\kappa}{2\lambda + (\alpha\kappa+\beta) + \gamma e^{-\lambda\tau_1} - \gamma\tau_1 e^{-\lambda\tau_1}}\right\}_{\lambda=i\omega}$$
$$= G(\gamma) + M(\gamma)i,$$
$$G(R) = -\frac{ac+bd}{a^2+b^2}, \qquad M(\gamma) = -\frac{ad-bc}{a^2+b^2},$$
$$a = \omega\sin\omega\tau_1 - \alpha\kappa, \qquad b = \omega\cos\omega\tau_1,$$
$$c = \alpha\kappa + \gamma\cos\omega\tau_1 - \gamma\tau_1\omega\sin\omega\tau_1,$$
$$d = 2\omega - \gamma\sin\omega\tau_1 - \gamma\tau_1\omega\cos\omega\tau_1. \quad (11)$$

Lemma 1. *The transversality conditions* $\mathfrak{R}e(d\lambda/d\gamma)_{\lambda=\omega i} = -ac - bd \neq 0$ *hold. If the choice of values for the model parameters is such that the inequality*

$$(\omega\sin\omega\tau_1 - \alpha\kappa)(\alpha\kappa + \gamma\cos\omega\tau_1 - \gamma\tau_1\omega\sin\omega\tau_1) + \omega\cos\omega\tau_1(2\omega - \gamma\sin\omega\tau_1 - \gamma\tau_1\omega\cos\omega\tau_1) < 0 \quad (12)$$

holds.

This means that the pair of eigenvalues $\lambda_{1,2} = v(\gamma) + i\omega(\gamma)$ of $\Delta(\lambda,\gamma) = 0$ with $v(\gamma_c) = 0$, $\omega(\gamma_c) \neq 0$ will cross the imaginary axis from left to right in the complex plane with a nonzero speed as the bifurcation parameter γ varies near its critical value γ_c. Also, it ensures that the steady-state stability of the system is lost at $\gamma = \gamma_c$, and the instability in the sense of supercritical and subcritical bifurcations will persist for larger values of γ. The exact nature of such bifurcations satisfying Hopf's conditions ((6)–(12)) will depend on the nonlinearity in the delay system.

Rewriting (10) as a quadratic in ω^2 and letting $\omega^2 = \widetilde{\omega}$, we have

$$\widetilde{\omega}^2 + \alpha_1\widetilde{\omega} + \alpha_2 = 0, \quad \alpha_1 = 2\alpha\gamma\kappa - \gamma^2, \quad \alpha_2 = \alpha^2\gamma^2\kappa^2 + (\alpha\kappa + \beta)^2, \tag{13}$$

whose real and positive solutions are of the form

$$\widetilde{\omega}_1 = \frac{1}{2}\left(-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}\right), \quad \widetilde{\omega}_2 = \frac{1}{2}\left(-\alpha_1 - \sqrt{\alpha_1^2 - 4\alpha_2}\right), \tag{14}$$

if the following holds

$$\begin{gathered} \alpha_1 < 0 \mid 2\alpha\kappa < \gamma, \\ \alpha_1^2 - 4\alpha_2 > 0 \mid \ \gamma^4 - 2\alpha\kappa\gamma^3 - 4(\alpha\kappa + \beta)^2 > 0, \\ \alpha_1 < 0 \mid 2\alpha\kappa < \gamma, \\ \alpha_1^2 - 4\alpha_2 = 0 \mid \gamma^4 = 2\alpha\kappa\gamma^3 + 4(\alpha\kappa + \beta)^2. \end{gathered} \tag{15}$$

In addition, one has to show that the substitution of $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ into (10) will ensure Hopf's transversality conditions, that is, $\mathfrak{Re}(d\lambda/d\gamma)_{\lambda=i\omega}$. Therefore, with $\widetilde{\omega}_1 = \omega^2$ and $\mathfrak{Re}(d\lambda/d\gamma)_{\lambda=i\omega}$, we obtain

$$\begin{gathered} 2\omega^2 + \alpha_1 = 0 \mid \omega = \frac{\sqrt{\gamma^2 - 2\alpha\gamma\kappa}}{2}, \\ \alpha_1^2 - 4\alpha_2 = 0 \mid \gamma^4 = 2\alpha\kappa\gamma^3 + 4(\alpha\kappa + \beta)^2. \end{gathered} \tag{16}$$

By setting $\omega_{\min} = \omega$ and $\gamma_{\min} = \gamma$, we have

$$\begin{gathered} \omega_{\min} = \frac{\sqrt{\gamma_{\min}^2 - 2\alpha\gamma_{\min}\kappa}}{2}, \\ \gamma_{\min}^4 = 2\alpha\kappa\gamma_{\min}^3 + 4(\alpha\kappa + \beta)^2. \end{gathered} \tag{17}$$

These are the minimum values separating the regimes of stable and unstable solutions of the linearized delay equation at the Hopf bifurcation point $\gamma = \gamma_c$, $\lambda = \pm\omega i(\gamma_{\min}) = \pm\sqrt{\gamma_{\min}^2 - 2\alpha\gamma_{\min}\kappa}/2$. Below and equal to these minimum values, all the infinite numbers of eigenvalues of the transcendental characteristic equation (6) will lie in the left-hand side of the complex plane.

Next, deriving an expression for τ_1 begins by further rewiring (8) as follows:

$$\tan\omega\tau_1 = \frac{\omega^2 + \gamma\alpha\kappa}{\alpha\kappa + \beta}, \tag{18}$$

thus, we obtain

$$\tau_{1,k} = \frac{1}{\omega}\left(\arctan\left(\frac{\omega^2 + \gamma\alpha\kappa}{\alpha\kappa + \beta}\right) + 2k\pi\right) \quad k = 0, 1, 2, \ldots, n, \ldots, \tag{19}$$

and by using the minimum value for $\omega_{\min} = \sqrt{\gamma_{\min}^2 - 2\alpha\gamma_{\min}\kappa}/2$, we get

$$\tau_{1,k,\min} = \frac{1}{\omega_{\min}}\left(\arctan\left(\frac{\omega_{\min}^2 + \gamma\alpha\kappa}{\alpha\kappa + \beta}\right) + 2k\pi\right) \quad k = 0, 1, 2, \ldots, n, \ldots. \tag{20}$$

With these minimum values for the gain $\omega_{\min}$, time delay $\tau_{1,k,\min}$, and the response frequency $\omega_{\min} = \sqrt{\gamma_{\min}^2 - 2\alpha\gamma_{\min}\kappa}/2$, one can describe all the critical settings of the model parameters in the delay equation (1) at which the unstable-free oscillation persists. Should unstable oscillations appear at some desired frequency such that $\omega_{\min} = \sqrt{\gamma_{\min}^2 - 2\alpha\gamma_{\min}\kappa}/2$, fine-tuning the model parameters in the vicinity of their critical settings can control such oscillations. To find out whether the bifurcations are supercritical and/or subcritical, we carry out a local center manifold analysis of the nonlinear delay equation (1) as presented in the preceding section.

2.1. The Single Hopf Bifurcation. To obtain the explicit analytical expressions for the stability condition of the Hopf bifurcation solutions (limit cycles), system (1) should be reduced to its center manifold [19–21]. While studying the critical infinite dimensional problem on a two-dimensional center manifold, we begin by defining the elements of the initial continuous function $\phi(\theta) \in \mathscr{C}$ and its projections $\phi^P(\theta)$, $\phi^Q(\theta)$ onto the center and stable subspaces $P, Q \in \mathscr{C}$. For the eigenvalues $\lambda = \pm i\omega$, we have $\phi^P(\theta) = \Phi(\theta)b$ with $\phi_k^P(\theta) = e^{\lambda_k\theta}$, $-\tau \le \theta \le 0$ of the adjoint-generalized eigenspace $P \in \mathscr{C}$ and $\psi^{\widehat{P}}(s) = \Psi\widehat{b}$, $\psi_j^{\widehat{P}}(s)e^{-\lambda_{js}}$, $0 \le s \le \tau$, $j, k = 1, 2$ of the adjoint-generalized eigenspace $\widehat{P} \in \widehat{\mathscr{C}}$. The values of the basis $\Phi(\theta) \in \mathscr{C}$ for P are thus given by

$$\Phi(\theta) = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \omega\sin\omega\theta & \omega\cos\omega\theta \end{bmatrix}\begin{bmatrix} b \\ b \end{bmatrix}, \tag{21}$$

and, similarly, the basis $\Psi(s) \in \widehat{\mathscr{C}}$ for $\widehat{P}$ is of the form

$$\Psi(\theta) = [\psi_1, \psi_2] = \begin{bmatrix} \cos\omega\theta & \omega\sin\omega\theta \\ -\sin\omega\theta & \omega\cos\omega\theta \end{bmatrix}, \quad 0 \le s \le R. \tag{22}$$

These bases form the inner product matrix

$$\begin{aligned} (\Psi(s), \Phi(\theta)) &= \begin{bmatrix} \psi_1(s) \\ \psi_2(s) \end{bmatrix}[\phi_1 \quad \phi_2] \\ &= \begin{bmatrix} (\psi_1(s)\phi_1(\theta)) & (\psi_1(s)\phi_2(\theta)) \\ (\psi_2(s)\phi_1(\theta)) & (\psi_2(s)\phi_2(\theta)) \end{bmatrix}, \end{aligned} \tag{23}$$

in which

$$\begin{aligned}
(\psi_1(s)\phi_1(\theta)) &= (\cos\omega s\cos\omega\theta + \omega^2\sin\omega s\sin\omega\theta),\\
(\psi_1(s)\phi_2(\theta)) &= -(\cos\omega s\sin\omega\theta - \omega^2\sin\omega s\cos\omega\theta),\\
(\psi_2(s)\phi_1(\theta)) &= -(\sin\omega s\cos\omega\theta - \omega^2\cos\omega s\sin\omega\theta),\\
(\psi_2(s)\phi_2(\theta)) &= (\sin\omega s\sin\omega\theta + \omega^2\cos\omega s\cos\omega\theta).
\end{aligned} \tag{24}$$

The substitution of the elements of $(\Psi,\Phi)_{nsg} \neq 0$

$$(\Psi,\Phi)_{nsg} = \begin{bmatrix}(\psi_1,\phi_1) & (\psi_1,\phi_2)\\ (\psi_2,\phi_1) & (\psi_2,\phi_2)\end{bmatrix}, \tag{25}$$

where

$$\begin{aligned}
(\psi_1,\phi_1) &= 1 - \frac{1}{2}\gamma\left\{\left(\frac{1}{\omega}-\omega\right)\sin\omega\tau_1 + \tau_1(1+\omega^2)\cos\omega\tau_1\right\},\\
(\psi_1,\phi_2) &= \gamma(1+\omega^2)\sin\omega\tau_1,\\
(\psi_2(s)\phi_1(\theta)) &= -\gamma(1+\omega^2)\sin\omega\tau_1,\\
(\psi_2(s)\phi_2(\theta)) &= \omega^2 + \gamma\left\{\left(\frac{1}{\omega}-\omega\right)\sin\omega\tau_1 - \tau_1(1+\omega^2)\cos\omega\tau_1\right\}.
\end{aligned} \tag{26}$$

Then, the basis $\Psi(s)$ for $\widehat{P} \in \widehat{\mathscr{C}}$ of the adjoint equation (2) is normalized to $\widetilde{\Psi} = [\widetilde{\psi}_1,\widetilde{\psi}_2]^\top \in \widehat{\mathscr{C}}$, computing $\widetilde{\Psi} = (\Psi,\Phi)^{-1}_{nsg}\Psi(s)$ to yield

$$\begin{aligned}
\widetilde{\Psi}(s) &= \begin{bmatrix}\widetilde{\psi}_{11}(s) & \widetilde{\psi}_{12}(s)\\ \widetilde{\psi}_{21}(s) & \widetilde{\psi}_{22}(s)\end{bmatrix}, \quad 0 \le s \le R,\\
\widetilde{\Psi}_{11}(s) &= \frac{1}{\det((\Psi,\Phi)^{-1}_{nsg})}\{(\psi_2,\phi_2)\cos\omega s + (\psi_1,\phi_2)\sin\omega s\},\\
\widetilde{\Psi}_{12}(s) &= \frac{\omega}{\det((\Psi,\Phi)^{-1}_{nsg})}\{(\psi_2,\phi_2)\sin\omega s - (\psi_1,\phi_2)\cos\omega s\},\\
\widetilde{\Psi}_{21}(s) &= \frac{\omega}{\det((\Psi,\Phi)^{-1}_{nsg})}\{(\psi_2,\phi_1)\cos\omega s + (\psi_1,\phi_1)\cos\omega s\},\\
\widetilde{\Psi}_{22}(s) &= \frac{\omega}{\det((\Psi,\Phi)^{-1}_{nsg})}\{(\psi_2,\phi_1)\sin\omega s - (\psi_1,\phi_1)\cos\omega s\},\\
\det((\Psi,\Phi)^{-1}_{nsg}) &= \{(\psi_1,\phi_1)(\psi_2,\phi_2) - (\psi_1,\phi_2)(\psi_2,\phi_1)\},
\end{aligned} \tag{27}$$

where the substitution of the new elements $(\psi_j(s),\phi_k(\theta))$, $j,k = 1,2$ into (3) will lead to the identity matrix

$$\begin{aligned}
(\Psi,\Phi)_{\text{id}} &= \frac{1}{\det((\Psi,\Phi)^{-1}_{nsg})}\begin{bmatrix}\det((\Psi,\Phi)^{-1}_{nsg}) & 0\\ 0 & \det((\Psi,\Phi)^{-1}_{nsg})\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.
\end{aligned} \tag{28}$$

Consequently, the constant matrices $B \in \mathscr{C}$ and $\widehat{B} \in \widehat{\mathscr{C}}$ are equivalent, and the elements of B at the Hopf bifurcation are $B = [[0,\omega]^\top, [-\omega,0]^\top]$.

With the algebraic simplifications

$$e^{B\theta} = \begin{bmatrix}\cos\omega\theta & -\sin\omega\theta\\ \sin\omega\theta & \cos\omega\theta\end{bmatrix}, \tag{29}$$

one can easily see that

$$\Phi(\theta) = \Phi(0)e^{B\theta} = \begin{bmatrix}\cos\omega\theta & -\sin\omega\theta\\ \omega\sin\omega\theta & \omega\cos\omega\theta\end{bmatrix}, \quad -R \le \theta \le 0, \tag{30}$$

and it follows that $J(t,\mu)\Phi(\theta) = \Phi(0)e^{B(\theta+t)}$ is indeed the solution operator of the linearized delay equations in $\mathscr{C}$. Similarly, we have $\overline{\Psi}(s) = \overline{\Psi}(0)e^{-\widehat{B}s}$, $0 \le s \le R$, and $\widehat{J}(\widehat{t},R)\overline{\Psi}(s) = \Phi(0)e^{B(\theta+t)}$ is the solution operator for the adjoint delay equations in $\widehat{\mathscr{C}}$.

By the change of variable $x_t^P(\theta) = \Phi(\theta)z(t) + x_t^Q(\theta)$ with $z(t) \in \mathfrak{R}^2$, $z(t) = (\widetilde{\Psi}(s),\phi^P(\theta))$, we obtain the following as a first-order approximation in ε, for $\theta = -\tau_1, -\tau_2$:

$$\begin{aligned}
\begin{bmatrix}x_1(t-\tau_1)\\ x_2(t-\tau_1)\end{bmatrix} &= \begin{bmatrix}z_1(t)\cos\omega\tau + z_2(t)\sin\omega\tau_1\\ -z_1(t)\omega\sin\omega\tau_1 + z_2(t)\omega\cos\omega\tau_1\end{bmatrix},\\
\begin{bmatrix}x_1(t-\tau_2)\\ x_2(t-\tau_2)\end{bmatrix} &= \begin{bmatrix}z_1(t)\cos\omega\tau + z_2(t)\sin\omega\tau_2\\ -z_1(t)\omega\sin\omega\tau_2 + z_2(t)\omega\cos\omega\tau_2\end{bmatrix},
\end{aligned} \tag{31}$$

and for $\theta = 0$,

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}z_1(t)\\ \omega z_2(t)\end{bmatrix}. \tag{32}$$

Therefore, we obtain (1) of the center manifold as follows:

$$\begin{aligned}
\dot{z}_1(t) &= -\omega z_2 + \Delta f^{(1)}(z_1,z_2,\gamma),\\
\dot{z}_2(t) &= \omega z_1 + \Delta f^{(2)}(z_1,z_2,\gamma),
\end{aligned} \tag{33}$$

where the perturbation functions $\Delta f^{(1)}(z_1,z_2,\gamma)$ and $\Delta f^{(2)}(z_1,z_2,\gamma)$ denote the following:

$$\begin{aligned}
\Delta f^{(1)}(z_1,z_2,\gamma) &= \widetilde{\gamma}(c_{10}z_1 + c_{100}z_2) + c_{11}z_1^3 + c_{12}z_1^2z_2\\
&\quad + c_{13}z_1z_2^2 + c_{14}z_2^3 + c_{15}z_1^2 + c_{16}z_1z_2 + c_{17}z_2^2,\\
\Delta f^{(2)}(z_1,z_2,\gamma) &= \widetilde{\gamma}(c_{20}z_1 + c_{200}z_2) + c_{21}z_1^3 + c_{22}z_1^2z_2\\
&\quad + c_{23}z_1z_2^2 + c_{24}z_2^3 + c_{25}z_1^2 + c_{26}z_1z_2 + c_{27}z_2^2.
\end{aligned} \tag{34}$$

Here, the lengthy expressions of c_{ij} are omitted here, and which in polar coordinate, $z_1 = a\sin\theta$, $z_2 = -a\cos\theta$, $\theta = \omega t + \varphi$ is equivalent to

$$\begin{aligned}\dot{a}(t) &= \nu(\gamma) a + \Pi^{(1)}(\tau_1, \tau_2, \gamma) a^3 + \bigcirc\left(a^5, \tau_1, \tau_2, \gamma\right),\\ \dot{\varphi}(t) &= \omega + \Pi^{(2)}(\tau_1, \tau_2, \gamma) a^2 + \bigcirc\left(a^4, \tau_1, \tau_2, \gamma\right).\end{aligned} \tag{35}$$

The so-called Poincaré-Lyapunov coefficient $\Pi^{(1)}(\tau_1, \tau_2, \gamma)$ is independently determined by the following formula:

$$\begin{aligned}&\Pi^{(1)}(a, \tau_1, \tau_2, \gamma)\\ &= \frac{1}{16}\left\{\Delta f^{(1)}_{z_1z_1z_1}(z_1, z_2, \gamma) + \Delta f^{(1)}_{z_1z_2z_2}(z_1, z_2, \gamma)\right.\\ &\qquad \left.+\Delta f^{(2)}_{z_1z_1z_2}(z_1, z_2, \gamma) + \Delta f^{(2)}_{z_2z_2z_2}(z_1, z_2, \gamma)\right\}\\ &\quad + \frac{1}{16\omega}\left\{\Delta f^{(1)}_{z_1z_2}(z_1, z_2, \gamma)\right.\\ &\qquad \times\left(\Delta f^{(1)}_{z_1z_1z_1}(z_1, z_2, \gamma) + \Delta f^{(1)}_{z_2z_2}(z_1, z_2, \gamma)\right)\\ &\qquad - \Delta f^{(2)}_{z_1z_2}(z_1, z_2, \gamma)\\ &\qquad \times\left(\Delta f^{(2)}_{z_1z_1}(z_1, z_2, \gamma) + \Delta f^{(2)}_{z_1z_2}(z_1, z_2, \gamma)\right)\\ &\qquad - \Delta f^{(1)}_{z_1z_1}(z_1, z_2, \gamma)\,\Delta f^{(2)}_{z_1z_1}(z_1, z_2, \gamma)\\ &\qquad \left.+ \Delta f^{(1)}_{z_2z_2}(z_1, z_2, \gamma)\,\Delta f^{(2)}_{z_2z_2}(z_1, z_2, \gamma)\right\}.\end{aligned} \tag{36}$$

As is well known [19–24], the amplitude equation (35) together with (36) can determine the stability behaviour of the center manifold ODEs (33), where all the partial derivatives of $\Delta f^{(1)}(z_1, z_2, \gamma)$, $\Delta f^{(2)}(z_1, z_2, \gamma)$, for example, $\Delta f^{(2)}_{z_1z_2} = \{\partial(\partial \Delta f^{(2)}_{z_1z_2}/\partial z_1)/\partial z_2\}_{(0,0,\gamma_c)}$, are evaluated at the bifurcation point $(z_1, z_2) \rightarrow (0, 0)$. Thus, the partial derivatives are given as follows:

$$\begin{aligned}\frac{\partial \Delta f^{(1)}(z_1, z_2, \gamma)}{\partial z_1} &= c_{10}\tilde{\gamma} + 3c_{11}z_1^2 + 2\left(c_{15} + c_{12}z_2\right) z_1\\ &\quad + \left(c_{16} + c_{13}z_2\right) z_2,\\ \frac{\partial \Delta f^{(1)}(z_1, z_2, \gamma)}{\partial z_2} &= c_{100}\tilde{\gamma} + c_{13}z_1^2 + \left(c_{16} + 2c_{13}z_2\right) z_1\\ &\quad + \left(2c_{17} + 3c_{14}z_2\right) z_2,\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial \Delta f^{(2)}(z_1, z_2, \gamma)}{\partial z_1} &= c_{20}\tilde{\gamma} + 3c_{21}z_1^2 + 2\left(c_{25} + c_{22}z_2\right) z_1\\ &\quad + \left(c_{26} + c_{23}z_2\right) z_2,\\ \frac{\partial \Delta f^{(2)}(z_1, z_2, \gamma)}{\partial z_2} &= c_{200}\tilde{\gamma} + c_{23}z_1^2 + \left(c_{26} + 2c_{23}z_2\right) z_1\\ &\quad + \left(2c_{27} + 3c_{24}z_2\right) z_2,\end{aligned} \tag{38}$$

from which we readily obtain the required values for the formula in (36) as

$$\begin{aligned}&\frac{\partial \Delta^3 f^{(1)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_1 \partial z_1} = 6c_{11}, \qquad \frac{\partial \Delta^3 f^{(1)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_2 \partial z_2} = 2c_{13},\\ &\frac{\partial \Delta^2 f^{(1)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_2} = 2\left(c_{12}z_1 + c_{13}z_2\right) + c_{16},\\ &\frac{\partial \Delta f^{(1)}(z_1, z_1, \gamma)}{\partial z_1 \partial z_1} = 2\left(3c_{11}z_1 + c_{12}z_2 + c_{15}\right),\\ &\frac{\partial \Delta f^{(1)}(z_2, z_2, \gamma)}{\partial z_2 \partial z_2} = 2\left(c_{13}z_1 + 3c_{14}z_2 + c_{17}\right),\\ &\frac{\partial \Delta^3 f^{(2)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_1 \partial z_1} = 6c_{24}, \qquad \frac{\partial \Delta^3 f^{(2)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_2 \partial z_2} = 2c_{23},\\ &\frac{\partial \Delta^2 f^{(2)}(z_1, z_2, \gamma)}{\partial z_1 \partial z_2} = 2\left(c_{22}z_1 + c_{23}z_2\right) + c_{26},\\ &\frac{\partial \Delta f^{(2)}(z_1, z_1, \gamma)}{\partial z_1 \partial z_1} = 2\left(3c_{21}z_1 + c_{22}z_2 + c_{25}\right),\\ &\frac{\partial \Delta f^{(2)}(z_1, z_2, \gamma)}{\partial z_2 \partial z_2} = 2\left(c_{23}z_1 + 3c_{24}z_2 + c_{27}\right).\end{aligned} \tag{39}$$

Substituting these quantities into the formulate in (36) will yield

$$\begin{aligned}&\Pi^{(1)}(a, \tau_1, \tau_2, \gamma)\\ &\quad = \frac{1}{8}\left\{3\left(c_{11} + c_{24}\right) + c_{13} + c_{23}\right\}\\ &\quad + \frac{1}{8\omega}\left\{c_{16}\left(c_{15} + c_{17}\right) - c_{26}\left(c_{25} + c_{27}\right)\right.\\ &\qquad \left.- c_{15}c_{25} + c_{17}c_{27}\right\}.\end{aligned} \tag{40}$$

This coefficient together with the amplitude bifurcation equation (35) produces the nonlinear bifurcations for multiple delays as shown in the following: (a) $\Pi^{(1)}(a, \tau_1, \tau_2, \gamma) < 0 \rightarrow$ supercritical bifurcation; (b) $\Pi^{(1)}(a, \tau_1, \tau_2, \gamma) > 0 \rightarrow$ subcritical bifurcation.

Theorem 2. *Let $ac + bd > 0$.*

(a) *If $\Pi^{(1)}(a, \tau_1, \tau_2, \gamma) < 0$, then system* (1) *occurs supercritical bifurcation.*

(b) *If $\Pi^{(1)}(a, \tau_1, \tau_2, \gamma) > 0$, then system* (1) *occurs subcritical bifurcation.*

2.2. The Double Hopf Bifurcation. For the eigenvalue $\lambda_{1,2} = \pm i\omega_1$, $\lambda_{1,2} = \pm i\omega_2$ of the equation $\Delta(\lambda, \gamma) = 0$ in (6), we have the base function $\Phi(\theta) = [\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)]$,

$-\tau \leq \theta \leq 0$, $\Psi(s) = [\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s)]^\top$, $0 \leq s \leq \tau$ in $\mathscr{C}([-\tau, 0], \mathfrak{R}^n)$ and $\mathscr{C}([-\tau, 0], \mathfrak{R}^n)$, accordingly,

$$\Phi(\theta) = \begin{bmatrix} \cos\omega_1\theta & -\sin\omega\theta_1 & \cos\omega_2\theta & -\sin\omega_2\theta \\ \sin\omega_1\theta & \cos\omega_1\theta & \sin\omega_2\theta & \cos\omega_2\theta \end{bmatrix},$$
$$\Psi(s) = \begin{bmatrix} \cos\omega_1 s & -\sin\omega s & \cos\omega_2 s & -\sin\omega_2 s \\ \sin\omega_1 s & \cos\omega_1 s & \sin\omega_2 s & \cos\omega_2 s \end{bmatrix}^\top, \quad (41)$$

and they form the inner product matrix $(\Psi(s), \Phi(\theta)) = (\psi_j(s)\phi_k(\theta))$, $j, k = 1, 2, 3, 4$, namely,

$$(\Psi(s), \Phi(\theta)) = \begin{bmatrix} (\psi_1(s)\phi_1(\theta)) & (\psi_1(s)\phi_2(\theta)) & (\psi_1(s)\phi_3(\theta)) & (\psi_1(s)\phi_4(\theta)) \\ (\psi_2(s)\phi_1(\theta)) & (\psi_2(s)\phi_2(\theta)) & (\psi_2(s)\phi_3(\theta)) & (\psi_2(s)\phi_4(\theta)) \\ (\psi_3(s)\phi_1(\theta)) & (\psi_3(s)\phi_2(\theta)) & (\psi_3(s)\phi_3(\theta)) & (\psi_3(s)\phi_4(\theta)) \\ (\psi_4(s)\phi_1(\theta)) & (\psi_4(s)\phi_2(\theta)) & (\psi_4(s)\phi_3(\theta)) & (\psi_4(s)\phi_4(\theta)) \end{bmatrix}. \quad (42)$$

The substitution of the elements $(\Psi(s), \Phi(\theta)) = (\psi_j(s)\phi_k(\theta))$ into the bilinear pairing (5) and integrating, we obtain the nonsingular matrix

$$(\Psi, \Phi)_{nsg} = \begin{bmatrix} \psi_{11} & -\psi_{12} & 0 & 0 \\ \psi_{21} & \psi_{22} & 0 & 0 \\ 0 & 0 & \psi_{23} & -\psi_{34} \\ 0 & 0 & \psi_{43} & \psi_{44} \end{bmatrix}, \quad (43)$$

where the constant values of this matrix are given by

$$\psi_{11} = \psi_{22} = 1 - \gamma\tau_1 \cos\omega_1\tau_1,$$
$$\psi_{12} = -\gamma\tau_1 \sin\omega_1\tau_1, \quad \psi_{12} = -\psi_{21},$$
$$\psi_{33} = \psi_{44} = 1 - \gamma\tau_1 \cos\omega_2\tau_1,$$
$$\psi_{34} = -\gamma\tau_1 \sin\omega_2\tau_1, \quad \psi_{43} = -\psi_{34}. \quad (44)$$

Then, $\Psi \in \widehat{\mathscr{C}}$ is normalized to the new basis $\widetilde{\Psi} = [\widetilde{\psi}_1(s), \widetilde{\psi}_3(s), \widetilde{\psi}_3(s), \widetilde{\psi}_4(s)]$, $0 \leq s \leq \tau$, where the elements $\widetilde{\psi}_j(s)$, $j = 1, 2, 3, 4$ of $\Psi \in \widehat{\mathscr{C}}$ are given by

$$\widetilde{\psi}_1 = \frac{1}{\psi_{11}^2 + \psi_{12}^2} \left\{ \left[\left(\psi_{22}^2 \cos\omega_1 s + \psi_{21}^2 \sin\omega_1 s \right), \left(\psi_{22}^2 \sin\omega_1 s - \psi_{21}^2 \cos\omega_1 s \right), 0, 0 \right]^\top \right\},$$
$$\widetilde{\psi}_2 = \frac{1}{\psi_{11}^2 + \psi_{12}^2} \left\{ \left[\left(\psi_{12}^2 \cos\omega_1 s - \psi_{11}^2 \sin\omega_1 s \right), \left(\psi_{12}^2 \sin\omega_1 s + \psi_{11}^2 \cos\omega_1 s \right), 0, 0 \right]^\top \right\},$$
$$\widetilde{\psi}_3 = \frac{1}{\psi_{33}^2 + \psi_{34}^2} \left\{ \left[0, 0, \left(\psi_{44}^2 \cos\omega_2 s + \psi_{43}^2 \sin\omega_2 s \right), \left(\psi_{44}^2 \sin\omega_2 s - \psi_{43}^2 \cos\omega_2 s \right) \right]^\top \right\},$$
$$\widetilde{\psi}_4 = \frac{1}{\psi_{11}^2 + \psi_{12}^2} \left\{ \left[0, 0, \left(\psi_{34}^2 \cos\omega_2 s - \psi_{33}^2 \sin\omega_2 s \right), \left(\psi_{34}^2 \sin\omega_2 s + \psi_{33}^2 \cos\omega_1 s \right) \right]^\top \right\}. \quad (45)$$

Again, the substitution of the elements $\widetilde{\psi}_j(s)$, $\phi_k(\theta)$, $j, k = 1, 2, 3, 4$ of $(\widetilde{\Psi}(s), \Phi(\theta))$ will yield the identity matrix, namely,

$$(\Psi, \Phi)_{nsg} = \frac{1}{\Lambda} \begin{bmatrix} \Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{bmatrix} = I,$$
$$\Lambda = \left(\psi_{11}^2 + \psi_{12}^2 \right) \left(\psi_{33}^2 + \psi_{34}^2 \right). \quad (46)$$

Then, the constant matrix B is given by

$$B = \begin{bmatrix} 0 & -\omega_1 & 0 & 0 \\ \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & \omega_2 & 0 \end{bmatrix}. \quad (47)$$

The change of variable $x_t^P(\phi(\theta), \gamma) = \Phi(\theta)z(t) + x_t^Q(\phi(\theta), \gamma)$, $z(t) \in \mathfrak{R}^4$, $z(t) = (\widetilde{\Psi}(\theta), \phi^P(\theta))$ will yield

$$\Phi(\theta) z(t) = \begin{bmatrix} \cos\omega_1\theta & -\sin\omega_1\theta & \cos\omega_2\theta & -\sin\omega_2\theta \\ \sin\omega_1\theta & \cos\omega_1\theta & \sin\omega_2\theta & \cos\omega_2\theta \end{bmatrix} \times \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix}, \quad (48)$$

thereby, for $-\tau \leq \theta < 0$ and $\tau_1 \leq \tau \leq \tau_2$, we have

$$\Phi(-\tau_1) z(t) = \begin{bmatrix} \cos\omega_1\tau_1 & \sin\omega_1\tau_1 & \cos\omega_2\tau_1 & \sin\omega_2\tau_1 \\ -\sin\omega_1\tau_1 & \cos\omega_1\tau_1 & \sin\omega_2\tau_1 & \cos\omega_2\tau_1 \end{bmatrix} \times \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix},$$
$$\Phi(-\tau_2) z(t) = \begin{bmatrix} \cos\omega_1\tau_2 & \sin\omega_1\tau_2 & \cos\omega_2\tau_2 & \sin\omega_2\tau_2 \\ -\sin\omega_1\tau_2 & \cos\omega_1\tau_2 & -\sin\omega_2\tau_2 & \cos\omega_2\tau_2 \end{bmatrix} \times \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix},$$
$$\Phi(0) z(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix}, \quad (49)$$

and so

$$\gamma x_1(t-\tau_1) = \gamma\left[z_1\cos\omega_1\tau_1 + z_2\sin\omega_1\tau_1 + z_3\cos\omega_2\tau_1 + z_4\sin\omega_2\tau_1\right],$$
$$\gamma x_1(t-\tau_1)x_2(t-\tau_2) = \gamma\left[z_1\cos\omega_1\tau_1 + z_2\sin\omega_1\tau_1 + z_3\cos\omega_2\tau_1 + z_4\sin\omega_2\tau_1\right] \times \left[z_1\cos\omega_1\tau_2 + z_2\sin\omega_1\tau_2 + z_3\cos\omega_2\tau_2 + z_4\sin\omega_2\tau_2\right]. \tag{50}$$

The ODEs of the center manifold $M_\gamma \in \mathscr{C}([-\tau,0],\mathfrak{R}^4)$ are given by as follows:

$$\begin{aligned}\dot{z}_1(t) &= -\omega_1 z_2 + \Delta g^1(z_1,z_2,z_3,z_4,\gamma),\\ \dot{z}_2(t) &= \omega_1 z_1 + \Delta g^2(z_1,z_2,z_3,z_4,\gamma),\\ \dot{z}_3(t) &= -\omega_2 z_2 + \Delta g^3(z_1,z_2,z_3,z_4,\gamma),\\ \dot{z}_4(t) &= \omega_2 z_2 + \Delta g^4(z_1,z_2,z_3,z_4,\gamma),\end{aligned} \tag{51}$$

where the perturbations in these equations denote

$$\begin{aligned}\Delta g^1&(z_1,z_2,z_3,z_4,\gamma)\\ &= c_{111}^{(1)}z_1^3 + c_{112}^{(1)}z_1^2z_2 + c_{113}^{(1)}z_1^2z_3 + c_{114}^{(1)}z_1^2z_2 + c_{115}^{(1)}z_1z_2^2\\ &\quad + c_{116}^{(1)}z_1z_3^2 + c_{117}^{(1)}z_1z_4^2 + c_{222}^{(1)}z_2^3 + c_{333}^{(1)}z_3^3 + c_{444}^{(1)}z_4^3\\ &\quad + d_{111}^{(1)}z_1^2 + d_{112}^{(1)}z_1z_2 + d_{113}^{(1)}z_1z_3 + d_{114}^{(1)}z_1z_4 + d_{222}^{(1)}z_2^2\\ &\quad + d_{333}^{(1)}z_3^2 + d_{444}^{(1)}z_4^2 + \cdots,\\ \Delta g^2&(z_1,z_2,z_3,z_4,\gamma)\\ &= c_{111}^{(2)}z_1^3 + c_{112}^{(2)}z_1^2z_2 + c_{113}^{(2)}z_1^2z_3 + c_{114}^{(2)}z_1^2z_2 + c_{115}^{(2)}z_1z_2^2\\ &\quad + c_{116}^{(2)}z_1z_3^2 + c_{117}^{(2)}z_1z_4^2 + c_{222}^{(2)}z_2^3 + c_{333}^{(1)}z_3^3 + c_{444}^{(1)}z_4^3\\ &\quad + d_{111}^{(2)}z_1^2 + d_{112}^{(2)}z_2z_2 + d_{113}^{(2)}z_1z_3 + d_{114}^{(2)}z_1z_4 + d_{222}^{(2)}z_2^2\\ &\quad + d_{333}^{(2)}z_3^2 + d_{444}^{(2)}z_4^2 + \cdots,\\ \Delta g^3&(z_1,z_2,z_3,z_4,\gamma)\\ &= c_{111}^{(3)}z_1^3 + c_{112}^{(3)}z_1^2z_2 + c_{113}^{(3)}z_1^2z_3 + c_{114}^{(3)}z_1^2z_2 + c_{115}^{(1)}z_1z_2^2\\ &\quad + c_{116}^{(3)}z_1z_3^2 + c_{117}^{(3)}z_1z_4^2 + c_{222}^{(3)}z_2^3 + c_{333}^{(3)}z_3^3 + c_{444}^{(1)}z_4^3\\ &\quad + d_{111}^{(3)}z_1^2 + d_{112}^{(3)}z_1z_2 + d_{113}^{(3)}z_1z_3 + d_{114}^{(3)}z_1z_4 + d_{222}^{(3)}z_2^2\\ &\quad + d_{333}^{(3)}z_3^2 + d_{444}^{(3)}z_4^2 + \cdots,\\ \Delta g^4&(z_1,z_2,z_3,z_4,\gamma)\\ &= c_{111}^{(4)}z_1^3 + c_{112}^{(4)}z_1^2z_2 + c_{113}^{(4)}z_1^2z_3 + c_{114}^{(4)}z_1^2z_2 + c_{115}^{(4)}z_1z_2^2\\ &\quad + c_{116}^{(4)}z_1z_3^2 + c_{117}^{(4)}z_1z_4^2 + c_{222}^{(4)}z_2^3 + c_{333}^{(4)}z_3^3 + c_{444}^{(4)}z_4^3\\ &\quad + d_{111}^{(4)}z_1^2 + d_{112}^{(4)}z_1z_2 + d_{113}^{(4)}z_1z_3 + d_{114}^{(4)}z_1z_4 + d_{222}^{(4)}z_2^2\\ &\quad + d_{333}^{(4)}z_3^2 + d_{444}^{(4)}z_4^2 + \cdots,\end{aligned} \tag{52}$$

and the above coefficients in these equations denote the constant values in (52). Here, the lengthy expressions of c_{ijl}^k and d_{ijl}^k are omitted here. These equations are further written into the desired standard form of amplitude and phase relations using the polar coordinates $z_j = a_j\sin\theta_j$, $z_j = a_j\cos\theta_j$, $\theta_j = \omega_j t + \varphi_j$, $j = 1,2,3,4$. Again, the application of the integral averaging method to the resulting amplitude and phase equations will yield

$$\begin{aligned}\frac{d\overline{a}_1}{dt} &= \beta_{111}^{(11)}a_1 + \Pi^{(11)}(a,\gamma)a_1^3 + \beta_{112}^{(11)}a_1a_2^2 + \cdots,\\ \frac{d\overline{\varphi}_1}{dt} &= \beta_{111}^{(12)}a_1 + \Pi^{(12)}(a,\gamma)a_1^3 + \beta_{112}^{(12)}a_1a_2^2 + \cdots,\\ \frac{d\overline{a}_2}{dt} &= \beta_{111}^{(21)}a_2 + \Pi^{(21)}(a,\gamma)a_2^3 + \beta_{112}^{(21)}a_1a_2^2 + \cdots,\\ \frac{d\overline{\varphi}_2}{dt} &= \beta_{111}^{(22)}a_1 + \Pi^{(22)}(a,\gamma)a_2^3 + \beta_{112}^{(22)}a_1a_2^2 + \cdots,\end{aligned} \tag{53}$$

where the coefficients in these equations are the resulting constant values after averaging. Similarly, the coefficients in these averaged equations, in particular the cubic ones $\Pi^{(11)}(a,\gamma)$, $\Pi^{(21)}(a,\gamma)$, will determine whether the double Hopf interactions are of the form (i) subsupercritical bifurcation type, (ii) supersubcritical bifurcation type, (iii) subsubcritical bifurcation type, and (iv) supersupercritical bifurcation type. Like the Poincaré-Lyapunov coefficient $\Pi(a,\gamma)$ in (36) for the single Hopf, there are also explicit formulas for computing the double Hopf coefficients $\Pi^{(11)}(a,\gamma)$, $\Pi^{(21)}(a,\gamma)$, which are extremely too long to be displayed here. For the account of these formulas for the double Hopf coefficients, again, the publications [22, 23] are excellent references. Classical studies of dynamical systems resulting to equations of the form (53) have demonstrated unprecedented dynamics ranging from global dynamic bifurcations to ultimately chaotic dynamics (see [22, 24]). There exist reliable and robust mathematical tools to examine the long-term behaviour of such dynamics. Campbell et al. [24], in particular, have shown numerically that equations of the form (53) exhibit four double Hopf's interactions, namely, two super-super-, one super-sub-, and one sub-sub-critical bifurcations. Other interesting dynamics such as large limit cycles, tori, and period doubling were also characterized by the authors.

3. Stochastic System

In the section, we present some preliminary results to be used in a subsequent section to establish the stochastic stability and stochastic bifurcation. Before proving the main theorem, we give some lemmas and definitions.

Definition 3 (see [1] (D-bifurcation)). Dynamical bifurcation is concerned with a family of random dynamical systems which is differential and has the invariant measure μ_α. If there exists a constant α_D satisfying in any neighborhood of α_D, there exists another constant α and the corresponding invariant measure $\nu_\alpha \neq \mu_\alpha$ satisfying $\nu_\alpha \to \mu_\alpha$ as $\alpha \to \alpha_D$. Then, the constant α_D is a point of dynamical bifurcation.

Definition 4 (see [1] (P-bifurcation)). Phenomenological bifurcation is concerned with the change in the shape of density (stationary probability density) of a family of random dynamical systems as the change of the parameter. If there exists a constant α_0 satisfying in any neighborhood of α_D, there exist other two constants α_1, α_2, and their corresponding stationary density functions p_{α_1}, p_{α_2} satisfying p_{α_1} and p_{α_2} are not equivalent. Then, the constant α_0 is a point of phenomenological bifurcation.

Definition 5 (see [1] (stochastic pitchfork bifurcation)). In the viewpoint of phenomenological bifurcation, the stationary solution of the Fokker-Planck-Kolmogorov (FPK) equation which is corresponded with the stochastic differential equation changes from one peak into two peaks.

In the viewpoint of dynamical bifurcation, there exists a constant μ_0 that satisfies the following conditions:

(i) when $\mu < \mu_0$, the stochastic differential equation has only one invariant measure ν_0; moreover, ν_0 is stable (i.e., the maximal Lyapunov exponent is negative).

(ii) when $\mu = \mu_0$, the invariant measure ν_0 loses its stability and becomes unstable (i.e., the maximal Lyapunov exponent is positive); moreover, the rotation number is zero.

(iii) when $\mu > \mu_0$, the stochastic differential equation has three invariant measures ν_0, ν_1, and ν_2; both ν_1 and ν_2 are stable (i.e., their maximal Lyapunov exponents are both negative).

(iv) the global attractors of the stochastic differential equation change from a single-point set into a collection of one-dimensional (the closure of the unstable manifold of the unstable invariant measure).

If the stochastic bifurcation of a stochastic differential equation has the above characters, then the stochastic differential equation admits stochastic pitchfork bifurcation at $\mu = \mu_0$.

Definition 6 (see [1] (the stochastic Hopf bifurcation)). In the viewpoint of phenomenological bifurcation, the stationary solution of the FPK equation which is corresponded to the stochastic differential equation changes from one peak into a crater.

In the viewpoint of dynamical bifurcation, the following can be noticed:

(i) if one of the invariant measures of the stochastic differential equation loses its stability and becomes unstable (i.e., two Lyapunov exponents are positive), then the rotation number is not zero. Meanwhile, there at least appears one new stable invariant measure,

(ii) the global attractors of the stochastic differential equation change from a single-point set into a random topological disk (the closure of the unstable manifold of the unstable invariant measure).

If the stochastic bifurcation of a stochastic differential equation has the above characters, then the stochastic differential equation admits the stochastic Hopf bifurcation.

Definition 7 (see [1] (stochastically stable)). The trivial solution $x(t, t_0, x_0)$ of stochastic differential equation is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $\alpha > 0$, there exists $\delta = \delta(\varepsilon, \delta, t_0) > 0$ such that

$$P\left\{\left|x\left(t, t_0, x_0\right)\right| < \alpha \ \forall t \geq t_0\right\} \geq 1 - \varepsilon, \tag{54}$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

In this section, taking some stochastic factors into account, we introduce randomness into the model (1) by replacing the parameters γ by $\gamma \to \gamma + \alpha\xi(t)$. This is only a first step in introducing stochasticity into the model. Ideally, we would also like to introduce stochastic environmental variation into the other parameters such as the transmission coefficient α and δ, but this would make the analysis much too difficult. Hence, we get the following system of stochastic differential equations:

$$\begin{aligned} \dot{u}(t) &= \alpha\kappa(1 - u - v) - \beta u - \gamma u\left(t - \tau_1\right) v\left(t - \tau_2\right) \\ &\quad + \left(\sigma_1 + \sigma_2 u + \sigma_3 v\right)\xi(t), \\ \dot{v}(t) &= \delta(1 - \kappa)(1 - u - v)^3 - \gamma u v \\ &\quad + \left(\sigma_4 + \sigma_5 u + \sigma_6 v\right)\xi(t). \end{aligned} \tag{55}$$

Here, $\xi(t)$ is the multiplicative random excitation and the external random excitation directly, and $\xi(t)$ is independent, in possession of zero mean value and standard variance Gaussian white noises, that is, $\mathscr{E}[\xi(t)] = 0$, $\mathscr{E}[\xi(t)\xi(t+\tau)] = \delta(\tau)$. Note that the intensity of the noise σ_i $(i = 1, 2, \ldots, 6)$, $\gamma = \gamma_c + \tilde{\gamma}$ is the bifurcation parameter. From (3)–(33), we obtain (55) of the stochastic center manifold:

$$\begin{aligned} \dot{z}_1(t) &= -\omega z_2 + c_{10} z_1 + c_{100} z_2 + c_{11} z_1^3 + c_{12} z_1^2 z_2 \\ &\quad + c_{13} z_1 y_2^2 + c_{14} z_2^3 + c_{15} z_1^2 + c_{16} z_2 z_1 \\ &\quad + c_{17} z_2^2 + \left(k_{11} z_1 + k_{12} z_2 + k_{13}\right)\xi(t), \\ \dot{z}_2(t) &= \omega z_1 + c_{20} z_1 + c_{200} z_2 + c_{21} z_1^3 + c_{22} z_1^2 z_2 \\ &\quad + c_{23} z_1 z_2^2 + c_{24} z_2^3 + c_{25} z_1^2 + c_{26} z_2 z_1 \\ &\quad + c_{27} z_2^2 + \left(k_{21} z_1 + k_{22} z_2 + k_{23}\right)\xi(t). \end{aligned} \tag{56}$$

Here, the lengthy expressions of c_{ij} and k_{ij} are omitted.

Setting the coordinate transformation $z_1 = r\cos\theta$, $z_2 = r\sin\theta$ and by substituting the variable in (34), we obtain

$$\begin{aligned}
\dot{r}(t) = {} & rc_{10}\cos^2\theta + r(c_{100}+c_{20})\sin\theta\cos\theta + rc_{200}\sin^2\theta \\
& + r^3\left[c_{11}\cos^4\theta + (c_{12}+c_{21})\cos^3\theta\sin\theta\right. \\
& + (c_{13}+c_{22})\cos^2\theta\sin^2\theta + (c_{14}+c_{23})\cos\theta\sin^3\theta \\
& \left.+c_{24}\sin^4\theta\right] + r^2\left[c_{15}\cos^3\theta\right. \\
& + (c_{16}+c_{25})\cos^2\theta\sin\theta \\
& \left.+(c_{17}+c_{26})\cos\theta\sin^2\theta + c_{27}\sin^3\theta\right] \\
& + r\left[k_{11}\cos^2\theta + (k_{12}+k_{21})\cos\theta\sin\theta\right. \\
& \left.+k_{22}\sin^2\theta\right]\xi(t) + \left[k_{13}\cos\theta + k_{23}\sin\theta\right]\xi(t), \\
\dot{\theta}(t) = {} & -\omega + c_{20}\cos^2\theta + (c_{200}-c_{10})\sin\theta\cos\theta - c_{100}\sin^2\theta \\
& + r^2\left[c_{21}\cos^4\theta + (c_{21}-c_{12})\cos^3\theta\sin\theta\right. \\
& + (c_{23}+c_{12})\cos^2\theta\sin^2\theta \\
& \left.+(c_{24}+c_{23})\cos\theta\sin^3\theta + c_{14}\sin^4\theta\right] \\
& + r\left[b_{25}\cos\theta + (c_{26}-c_{15})\cos^2\theta\sin\theta\right. \\
& \left.+(c_{27}-c_{16})\cos\theta\sin^2\theta - c_{17}\sin^3\theta\right] \\
& + \left[k_{21}\cos^2\theta + (k_{22}-k_{11})\cos\theta\sin\theta - k_{12}\sin^2\theta\right]\xi(t) \\
& + \frac{1}{r}\left[k_{23}\cos\theta - k_{13}\sin\theta\right]\xi(t).
\end{aligned} \tag{57}$$

It is difficult to calculate the exact solution for system (57) today. According to the Khasminskii limit theorem, when the intensities of the white noises σ_i $(i = 1,\ldots,6)$ are small enough, the response process $\{r(t),\theta(t)\}$ weakly converged to the two-dimensional Markov diffusion process [1–5]. Through the stochastic averaging method, we obtained the Itô stochastic differential equation (55) as follows:

$$\begin{aligned}
dr = {} & \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_3}{r} + \frac{\mu_7}{8}r^3\right]dt \\
& + \left(\mu_3 + \frac{\mu_4}{8}r^2\right)^{1/2}dW_r + (r\mu_5)^{1/2}dW_\theta, \\
d\theta = {} & \left(\mu_{10} + \frac{\mu_8}{8}r^2\right)dt + (r\mu_5)^{1/2}dW_r \\
& + \left(\frac{\mu_3}{r^2} + \frac{\mu_6}{8}\right)^{1/2}dW_\theta,
\end{aligned} \tag{58}$$

where $W_r(t)$ and W_θ are the independent and standard Wiener processes. Set the parameter as follows:

$$\begin{aligned}
& \mu_1 = \frac{1}{2}(c_{10}+c_{200}), \qquad \mu_{10} = -\omega + \frac{1}{2}(c_{20}+c_{100}), \\
& \mu_2 = 5k_{11}^2 + 5k_{22}^2 + 3k_{21}^2 + 3k_{12}^2 + 6k_{12}k_{21} - 2k_{11}k_{22}, \\
& \mu_3 = \frac{1}{2}\left(k_{13}^2 + k_{23}^2\right), \\
& \mu_4 = 3k_{11}^2 + 3k_{22}^2 + k_{12}^2 + k_{21}^2 + 2k_{12}k_{21} + 2k_{11}k_{22}, \\
& \mu_5 = \frac{1}{4}(k_{11}+k_{22})(k_{21}-k_{12}), \\
& \mu_6 = k_{11}^2 + k_{22}^2 + 3k_{12}^2 + 3k_{21}^2 - 2k_{12}k_{21} - 2k_{11}k_{22}, \\
& \mu_7 = 3c_{11} + 3c_{21} + c_{13} + c_{22}, \qquad \mu_8 = 3c_{21} - 3c_{14} + c_{23} - c_{12}.
\end{aligned} \tag{59}$$

3.1. Stochastic D-Bifurcation. In the section, we will see how the introduction of randomness changes the stochastic behavior significantly from both the dynamical and phenomenological points of view.

Theorem 8 (D-bifurcation). *Let $\mu_3 = 0$, $\mu_7 = 0$. Then, system (55) undergoes stochastic D-bifurcation.*

Proof. When $\mu_3 = 0$, $\mu_7 = 0$, Then, system (58) becomes

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{1/2}dW_r. \tag{60}$$

When $\mu_4 = 0$, (60) is a deterministic system, and there is no bifurcation phenomenon. Here, we discuss the situation $\mu_4 \neq 0$, let

$$m(r) = \left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)r, \qquad \sigma(r) = \left(\frac{\mu_4}{8}\right)^{1/2}r. \tag{61}$$

The continuous random dynamical system generated by (60) is

$$\varphi(t)x = x + \int_0^t m(\varphi(s)x)\,ds + \int_0^t \sigma(\varphi(s)x)\circ dW_r, \tag{62}$$

where $\circ dW_r$ is the differential in the sense of Statonovich, and it is the unique strong solution of (60) with the initial value x. And $m(0) = 0$, $\sigma(0) = 0$, so 0 is a fixed point of φ. Since $m(r)$ is bounded and for any $r \neq 0$, it satisfies the ellipticity condition $\sigma(r) \neq 0$. This ensures that there is at most one stationary probability density. According to the Itô equation of amplitude $r(t)$, we obtain its Fokker-Planck-Kolmogorov equation corresponding to (60) as follows:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r\right]p\right\} + \frac{\partial^2}{\partial r^2}\left\{\left[\frac{\mu_4}{8}r^2\right]p\right\}. \tag{63}$$

Let $\partial p/\partial t = 0$, then we obtain the solution of system (63)

$$p(t) = c\left|\sigma^{-1}(t)\right|\exp\left(\int_0^t \frac{2m(u)}{\sigma^2(u)}du\right). \tag{64}$$

The above equation (63) has two kinds of equilibrium states: fixed point and nonstationary motion. The invariant measure of the former is δ_0, and its probability density is δ_x. The invariant measure of the latter is ν, and its probability density is (64). In the following, we calculate the Lyapunov exponent of the two invariant measures.

Using the solution of linear Itô stochastic differential equation, we obtain the solution of system (60):

$$r(t) = r(0)\exp\left(\int_0^t \left[m'(0) + \frac{\sigma(0)\sigma'(0)}{2}\right] ds + \int_0^t \sigma'(0)\, dW_r\right). \tag{65}$$

The Lyapunov exponent with regard to μ of the dynamical system φ is defined as:

$$\lambda_\varphi(\mu) = \lim_{t\to+\infty}\frac{1}{t}\ln\|r(t)\|, \tag{66}$$

substituting (65) into (66), note that $\sigma(0) = 0$, $\sigma'(0) = 0$, we obtain the Lyapunov exponent of the fixed point:

$$\begin{aligned}\lambda_\varphi(\delta_0) &= \lim_{t\to+\infty}\frac{1}{t}\left(\ln\|r(0)\| + m'(0)\int_0^t ds + \sigma'(0)\int_0^t dW_r(s)\right)\\ &= m'(0) + \sigma'(0)\lim_{t\to+\infty}\frac{W_r(t)}{t}\\ &= m'(0)\\ &= \mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}.\end{aligned} \tag{67}$$

For the invariant measure which regard (65) as its density, we obtain the Lyapunov exponent:

$$\begin{aligned}\lambda_\varphi(\nu) &= \lim_{t\to+\infty}\frac{1}{t}\int_0^t \left[m'(r) + \sigma(r)\sigma'(r)\right] ds\\ &= \int_R \left[m'(r) + \frac{\sigma(r)\sigma'(r)}{2}\right] p(r)\, dr\\ &= -2\int_R \left[\frac{m(r)}{\sigma(r)}\right]^2 p(r)\, dr\\ &= -32\sqrt{2}\mu_4^{3/2} m(r)^2 \exp\left[\frac{16}{\mu_4} m(r)\right]\\ &= -32\sqrt{2}\mu_4^{3/2}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)^2\\ &\quad\times \exp\left[\frac{16}{\mu_4}\left(\mu_1 + \frac{\mu_2}{8} - \frac{\mu_4}{16}\right)\right].\end{aligned} \tag{68}$$

Let $\alpha = \mu_1 + \mu_2/8 - \mu_4/16$. We can obtain that the invariant measure of the fixed point is stable when $\alpha < 0$, but the invariant measure of the nonstationary motion is stable when $\alpha > 0$, so $\alpha = \alpha_D = 0$ is a point of D-bifurcation. □

Theorem 9. *Let $\mu_3 = 0$, $\mu_7 = 0$. Then, system* (55) *does not undergo stochastic P-bifurcation.*

Proof. By simplifying (64), we can obtain

$$p_{\text{st}}(r) = c r^{2(8\mu_1+\mu_2-\mu_4)/\mu_4}, \tag{69}$$

where c is a normalization constant; thus, we have

$$p_{\text{st}}(r) = o(r^{\upsilon}) \quad r \longrightarrow 0, \tag{70}$$

where $\upsilon = 2(8\mu_1+\mu_2-\mu_4)/\mu_4$. Obviously, when $\upsilon < -1$, that is, $\mu_1+\mu_2/8-\mu_4/16 < 0$, $p_{\text{st}}(r)$ is a δ function. When $-1 < \upsilon < 0$, that is, $\mu_1 + \mu_2/8 - \mu_4/16 > 0$, $r = 0$ is a maximum point of $p_{\text{st}}(r)$ in the state space; thus, the system admits D-bifurcation when $\upsilon = -1$, that is, $\mu_1 + \mu_2/8 - \mu_4/16 = 0$, is the critical condition of D-bifurcation at the equilibrium point. When $\upsilon > 0$, there is no point that makes $p_{\text{st}}(r)$ has the maximum value; thus, the system does not admits P-bifurcation. □

Remark 10. Unfortunately, these two definitions (D-bifurcation and P-bifurcation) do not necessarily yield the same result.

Theorem 11 (stochastic pitchfork bifurcation). *Let $\gamma\alpha\kappa = 0$, $\mu_3 = 0$, $\mu_7 < 0$. Then, system* (55) *undergoes stochastic pitchfork bifurcation.*

Proof. When $\mu_3 = 0$, $\mu_7 \neq 0$, then (58) can be rewritten as

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right) r + \frac{\mu_7}{8} r^3\right] dt + \left(\frac{\mu_4}{8} r^2\right)^{1/2} dW_r. \tag{71}$$

Let $\alpha = \mu_1 + \mu_2/8$, $\phi = (\sqrt{-\mu_7/8})r$, $\mu_7 < 0$, then the system (71) becomes

$$d\phi = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)\phi - \phi^3\right] dt + \left(\frac{\mu_4}{8}\right)^{1/2} \phi \circ dW_t, \tag{72}$$

which is solved by

$$\begin{aligned}\phi &\longrightarrow \psi_\alpha(t,\omega)\phi\\ &= \left(\phi \exp\left(\left(\mu_1 + \frac{\mu_2}{8}\right) t + \left(\frac{\mu_4}{8}\right)^{1/2} W_t(\omega)\right)\right)\\ &\quad\times\left(\left(1 + 2\phi^2 \int_0^t \exp\left(2\left(\left(\mu_1 + \frac{\mu_2}{8}\right) t + \left(\frac{\mu_4}{8}\right)^{1/2} W_s(\omega)\right)\right) ds\right)^{1/2}\right)^{-1}.\end{aligned} \tag{73}$$

We now determine the domain $D_\alpha(t,\omega)$, where $D_\alpha(t,\omega) := \{\phi \in \mathfrak{R} : (t,\omega,\phi) \in D\}$ ($D = \mathfrak{R} \times \Omega \times X$) is (in general possibly empty) the set of initial values $\phi \in \mathfrak{R}$ for which the trajectories still exist at time t and the range $R_\alpha(t,\omega)$ of $\psi_\alpha(t,\omega) : D_\alpha(t,\omega) \to R_\alpha(t,\omega)$.

We have

$$D_\alpha(t,\omega) = \begin{cases} \mathfrak{R}, & t \geq 0,\\ (-d_\alpha(t,\omega), d_\alpha(t,\omega)), & t < 0,\end{cases} \tag{74}$$

where

$$d_\alpha(t,\omega) = 1\times\left(\left(2\left|\int_0^t \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{1/2} W_s(\omega)\right)ds\right|\right)^{1/2}\right)^{-1} > 0,$$

$$R_\alpha(t,\omega) = D_\alpha(-t,\vartheta(t)\omega) = \begin{cases} (-r_\alpha(t,\omega),\ r_\alpha(t,\omega)), & t>0,\\ \mathfrak{R}, & t\le 0,\end{cases} \tag{75}$$

where

$$r_\alpha(t,\omega) = d_\alpha(-t,\vartheta(t)\omega) = \left(\exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{1/2}W_t(\omega)\right)\right) \times\left(\left(2\left|\int_0^t \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{1/2} W_s(\omega)\right)ds\right|\right)^{1/2}\right)^{-1} > 0. \tag{76}$$

We can now determine that

$$E_\alpha(\omega) := \bigcap_{t\in\mathfrak{R}} D_\alpha(t,\omega) \tag{77}$$

and obtain

$$E_\alpha(\omega) = \begin{cases} (-d_\alpha^-(t,\omega), d_\alpha^-(t,\omega)), & \alpha = \mu_1+\dfrac{\mu_2}{8} > 0,\\ \{0\}, & \alpha = \mu_1+\dfrac{\mu_2}{8}\le 0,\end{cases} \tag{78}$$

where

$$0 < d_\alpha^\pm(t,\omega) = 1\times\left(\left(2\left|\int_0^{\pm\infty} \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{1/2} W_s(\omega)\right)ds\right|\right)^{1/2}\right)^{-1} < \infty. \tag{79}$$

The ergodic invariant measures of system (71) are as follows.

(i) For $\alpha\le 0$, the only invariant measure is $\mu_\omega^\alpha = \delta_0$.

(ii) For $\alpha > 0$, we have the three invariant forward Markov measures, $\mu_\omega^\alpha = \delta_0$ and $\upsilon_{\pm,\omega}^\alpha = \delta_{\pm k_\alpha(\omega)}$, where

$$k_\alpha(\omega) := \left(2\int_{-\infty}^0 \exp\left(2\left(\mu_1+\frac{\mu_2}{8}\right)t + 2\left(\frac{\mu_4}{8}\right)^{1/2}W_t(\omega)\right)ds\right)^{-1/2}. \tag{80}$$

We have $\mathbb{E}k_\alpha^2(\omega)=\alpha$. Solving the forward Fokker-Planck-Kolmogorov equation

$$L^* p_\alpha = -\left(\left(\left(\mu_1+\frac{\mu_2}{8}\right)\phi + \frac{\mu_4}{16}\phi - \phi^3\right)P_\alpha(\phi)\right)' + \frac{\mu_4}{16}\left(\phi^2 P_{\nu_1}(\phi)\right)'' = 0, \tag{81}$$

yield

(i) $p_\alpha = \delta_0$ for all α,

(ii) for $p_\alpha > 0$,

$$q_\alpha^+(\phi) = \begin{cases} N_\alpha \phi^{(2(\mu_1+\mu_2/8)/\mu_4)-1}\exp\left(\dfrac{16\phi^2}{\mu_4}\right), & \phi>0,\\ 0, & \phi\le 0,\end{cases} \tag{82}$$

and $q_\alpha^+(\phi) = q_\alpha^+(-\phi)$, where $N_\alpha^- = \Gamma(\nu_1/\mu_4)(\mu_4/8)^{(\mu_1+\mu_2/8)/\mu_4}$.

Naturally, the invariant measures $\upsilon_{\pm,\omega}^\alpha = \delta_{\pm k_\alpha(\omega)}$ are those corresponding to the stationary measures q_α^+. Hence, all invariant measures are Markov measures.

We determine all invariant measures (necessarily Dirac measure) of local RDS χ generated by the SDE

$$d\phi = \left[\left(\mu_1+\frac{\mu_2}{8}\right)\phi - \phi^3\right]dt + \left(\frac{\mu_4}{8}\right)^{1/2}\phi\circ dW, \tag{83}$$

on the state space $\mathfrak{R}$, $\mu_1+\mu_2/8\in\mathfrak{R}$. We now calculate the Lyapunov exponent for each of these measures.

The linearized RDS $\chi_t = D\Upsilon(t,\omega,\phi)\chi$ satisfies the linearized SDE

$$d\chi_t = \left[\left(\mu_1+\frac{\mu_2}{8}\right) - 3(\Upsilon(t,\omega,\phi))^2\chi_t\right]dt + \left(\frac{\mu_4}{8}\right)^{1/2}\chi_t\circ dW. \tag{84}$$

Hence,

$$D\Upsilon(t,\omega,\phi)\chi = \chi\exp\left(\left(\mu_1+\frac{\mu_2}{8}\right)t + \left(\frac{\mu_4}{8}\right)^{1/2}W_t(\omega) - 3\int_0^t(\Upsilon(s,\omega,\phi))^2 ds\right). \tag{85}$$

Thus, if $\nu_\omega = \delta_{\phi_0(\omega)}$ is a Υ-invariant measure, its Lyapunov exponent is

$$\begin{aligned}\lambda(\mu) &= \lim_{t\to\infty}\frac{1}{t}\log\|D\Upsilon(t,\omega,\phi)\chi\|\\ &= \left(\mu_1+\frac{\mu_2}{8}\right) - 3\lim_{t\to\infty}\int_0^t(\Upsilon(s,\omega,\phi))^2ds\\ &= \left(\mu_1+\frac{\mu_2}{8}\right) - 3\mathbb{E}\phi_0^2,\end{aligned} \tag{86}$$

providing the IC $\phi_0^2\in L^1(\mathbb{P})$ is satisfied.

(i) For $\mu_1 + \mu_2/8 \in \mathfrak{R}$, the IC for $\nu^\alpha_{\pm,\omega} = \delta_0$ is trivially satisfied, and we obtain

$$\lambda(\nu^\alpha_1) = \mu_1 + \frac{\mu_2}{8}. \quad (87)$$

So, ν^α_1 is stable for $\mu_1 + \mu_2/8 < 0$ and unstable for $\mu_1 + \mu_2/8 > 0$.

(ii) For $\mu_1 + \mu_2/8 > 0$, $\nu^\alpha_{2,\omega} = \delta_{d^\alpha_\omega}$ is $\mathscr{F}^0_{-\infty}$ measurable; hence, the density p^α of $\rho^\alpha = \mathbb{E}\nu^\alpha_2$ satisfies the Fokker-Planck-Kolmogorov equation

$$\begin{aligned} L^*_{\nu_1} = & -\left(\left(\left(\mu_1 + \frac{\mu_2}{8}\right)\phi + \frac{\mu_4}{16}\phi - \phi^3\right)p_\alpha(\phi)\right)' \\ & + \frac{\mu_4}{16}\left(\phi^2 p_\alpha(\phi)\right)'' = 0, \end{aligned} \quad (88)$$

which has the unique probability density solution

$$P^\alpha(\phi) = N_\alpha \phi^{(2(\mu_1+\mu_2/8)/\mu_4)-1}\exp\left(\frac{\phi^2 8}{\mu_4}\right), \quad \phi > 0. \quad (89)$$

Since

$$\mathbb{E}_{\nu^\alpha_2}\phi^2 = \mathbb{E}(d^\alpha_-)^2 = \int_0^\infty \phi^2 p^\alpha(\phi)\,d\phi < \infty, \quad (90)$$

the IC is satisfied. The calculation of the Lyapunov exponent is accomplished by observing that

$$\begin{aligned} & d^\alpha_-(\vartheta_t\omega)^2 \\ & = \frac{\exp\left(2(\mu_1+\mu_2/8)t + 2(\mu_4/8)^{1/2}W_t(\omega)\right)}{2\int_{-\infty}^t \exp\left(2(\mu_1+\mu_2/8)s + 2(\mu_4/8)^{1/2}W_s(\omega)\right)ds} \\ & = \frac{\Psi'(t)}{2\Psi}, \end{aligned} \quad (91)$$

where

$$\Psi(t) = \int_{-\infty}^t \exp\left(\left(\mu_1 + \frac{\mu_2}{8}\right)s + \left(\frac{\mu_4}{8}\right)^{1/2}W_s(\omega)\right)ds. \quad (92)$$

Hence, by the ergodic theorem

$$\mathbb{E}(d^\alpha_-)^2 = \frac{1}{2}\lim_{t\to\infty}\frac{1}{t}\log\Psi(t) = \mu_1 + \frac{\mu_2}{8}, \quad (93)$$

finally

$$\lambda(\nu^\alpha_2) = -2\left(\mu_1 + \frac{\mu_2}{8}\right) < 0. \quad (94)$$

(iii) For $(\mu_1 + \mu_2/8) > 0$, $\nu^\alpha_{2,\omega} = \delta_{d^\alpha_\omega}$ is $\mathscr{F}^0_{-\infty}$ measurable. Since $\mathscr{L}(d^\alpha_+) = \mathscr{L}(d^\alpha_-)$

$$\mathbb{E}(-d^\alpha_-)^2 = \mathbb{E}(d^\alpha_-)^2 = \mu_1 + \frac{\mu_2}{8}, \quad (95)$$

thus

$$\lambda(\nu^\alpha_2) = -2\left(\mu_1 + \frac{\mu_2}{8}\right) < 0. \quad (96)$$

The two families of densities $(q^+_\alpha)_{\alpha>0}$ clearly undergo a P-bifurcation at the parameter value $\alpha_P = \mu_4/8$, which is the same value as the transcritical case, since the SDE linearized at $\phi = 0$ is the same in the next section. In both cases, we have a D-bifurcation of the trivial reference measure δ_0 at $\alpha_D = 0$ and a P-bifurcation of $\alpha_P = (\mu_1 + \mu_2/8)^2/2$. Then, system (55) has stochastic pitchfork bifurcation. □

3.2. P-Bifurcation. In the following, we investigate the steady-state probability density $p_{st}(r)$ of the linear Itô stochastic differential equation. Calculating extreme values of the steady-state probability density is one of the most popular efficient methods in studying the bifurcation of a nonlinear dynamical system. The steady-state probability density is an important characteristic value of stochastic bifurcation.

Case I. When $\mu_3 = 0$, $\mu_7 \neq 0$, then (38) can be rewritten as

$$dr = \left[\left(\mu_1 + \frac{\mu_2}{8}\right)r + \frac{\mu_7}{8}r^3\right]dt + \left(\frac{\mu_4}{8}r^2\right)^{1/2}dW_r. \quad (97)$$

According to the Itô equation of amplitude $r(t)$, we obtain its Fokker-Planck-Kolmogorov equation form (97) as follows:

$$\begin{aligned} \frac{\partial p}{\partial t} = & -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r - \frac{\mu_4}{2\mu_7}r - r^3\right]p(r)\right\} \\ & - \frac{\mu_4}{2\mu_7}\frac{\partial^2}{\partial r^2}\left(r^2 p(r)\right), \end{aligned} \quad (98)$$

with the initial value condition $\mu_7 = 0$, $p(r,t|r_0,t_0) \to \delta(r - r_0)$, $t \to t_0$, where $p(r,t|r_0,t_0)$ is the transition probability density of diffusion process $r(t)$. The invariant measure of $r(t)$ is the steady-state probability density $p_{st}(r)$ which is the solution of the degenerate system as follows:

$$\begin{aligned} 0 = & -\frac{\partial}{\partial r}\left\{\left[\left(\mu_1 + \frac{\mu_2}{8}\right)r - \frac{\mu_4}{2\mu_7}r - r^3\right]p(r)\right\} \\ & - \frac{\mu_4}{2\mu_7}\frac{\partial^2}{\partial r^2}\left(r^2 p(r)\right). \end{aligned} \quad (99)$$

Through calculation, we can obtain

$$p_{st}(r) = \frac{\exp\left(r^2\mu_7/\mu_4\right)r^{-1-(2(\mu_1+\mu_2/8)\mu_7/\mu_4)}}{\Gamma\left[-(\mu_1+\mu_2/8)\mu_7/\mu_4\right]\left(-\mu_7/\mu_4\right)^{(\mu_1+\mu_2/8)\mu_7/\mu_4}}. \quad (100)$$

According to Namachivaya's theory, the extreme value of an steady-state probability density contains the most important essence of the nonlinear stochastic system. In other words, the steady-state probability density can uncover the characteristic information of the steady state. When the intension of the noise tends to zero, the extreme values of

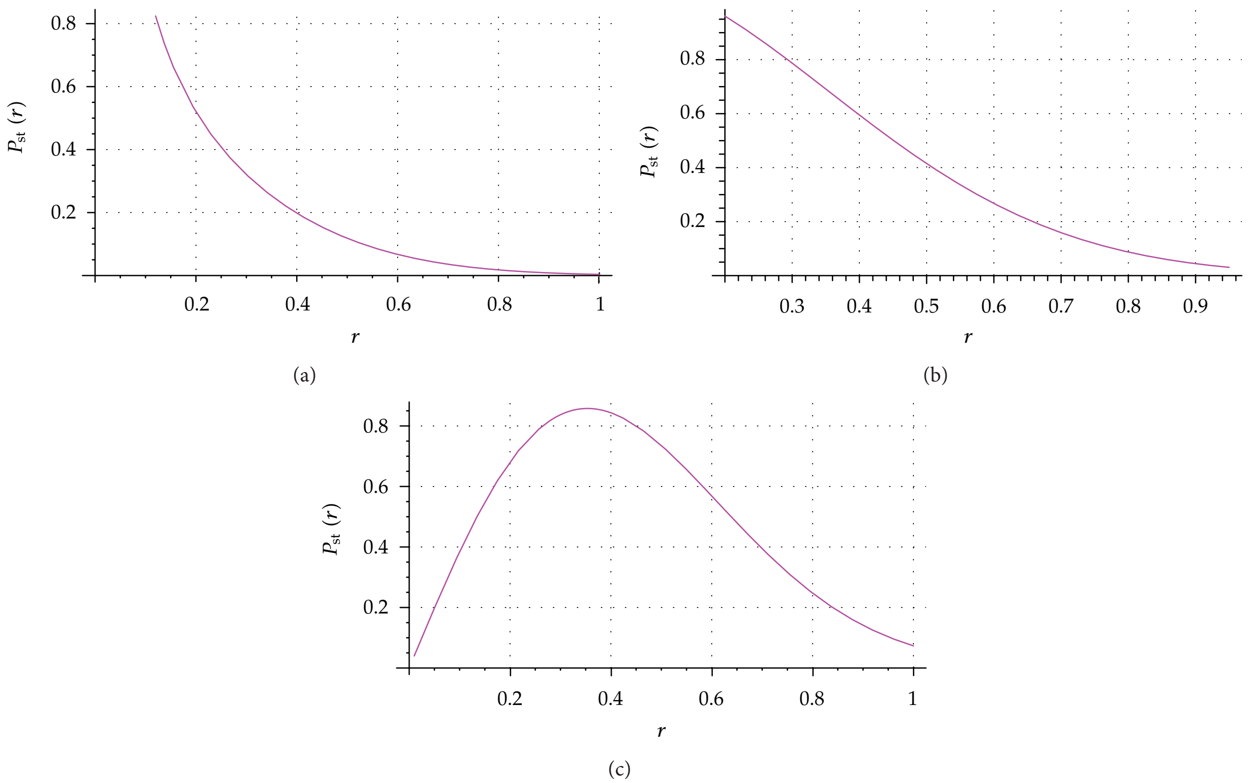

Figure 1: P-Bifurcation of system (55) with $\mu_1 = -0.2$, $\mu_3 = 0$, $\mu_4 = 5.1183$, $\mu_7 = -20.4732$; (a) P-bifurcation of $p_\alpha(r)$ at $\alpha_p < \sigma^2/2$, $\mu_2 = 5.6$, where $\alpha = \mu_1 + \mu_2/8$, $\sigma^2 = -\mu_4/\mu_7$; (b) P-bifurcation of $p_\alpha(r)$ at $\alpha_p = \sigma^2/2$, $\mu_2 = 9.5907$, where $\alpha = (\mu_1 + \mu_2/8)$, $\sigma^2 = -\mu_4/\mu_7$; (c) P-bifurcation of $p_\alpha(r)$ at $\alpha_p > \sigma^2/2$, $\mu_2 = 17.6$, where $\alpha = \nu_1/8$, $\sigma^2 = -\nu_3/\nu_2$.

$p_{st}(r)$ approximate to show the behavior of the deterministic system. If the process $r(t)$ is ergodic, then $p_{st}(r)$ can be regarded as the time measurement for staying in the neighborhood of $r(t)$ according to the Oseledec ergodic theorem.

From the analysis above, if $p_{st}(r)$ has a maximum value at r^*, the sample trajectory will stay for a longer time in the neighborhood of r^*, that is, r^* is stable in the meaning of probability (with a bigger probability). If $p_{st}(r)$ has a minimum value (zero), it is just the opposite.

We now calculate the most possible amplitude r^* of system (97), that is, $p_{st}(r)$ has a maximum value at r^*. So, we have

$$\left.\frac{dp_{st}(r)}{dr}\right|_{r=r^*} = 0, \qquad \left.\frac{d^2 p_{st}(r)}{dr^2}\right|_{r=r^*} < 0 \tag{101}$$

and the solution $r = \tilde{r} = \sqrt{(\nu_1\nu_2 + 4\nu_3)/8\nu_2}$. The probabilities and the positions of the Hopf bifurcation occur with different parameter.

Since

$$\begin{aligned}\left.\frac{d^2 p_{st}(r)}{dr^2}\right|_{=\tilde{r}} &= \left(2^{4+(3(\mu_1+\mu_2/8)\mu_2/\mu_3)} \exp\left(\frac{1}{8}\left(4+\frac{\mu_7}{\mu_3}\right)\right)\right. \\ &\quad \times \mu_7\left(-\frac{\mu_7}{\mu_4}\right)^{-(\mu_1+\mu_2/8)\mu_7/\nu_4} \\ &\quad \times \left.\left(\sqrt{\frac{(8(\mu_1+\mu_2/8)\mu_7+4\mu_4)}{\mu_7}}\right)^{-2(\mu_1+\mu_2/8)\mu_7/4\mu_4}\right) \\ &\quad \times \left(\Gamma\left[-\frac{(\mu_1+\mu_2/8)\mu_7}{\mu_4}\right]\left(-\frac{\mu_7}{\mu_4}\right)^{(\mu_1+\mu_2/8)\mu_7/\mu_4}\right)^{-1} \\ &< 0,\end{aligned} \tag{102}$$

thus, what we need is $r^* = \tilde{r}$. The conclusion is to go all the way with what has been obtained by the singular boundary theory. The original nonlinear stochastic system has a stochastic Hopf bifurcation at $r = \tilde{r}$, where

$$x_1^2 + x_2^2 = \frac{8(\mu_1 + \mu_2/8)\mu_7 + 4\mu_4}{8\mu_7}. \tag{103}$$

We now choose some values of the parameters in the equations, draw the graphics of $p_{st}(r)$ (Figure 1). It is worth putting forward that calculating the Hopf bifurcation with the parameters in the original system is necessary. If we now have values of the original parameters in system (55), then $\alpha = 0.778205$, $\beta = 0.024530$, $\kappa = 0.08$, $\gamma = 5.640593$, $\delta = 0.200101$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.2$, $\sigma_4 = 0.3$, $\sigma_5 = 0.1$, $\sigma_6 =$

0.5. After further calculations, we obtain $\mu_1 = -0.2$, $\mu_2 = 9.5907$, $\mu_3 = 0$, $\mu_4 = 5.1183$, $\mu_7 = -20.4732$, then

$$p(r) = 0.28209e^{-4r^2}. \quad (104)$$

What is more is that $\tilde{r} = 0.28209$, where $p_{st}(r)$ has the maximum value.

4. Conclusion

In this work, we investigate both the deterministic and stochastic bifurcations of the catalytic CO oxidation on Ir(111) surfaces with multiple delays. By replacing $\tilde{\gamma}$ with $\gamma - \gamma_c$, it can be readily seen that the addition of the noise gives rise to a shift in the critical bifurcation point γ_c at the Hopf bifurcation. The results of our computational studies using the Lyapunov exponents and the integral stochastic averaging combined with the Hopf bifurcation produced stability boundaries for structural systems that were much sharper and well defined than those obtained in the earlier stability studies of the same systems. The influence of noise on the reaction-diffusion system model has been studied by the authors [11–18], but explicit and derived conditions for stochastic stability and bifurcation, as the ones displayed here, were not transparent in their publications.

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