Research Article

Symmetry Reduced and New Exact Nontraveling Wave Solutions of (2+1)-Dimensional Potential Boiti-Leon-Manna-Pempinelli Equation

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With the aid of Maple symbolic computation and Lie group method, (2+1)-dimensional PBLMP equation is reduced to some (1+1)-dimensional PDE with constant coefficients. Using the homoclinic test technique and auxiliary equation methods, we obtain new exact nontraveling solution with arbitrary functions for the PBLMP equation.

1. Introduction

In this paper, we will consider the potential Boiti-Leon-Manna-Pempinelli (PBLMP) equation

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_xu_{xy} = 0, \tag{1}$$

where $u : R_x \times R_y \times R_t^+ \rightarrow R$. By some transformations, the PBLMP equation (1) can be equivalent to the asymmetric Nizhnik-Novikov-Veselov (ANNV) system. In fact, the ANNV equation can be obtained from the inner parameter-dependent symmetry constraint of the KP equation [1] and may be considered as a model for an incompressible fluid [2]. The Painlevé analysis, Lax pair, and some exact solutions have been studied for the PBLMP equation [3]. Tang and Lou obtained the bilinear form of (1) and variable separation solutions including two arbitrary functions by the multilinear variable separation approach [4, 5].

In this paper, by means of Maple symbolic computation, we will use the Lie group method [6, 7], homoclinic test technique [8, 9] and so forth to reduce and solve the PBLMP equation. First, we will derive symmetry of (1). Then we use the symmetry to reduce (1) to some (1 + 1)-dimensional PDE with constant coefficients. Finally, solving the reduced PDE by Homoclinic test technique and auxiliary equation methods [10, 11] implies abundant exact nontraveling wave periodic solutions for the PBLMP equation.

2. Symmetry of (1)

This section is devoted to Lie point group symmetries of (1). Let

$$\sigma = \sigma\left(x, y, t, u, u_x, u_y, u_t\right) \tag{2}$$

be the symmetry of (1). Based on Lie group theory [6], σ satisfies the following symmetry equation:

$$\sigma_{yt} + \sigma_{yxxx} - 3u_{xx}\sigma_y - 3u_y\sigma_{xx} - 3u_x\sigma_{xy} - 3u_{xy}\sigma_x = 0. \tag{3}$$

To get some symmetries of (1), we take the function σ in the form

$$\sigma = a\left(x, y, t\right)u_x + b\left(x, y, t\right)u_y + c\left(x, y, t\right)u_t \\ + d\left(x, y, t\right)u + e\left(x, y, t\right), \tag{4}$$

where a, b, c, d, e are functions of x, y, t to be determined, and $u(x, y, t)$ satisfies (1). Substituting (4) and (1) into (3), one can get

$$a = \frac{1}{3}k_1x + \lambda\left(t\right), \qquad b = \mu\left(y\right),$$
$$c = k_1t + k_2, \qquad d = \frac{1}{3}k_1, \qquad e = \frac{1}{3}\lambda'\left(t\right)x + \xi\left(t\right), \tag{5}$$

where k_1, k_2 are arbitrary constants. $\lambda(t), \xi(t)$ are arbitrary functions of t. $\mu(y)$ is a arbitrary function of y. Substituting (5) into (4), we obtain the symmetries of (1) as follows:

$$
\sigma = \left(\frac{1}{3} k_1 x + \lambda(t) \right) u_x + \mu(y) u_y + (k_1 t + k_2) u_t + \frac{1}{3} k_1 u + \frac{1}{3} \lambda'(t) x + \xi(t).
$$
(6)

3. Symmetry Reduction of (1)

Based on the integrability of reduced equation of the symmetry (6), we consider the following three cases.

Case 1. Let $k_1 = k_2 = 0$, $\lambda(t) = r$, $\xi(t) = 1$, $\mu(y) = -1/\tau(y)$ in (6), then

$$
\sigma = \tau(y)^{-1} \left(r\tau(y) u_x - u_y + \tau(y) \right),
$$
(7)

where r is an arbitrary nonzero constant, $\tau(y) \neq 0$. Solving the differential equation for $\sigma = 0$, one gets

$$
u = \int \tau(y) \, dy + w(\theta, t), \qquad \theta = x + \int r\tau(y) \, dy.
$$
(8)

Substituting (8) into (1), we get the following $(1+1)$-dimensional nonlinear PDE with constant coefficients:

$$
r w_{\theta\theta\theta\theta} - 6r w_\theta w_{\theta\theta} + r w_{\theta t} - 3 w_{\theta\theta} = 0.
$$
(9)

Integrating (9) once with respect to θ and taking integration constant to zero yield

$$
r w_{\theta\theta\theta} - 3r w_\theta^2 + r w_t - 3 w_\theta = 0.
$$
(10)

Case 2. Taking $k_1 = 0$, $k_2 = 1$, $\lambda(t) = 0$, $\xi(t) = 0$, $\mu(y) = 1/\tau(y)$ in (6) yields

$$
\sigma = \tau(y)^{-1} \left(u_y + r\tau(y) u_t \right).
$$
(11)

Solving the differential equation for $\sigma = 0$, one gets

$$
u = w(x, \theta), \qquad \theta = t - \int \tau(y) \, dy.
$$
(12)

Substituting (12) into (1), we have the function $w(x, \theta)$ which must satisfy the following PDE:

$$
w_{xxx\theta} - 3 w_{xx} w_\theta - 3 w_x w_{x\theta} + w_{\theta\theta} = 0.
$$
(13)

Case 3. Let $k_1 = k_2 = 0$, $\lambda(t) = 1$, $\xi(t) = 0$, $\mu(y) = -1/\tau(y)$ in (6), then

$$
\sigma = \tau(y)^{-1} \left(\tau(y) u_x - u_y \right).
$$
(14)

Solving the equation for $\sigma = 0$, we obtain

$$
u = w(\theta, t), \qquad \theta = x + \int \tau(y) \, dy.
$$
(15)

Substituting (15) into (1) yields a reduced PDE of (1) with constant coefficients:

$$
w_{\theta\theta\theta\theta} - 6 w_{\theta\theta} w_\theta + w_{\theta t} = 0.
$$
(16)

Integrating (16) once with respect to θ and taking integration constant to zero yield

$$
w_{\theta\theta\theta} - 3 w_\theta^2 + w_t = 0.
$$
(17)

Combining the above results, we obtain some reduced equations of (1) expressed by (10), (13), and (17), respectively. Meanwhile many new explicit solutions of (1) from these reduced Equations. can be achieved. We omit other cases based on symmetries (6) here.

4. Solve Reduced PDE and Get Exact Nontraveling Wave Solutions of (1)

In this section, we seek exact nontraveling wave solutions of (1) by using some appropriate methods to solve reduced equations (10), (13), and (17).

4.1. Solve Reduced PDE (10). Now, we seek solutions of (10) by auxiliary equation method. Make transformation as follows:

$$
w(\theta, t) = \varphi(\xi), \qquad \xi = p\theta + qt,
$$
(18)

where p, q are nonzero constants. Substituting (18) into (10) obtains an ordinary differential equation for $\varphi(\xi)$ as follows:

$$
p^3 r \varphi''' - 3r p^2 \varphi'^2 + (qr - 3p) \varphi' = 0,
$$
(19)

where $\varphi' = d\varphi/d\xi$. Let $\varphi' = f$, then (19) can be written as

$$
p^3 r f'' - 3r p^2 f^2 + (qr - 3p) f = 0.
$$
(20)

This is the fourth type of ellipse equation (12), its solutions are as follows:

$$
f = \begin{cases} -\dfrac{3p - qr}{2p^2 r} \operatorname{sech}^2 \left[\sqrt{\dfrac{3p - qr}{4p^3 r}} (\xi - \xi_0) \right], \\ \qquad\qquad pr(3p - qr) > 0, \\[2mm] \dfrac{3p - qr}{2p^2 r} \operatorname{csch}^2 \left[\sqrt{\dfrac{3p - qr}{4p^3 r}} (\xi - \xi_0) \right], \\ \qquad\qquad pr(3p - qr) > 0, \\[2mm] -\dfrac{3p - qr}{2p^2 r} \sec^2 \left[\sqrt{-\dfrac{3p - qr}{4p^3 r}} (\xi - \xi_0) \right], \\ \qquad\qquad pr(3p - qr) < 0, \end{cases}
$$
(21)

where ξ_0 is the integration constant. From the result of (21), some new exact solutions u_1 through u_3 of (1) can be obtained:

$$u_1 = \int \tau(y)\,dy - \sqrt{\frac{3p - qr}{pr}}$$

$$\times \tanh\left[\sqrt{\frac{3p - qr}{4p^3r}}\left(p\left(x + r\int \tau(y)\,dy\right) + qt - \xi_0\right)\right],$$

$$pr(3p - qr) > 0,$$

$$u_2 = \int \tau(y)\,dy - \sqrt{\frac{3p - qr}{pr}}$$

$$\times \coth\left[\sqrt{\frac{3p - qr}{4p^3r}}\left(p\left(x + r\int \tau(y)\,dy\right) + qt - \xi_0\right)\right],$$

$$pr(3p - qr) > 0,$$

$$u_3 = \int \tau(y)\,dy - \sqrt{\frac{qr - 3p}{pr}}$$

$$\times \tan\left[\sqrt{\frac{qr - 3p}{4p^3r}}\left(p\left(x + r\int \tau(y)\,dy\right) + qt - \xi_0\right)\right],$$

$$pr(qr - 3p) < 0. \tag{22}$$

Particularly, we assume $p = q = 1$, $r = 2$, $\tau(y) = \sin(y)$, $\xi_0 = 0$, $x = \mathrm{sech}(t)$, then the solution u_1 can be depicted by Figure 1(a). If $p = -1$, $q = 1$, $r = -1$, $\tau(y) = \mp\cos(y)$, $\xi_0 = 0$, $x = \sin(t)$, then u_3 can be depicted by Figures 1(b) and 2(a).

4.2. Solve Reduced PDE (13). Make transformation to (13) as follows:

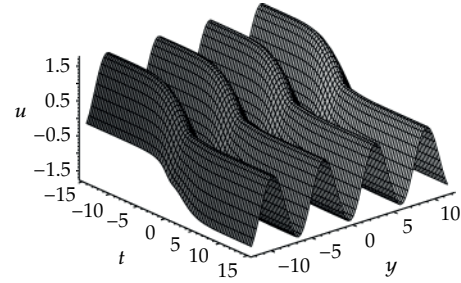$$w(x, \theta) = \varphi(\xi), \qquad \xi = kx + c\theta, \tag{23}$$

where k, c are non-zero constants. Substituting (23) into (13) then we have

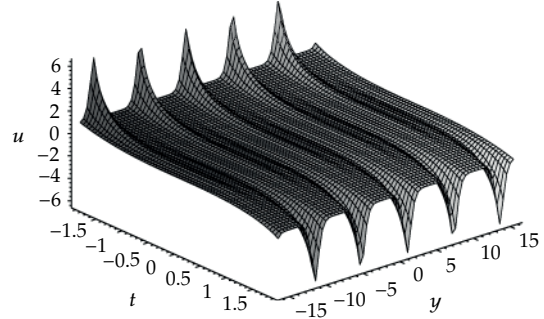$$c\varphi' + k^3\varphi''' - 3K^2\varphi'^2 = 0. \tag{24}$$

It is equivalent to (19). Based on the above accordant idea, we can get

$$u_4 = \sqrt{-\frac{c}{2k}}$$

$$\times \tanh\left[\sqrt{\frac{c}{2k^3}}\left(kx + c\left(t - \int \tau(y)\,dy\right) - \xi_0\right)\right], \quad kc < 0,$$

$$u_5 = \sqrt{-\frac{c}{2k}}$$

$$\times \coth\left[\sqrt{\frac{c}{2k^3}}\left(kx + c\left(t - \int \tau(y)\,dy\right) - \xi_0\right)\right], \quad kc < 0,$$
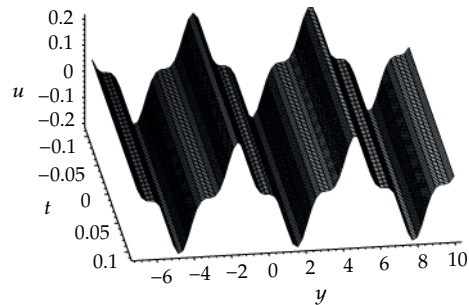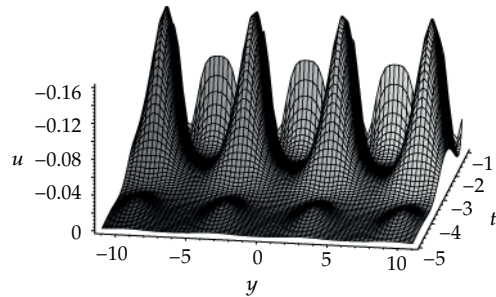

(a)

(b)

FIGURE 1: (a) The figure of u_1 as $p = 1$, $q = 1$, $r = 2$, $\tau(y) = \sin(y)$, $\xi_0 = 0$, $x = \mathrm{sech}(t)$. (b) The figure of u_3 as $p = -1$, $q = 1$, $r = -1$, $\tau(y) = -\cos(y)$, $\xi_0 = 0$, $x = \sin(t)$.

(a)

(b)

FIGURE 2: (a) The figure of u_3 as $p = -1$, $q = 1$, $r = -1$, $\tau(y) = \cos(y)$, $\xi_0 = 0$, $x = \sin(t)$. (b) The figure of u_9 as $p_1 = 1$, $c_1 = 1$, $p_2 = 1$, $\tau(y) = \sin(y)$, $x = \sin(t)$.

$$u_6 = \sqrt{\frac{c}{2k}}$$

$$\times \tan\left[\sqrt{\frac{c}{2k^3}}\left(kx+c\left(t-\int \tau(y)\,dy\right)-\xi_0\right)\right], \quad kc>0. \tag{25}$$

4.3. Solve Reduced PDE (17). In this section, we use homoclinic test technique [8, 9] to (17) and transform the unknown function as follows:

$$w(\theta, t) = -2(\ln f(\theta, t))_\theta. \tag{26}$$

Substituting (26) into (17) and using the bilinear form, we can get

$$\left(D_\theta D_t + D_\theta^4\right)(f \cdot f) = 0, \tag{27}$$

where the Hirota operator D is defined in [12]. In this case we choose extended homoclinic test function

$$f = e^{-p_1(\theta-\omega_1 t)} + c_1 \cos\left(p_2(\theta+\omega_2 t)\right) + c_2 e^{p_1(\theta-\omega_1 t)}, \tag{28}$$

where $p_2, \omega_1, \omega_2, c_1,$ and c_2 are real constants to be determined. Substituting (28) into (27) yields a set of algebraic equations as follows:

$$p_1 c_1 p_2\left(4\left(p_1^2 - p_2^2\right) + \omega_2 - \omega_1\right) = 0,$$

$$c_1\left(\left(p_1^4 + p_4^4 - 6p_1^2 p_2^2\right) - p_1^2\omega_1 - p_2^2\omega_2\right) = 0,$$

$$p_1 p_2 c_1 c_2\left(4\left(p_1^2 - p_2^2\right) + \omega_2 - \omega_1\right) = 0, \tag{29}$$

$$c_1 c_2\left(\left(p_1^4 + p_4^4 - 6p_1^2 p_2^2\right) - p_1^2\omega_1 - p_2^2\omega_2\right) = 0,$$

$$4\left(4p_1^4 c_2 + c_1^2 p_2^4\right) - 4p_1^2\omega_1 c_2 - c_1^2 p_2^2\omega_2 = 0.$$

Solving the above equations (29) yields

$$(1)\begin{cases} p_1 = p_1, & p_2 = p_2, & c_1 = 0, & c_2 = c_2, \\ \omega_1 = 4p_1^2, & \omega_2 = \omega_2, \end{cases} \tag{30}$$

$$(2)\begin{cases} p_1 = p_1, & p_2 = p_2, & c_1 = c_1, & c_2 = -\dfrac{c_1^2 p_2^2}{p_1^2}, \\ \omega_1 = -3p_2^2 + p_1^2, & \omega_2 = -3p_1^2 + p_2^2, \end{cases} \tag{31}$$

$$(3)\begin{cases} p_1 = p_2 i, & p_2 = p_2, & c_1 = c_1, & c_2 = c_2, \\ \omega_1 = -4p_2^2, & \omega_2 = 4p_2^2, \end{cases} \tag{32}$$

$$(4)\begin{cases} p_1 = p_2 i, & p_2 = p_2, & c_1 = c_1, & c_2 = \dfrac{1}{4}c_1^2, \\ \omega_1 = \omega_2 - 8p_2^2, & \omega_2 = \omega_2, \end{cases} \tag{33}$$

where $i^2 = -1$. Substituting (30)–(33) into (28) yields the solutions u_7 through u_{11} of (1) as follows:

$$u_7 = -2p_1 \tanh\left(p_1\left(x + \int \tau(y)\,dy\right) - 4p_1^2 t + \frac{1}{2}\ln c_2\right), \tag{34}$$

when $c_2 > 0$ in (30);

$$u_8 = -2p_1 \coth\left(p_1\left(x + \int \tau(y)\,dy\right) - 4p_1^2 t + \frac{1}{2}\ln(-c_2)\right), \tag{35}$$

when $c_2 < 0$ in (30);

$$u_9 = -2p_1 p_2$$

$$\times \left(\coth\left(p_1\left(x + \int \tau(y)\,dy\right)\right.\right.$$

$$\left. - \left(p_1^2 - 3p_2^2\right)t + \ln\frac{c_1 p_2}{p_1}\right)$$

$$\left. + \sin\left(p_2\left(x + \int \tau(y)\,dy\right) - \left(3p_1^2 - p_2^2\right)t\right)\right)$$

$$\times \left(p_2 \sinh\left(p_1\left(x + \int \tau(y)\,dy\right)\right.\right.$$

$$\left. - \left(p_1^2 - 3p_2^2\right)t + \ln\frac{c_1 p_2}{p_1}\right)$$

$$\left. + p_1 \cos\left(p_2\left(x + \int \tau(y)\,dy\right) - \left(3p_1^2 - p_2^2\right)t\right)\right)^{-1}, \tag{36}$$

when $c_1 p_1 p_2 > 0$ in (31) (see Figure 2(b));

$$u_{10}(x, y, t) = p_2 \tan\left(p_2\left(x + \int \tau(y)\,dy\right) + 4p_2^2 t\right), \tag{37}$$

when $c_2 = 1$ in (32);

$$u_{11}(x, y, t)$$

$$= -2p_2$$

$$\times \frac{\sin\left(p_2\left(x+\int \tau(y)\,dy\right)+\left(8p_2^2-\omega_2\right)t\right)+\sin\left(p_2\left(x+\int \tau(y)\,dy\right)+\omega_2 t\right)}{\cos\left(p_2\left(x+\int \tau(y)\,dy\right)+\left(8p_2^2-\omega_2\right)t\right)+\cos\left(p_2\left(x+\int \tau(y)\,dy\right)+\omega_2 t\right)}, \tag{38}$$

when $c_1 = 2$ in (33).

Remark 1. If one lets $w_\theta = v$ in (16), then (16) can be written as

$$v_t - 6vv_\theta + v_{\theta\theta\theta} = 0. \tag{39}$$

This is the famous KdV equation.

5. Conclusions

In this paper, a combination of Lie group method and homoclinic test technique and so forth is applied and thus the

symmetries (6) are obtained. The (2+1)-dimensional potential Boiti-Leon-Manna-Pempinelli equation (1) is reduced to $(1 + 1)$-dimensional nonlinear PDE of constant coefficients (10), (13), and (17). Further auxiliary equation method and homoclinic test technique are used and some new exact nontraveling wave solutions are obtained. And they include some special and strange structures to be further studied and other relevant solutions about symmetry (6) will be discussed later in another paper. Our results show that combining the Lie group method with homoclinic test technique and so forth is effective in finding nontraveling wave exact solutions of nonlinear evolution equations.

Acknowledgments

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