*Research Article*

# Almost Sequence Spaces Derived by the Domain of the Matrix $A^r$

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By using $A^r$, we introduce the sequence spaces $a_f^r$, $a_{f_0}^r$, and $a_{fs}^r$ of normed space and $BK$-space and prove that $a_f^r, a_{f_0}^r$, and $a_{fs}^r$ are linearly isomorphic to the sequence spaces $f$, $f_0$, and $fs$, respectively. Further, we give some inclusion relations concerning the spaces $a_f^r$, $a_{f_0}^r$, and the nonexistence of Schauder basis of the spaces $fs$ and $a_{fs}^r$ is shown. Finally, we determine the $\beta$- and $\gamma$-duals of the spaces $a_f^r$ and $a_{fs}^r$. Furthermore, the characterization of certain matrix classes on new almost convergent sequence and series spaces has exhaustively been examined.

## 1. Preliminaries, Background and Notation

By $w$, we will denote the space of all real or complex valued sequences. Any vector subspace of $w$ is called sequence space. We will write $\ell_\infty$, $c_0$, $c$, and $\ell_p$ for the spaces of all bounded, null, convergent, and absolutely $p$-summable sequences, respectively, which are $BK$-space with the usual sup-norm defined by $\|x\|_\infty = \sup_k |x_k|$ and $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$, for $1 < p < \infty$, where, here and in what follows, the summation without limits runs from 0 to $\infty$. Further, we will write $bs$, $cs$ for the spaces of all sequences associated with bounded and convergent series, respectively, which are $BK$-spaces with their natural norm [1].

Let $\mu$ and $\gamma$ be two sequence spaces and $A = (a_{nk})$ an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $\mu$ into $\gamma$ and we denote it by writing that $A : \mu \to \gamma$ and if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = (Ax)_n$, the $A$-transform of $x$ is in $\gamma$, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{1}$$

The notation $(\mu : \gamma)$ denotes the class of all matrices $A$ such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right hand side of (1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$ and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \gamma$ for all $x \in \mu$.

The matrix domain $\mu_A$ of an infinite matrix $A$ in a sequence space $\mu$ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\}. \tag{2}$$

The approach constructing a new sequence space by means of the matrix domain of a particular triangle has recently been employed by several authors in many research papers. For example, they introduced the sequence spaces $(c)_{C_1} = \tilde{c}$ in [2], $(\ell_p)_{A^r} = a_p^r$ and $(\ell_\infty)_{A^r} = a_\infty^r$ in [3], $\mu_G = Z(u, v; \mu)$ in [4], $(c_0)_\Lambda = c_0^\lambda$ and $c_\Lambda = c^\lambda$ in [5], and $(\ell_p)_{E^r} = e_p^r$ and $(\ell_\infty)_{E^r} = e_\infty^r$ in [6]. Recently, matrix domains of the generalized difference matrix $B(r, s)$ and triple band matrix $B(r, s, t)$ in the sets of almost null and almost convergent sequences have been investigated by Başar and Kirişçi [7] and Sönmez [8], respectively. Later, Kayaduman and Şengönül introduced some almost convergent spaces which are the matrix domains of the Riesz matrix and Cesàro matrix of order 1 in the sets of almost null and almost convergent sequences (see [9, 10]).

We now focus on the sets of almost convergent sequences. A continuous linear functional $\phi$ on $\ell_\infty$ is called a Banach limit if (i) $\phi(x) \geq 0$ for $x = (x_k)$ and $x_k \geq 0$ for every $k$, (ii) $\phi(x_{\sigma(k)}) = \phi(x_k)$, where $\sigma$ is shift operator which is defined on $\omega$ by $\sigma(k) = k + 1$, and (iii) $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to

the generalized limit $\alpha$ if all Banach limits of $x$ are $\alpha$ [11] and denoted by $f - \lim x = \alpha$. In other words, $f - \lim x_k = \alpha$ uniformly in $n$ if and only if

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{k=0}^{m} x_{k+n} \quad \text{uniformly in } n. \tag{3}$$

The characterization given above was proved by Lorentz in [11]. We denote the sets of all almost convergent sequences $f$ and series $fs$ by

$$f = \left\{ x = (x_k) \in \omega : \lim_{m \to \infty} t_{mn}(x) = \alpha \quad \text{uniformly in } n \right\}, \tag{4}$$

where

$$t_{mn}(x) = \sum_{k=0}^{m} \frac{1}{m+1} x_{k+n}, \quad t_{-1,n} = 0,$$

$$fs = \left\{ x = (x_k) \in \omega : \right.$$

$$\exists l \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \frac{x_j}{m+1} = l$$

$$\left. \text{uniformly in } n \right\}. \tag{5}$$

We know that the inclusions $c \subset f \subset \ell_\infty$ strictly hold. Because of these inclusions, norms $\| \cdot \|_f$ and $\| \cdot \|_\infty$ of the spaces $f$ and $\ell_\infty$ are equivalent. So the sets $f$ and $f_0$ are BK-spaces with the norm $\|x\|_f = \sup_{m,n} |t_{mn}(x)|$.

The rest of this paper is organized, as follows. We give foreknowledge on the main argument of this study and notations in this section. In Section 2, we introduce the almost convergent sequence and series spaces $a_{fs}^r$ and $a_f^r$ which are the matrix domains of the $A^r$ matrix in the almost convergent sequence and series spaces $fs$ and $f$, respectively. In addition, we give some inclusion relations concerning the spaces $a_f^r$, $a_{f_0}^r$, and the non-existence of Schauder basis of the spaces $fs$ and $a_{fs}^r$ is shown to give certain theorems related to behavior of some sequences. In Section 3, we determine the beta- and gamma-duals of the spaces $a_f^r$ and $a_{fs}^r$ and characterize the classes $(\gamma : a_f^r)$, $(a_f^r : \mu)$, $(\delta : a_{fs}^r)$ and $(a_f^r : \theta)$, where $\gamma \in \{c(p), c_0(p), \ell_\infty(p), cs, bs, fs, f, c, \ell_\infty\}$, $\mu \in \{cs, c, \ell_\infty\}$, $\delta \in \{cs, fs, bs\}$, and $\theta \in \{f, c, fs, \ell_\infty\}$, where $c(p), c_0(p)$, and $\ell_\infty(p)$ denote the space of Maddox convergent, null and bounded sequence spaces defined by Maddox [12].

**Lemma 1** (see [13]). *The set $fs$ has no Schauder basis.*

## 2. The Sequence Spaces $a_f^r$, $a_{f_0}^r$, and $a_{fs}^r$ Derived by the Domain of the Matrix $A^r$

In the present section, we introduce the sequence spaces $a_f^r$, $a_{f_0}^r$, and $a_{fs}^r$ as the set of all sequences such that $A^r$-transforms

of them are in the spaces $f$, $f_0$, and $fs$, respectively. Further, this section is devoted to examination of the basic topological properties of the sets $a_f^r$, $a_{f_0}^r$, and $a_{fs}^r$. Recently, Aydın and Başar [14] studied the sequence spaces $a_c^r$ and $a_0^r$:

$$a_c^r = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( 1 + r^k \right) x_k \text{ exists} \right\},$$

$$a_0^r = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \left( 1 + r^k \right) x_k = 0 \right\}, \tag{6}$$

where $A^r$ denotes the matrix $A^r = (a_{nk}^r)$ defined by

$$a_{nk}^r = \begin{cases} \dfrac{1 + r^k}{n+1} & (0 \leqslant k \leqslant n), \\ 0 & (k > n). \end{cases} \tag{7}$$

Now we introduce the sequence spaces $a_f^r$, $a_f^r$, and $a_{fs}^r$ as the sets of all sequences such that their $A^r$-transforms are in the spaces $f$, $f_0$, and $fs$, respectively; that is,

$$a_f^r = \left\{ x = (x_k) \in \omega : \right.$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} \frac{1}{k+1} \left( 1 + r^i \right) x_{n+i}$$

$$\left. = \alpha \quad \text{uniformly in } n \right\},$$

$$a_{f_0}^r = \left\{ x = (x_k) \in \omega : \right.$$

$$\lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} \frac{1 + r^i}{k+1} x_{n+i} = 0$$

$$\left. \text{uniformly in } n \right\},$$

$$a_{fs}^r = \left\{ x = (x_k) \in \omega : \right.$$

$$\exists \beta \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} \sum_{j=0}^{n+k} \sum_{i=0}^{j} \frac{1 + r^j}{i+1} x_{n+j}$$

$$\left. = \beta \quad \text{uniformly in } n \right\}. \tag{8}$$

We can redefine the spaces $a_{fs}^r$, $a_f^r$, and $a_{f_0}^r$ by the notation of (2):

$$a_{f_0}^r = (f_0)_{A^r}, \qquad a_f^r = f_{A^r}, \qquad a_{fs}^r = (fs)_{A^r}. \tag{9}$$

It is known by Başar [15] that the method is regular for $0 < r < 1$. We assume unless stated otherwise that $0 < r < 1$.

Define the sequence $y = (y_k)$, which will be frequently used, as the $A^r$-transform of a sequence $x = (x_k)$; that is,

$$y_k(r) = \sum_{i=0}^{k} \frac{1+r^i}{k+1} x_i \ (k \in \mathbb{N}). \tag{10}$$

**Theorem 2.** *The spaces $a_f^r$ and $a_{fs}^r$ have no Schauder basis.*

*Proof.* Since it is known that the matrix domain $\mu_A$ of a normed sequence space $\mu$ has a basis if and only if $\mu$ has a basis whenever $A = (a_{nk})$ is a triangle [16, Remark 2.4] and the space $f$ has no Schauder basis by [7, Corollary 3.3], we have that $a_f^r$ has no Schauder basis. Since the set $fs$ has no basis in Lemma 1, $a_{fs}^r$ has no Schauder basis. □

**Theorem 3.** *The following statements hold.*

(i) *The sets $a_f^r$ and $a_{f_0}^r$ are linear spaces with the coordinatewise addition and scalar multiplication which are BK-spaces with the norm*

$$\|x\|_{a_f^r} = \sup_m \left| \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} \frac{1+r^i}{k+1} x_{i+n} \right|. \tag{11}$$

(ii) *The set $a_{fs}^r$ is a linear space with the coordinatewise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\|_{a_{fs}^r} = \sup_m \left| \sum_{k=0}^{m} \frac{1}{m+1} \sum_{j=0}^{k+n} \sum_{i=0}^{j} \frac{1+r^i}{j+1} x_i \right|. \tag{12}$$

*Proof.* Since the second part can be similarly proved, we only focus on the first part. Since the sequence spaces $f$ and $f_0$ endowed with the norm $\| \cdot \|_\infty$ are BK-spaces (see [1, Example 7.3.2(b)]) and the matrix $A^r = (a_{nk}^r)$ is normal, Theorem 4.3.2 of Wilansky [17, p.61] gives the fact that the spaces $a_f^r$ and $a_{f_0}^r$ are BK-spaces with the norm in (11). □

Now, we may give the following theorem concerning the isomorphism between our spaces and the sets $f$, $f_0$, and $fs$.

**Theorem 4.** *The sequence spaces $a_f^r$, $a_{f_0}^r$, and $a_{fs}^r$ are linearly isomorphic to the sequence spaces $f$, $f_0$, and $fs$, respectively; that is, $a_f^r \cong f$, $a_{f_0}^r \cong f_0$, and $a_{fs}^r \cong fs$.*

*Proof.* To prove the fact that $a_f^r \cong f$, we should show the existence of a linear bijection between the spaces $a_f^r$ and $f$. Consider the transformation $T$ defined with the notation of (2) from $a_f^r$ to $f$ by $x \mapsto y = Tx = A^r x$. The linearity of $T$ is clear. Further, it is clear that $x = \theta$ whenever $Tx = \theta$, and hence, $T$ is injective.

Let $y = (y_k) \in a_f^r$, and define the sequence $x = (x_k(r))$ by

$$x_k = \frac{1}{1+r^k} \left[ (k+1) y_k - k y_{k-1} \right] \quad \text{for each } k \in \mathbb{N}, \tag{13}$$

whence

$$f_{A^r} - \lim x$$

$$= \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1}$$

$$\times \sum_{i=0}^{k} \frac{(1+r^i) x_{i+n}}{1+k} \quad \text{uniformly in } n$$

$$= \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1}$$

$$\times \sum_{i=0}^{k} \frac{(1+r^i) \left[ 1/(1+r^i) (y_{i+n}(k+1) - y_{i+n-1}k) \right]}{1+k}$$

$$\text{uniformly in } n$$

$$= \lim_{m \to \infty} \frac{1}{m+1} \sum_{k=0}^{m} y_{k+n} \quad \text{uniformly in } n$$

$$= f - \lim y \tag{14}$$

which implies that $x \in a_f^r$. As a result, $T$ is surjective. Hence, $T$ is a linear bijection which implies that the spaces $a_f^r$ and $f$ are linearly isomorphic, as desired. Similarly, the isomorphisms $a_{f_0}^r \cong f_0$ and $a_{fs}^r \cong fs$ can be proved. □

**Theorem 5.** *The inclusion $f \subset a_f^r$ strictly holds.*

*Proof.* Let $x = (x_k) \in c$. Since $c \subset f$, $x \in f$. Because $A^r$ is regular for $0 < r < 1$, $A^r x \in c$. Therefore, since $\lim A^r x = f - \lim A^r x$, we see that $x \in a_f^r$. So we have that the inclusion $f \subset a_f^r$ holds. Further, consider the sequence $t = (t_k(r))$ defined by $t_k(r) = (2k+1)/(1+r^k)(-1)^k \ \forall k \in \mathbb{N}$. Then, since $A^r t = (-1)^n \in f$, $x \in a_f^r$. One can easily see that $t \notin f$. Thus, $t \in a_f^r \setminus f$, and this completes the proof. □

**Theorem 6.** *The sequence spaces $a_f^r$ and $\ell_\infty$ overlap, but neither of them contains the other.*

*Proof.* Let us consider the sequence $u = (u_k(r))$ defined by $u_k(r) = 1/(1+r^k)$ for all $\mathbb{N}$. Then, since $A^r u = e \in f$, $u \in a_f^r$. It is clear that $u \in \ell_\infty$. This means that the sequence spaces $a_f^r$ and $\ell_\infty$ are not disjoint. Now, we show that the sequence space $a_f^r$ and $\ell_\infty$ do not include each other. Let us consider the sequence $t = (t_k(r))$ defined as in proof of Theorem 5 above and $z = (z_k(r)) = (0, \ldots, 0, 1/(1 + r^{101}), \ldots, 1/(1 + r^{110}), 0, \ldots, 0, 1/(1 + r^{211}), \ldots, 1/(1 + r^{231}), 0, \ldots, 0, \ldots)$ where the blocks of 0's are increasing by factors of 100 and the blocks of $1/(1 + r^k)$'s are increasing by factors of 10. Then, since $A^r t = (-1)^n \in f$, $t \in a_f^r$, but $t \notin \ell_\infty$. Therefore, $t \in a_f^r \setminus \ell_\infty$. Also, the sequence $z \notin a_f^r$ since $A^r z = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, \ldots) \notin f$ where the blocks of 0's are increasing by factors of 100 and the blocks of 1's are increasing by factors of 10, but $z$ is bounded. This means that $z \in \ell_\infty \setminus a_f^r$. Hence, the

sequence spaces $a_f^r$ and $\ell_\infty$ overlap, but neither of them contains the other. This completes the proof. ∎

**Theorem 7.** *Let the spaces $a_{f_0}^r$, $a_c^r$, and $a_f^r$ be given. Then,*

(i) $a_{f_0}^r \subset a_f^r$ *strictly hold;*

(ii) $a_c^r \subset a_f^r$ *strictly hold.*

*Proof.* (i) Let $x = (x_k) \in a_{f_0}^r$ which means that $A^r x \in f_0$. Since $f_0 \subset f$, $A^r x \in f$. This implies that $x \in a_f^r$. Thus, we have $a_{f_0}^r \subset a_f^r$.

Now, we show that this inclusion is strict. Let us consider the sequence $u = (u_k(r))$ defined as in proof of Theorem 6 for all $k \in \mathbb{N}$. Consider the following:

$$f_{A^r} - \lim u = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1} \sum_{i=0}^{k} \frac{1+r^i}{k+1} u_{i+n}$$

$$= \lim_{m \to \infty} \sum_{k=0}^{m} \frac{1}{m+1} = e = (1, 1, \ldots) \tag{15}$$

which means that $u \in a_f^r \setminus a_{f_0}^r$; that is to say, the inclusion is strict.

(ii) Let $x = (x_k) \in a_c^r$ which means that $A^r x \in c$. Since $c \subset f$, $A^r x \in f$. This implies that $x \in a_f^r$. Thus, we have $a_c^r \subset a_f^r$. Furthermore, let us consider the sequence $t = \{t_k(r)\}$ defined as in proof of Theorem 5 for all $k \in \mathbb{N}$. Then, since $A^r t = (-1)^n \in f \setminus c$, $t \in a_f^r \setminus a_c^r$. This completes the proof.

∎

## 3. Certain Matrix Mappings on the Sets $a_f^r$, $a_{fs}^r$ and Some Duals

In this section, we will characterize some matrix transformations between the spaces of $A^r$ almost convergent sequence and almost convergent series in addition to paranormed and classical sequence spaces after giving $\beta$- and $\gamma$-duals of the spaces $a_{fs}^r$ and $a_f^r$. We start with the definition of the beta- and gamma-duals.

If $x$ and $y$ are sequences and $X$ and $Y$ are subsets of $\omega$, then we write $x \cdot y = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : a \cdot x \in Y \;\forall x \in X\} \tag{16}$$

for the multiplier space of $X$ and $Y$. One can easily observe for a sequence space $Z$ with $Y \subset Z$ and $Z \subset X$ that inclusions $M(X, Y) \subset M(X, Z)$ and $M(X, Y) \subset M(Z, Y)$ hold, respectively. The $\alpha$-, $\beta$-, and $\gamma$-duals of a sequence space, which are, respectively, denoted by $X^\alpha$, $X^\beta$, and $X^\gamma$, are defined by

$$X^\alpha = M(X, \ell_1), \qquad X^\beta = M(X, cs), \qquad X^\gamma = M(X, bs). \tag{17}$$

It is obvious that $X^\alpha \subset X^\beta \subset X^\gamma$. Also, it can easily be seen that the inclusions $X^\alpha \subset Y^\alpha$, $X^\beta \subset Y^\beta$, and $X^\gamma \subset Y^\gamma$ hold whenever $Y \subset X$.

**Lemma 8** (see [18]). *$A = (a_{nk}) \in (f : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{18}$$

**Lemma 9** (see [18]). *$A = (a_{nk}) \in (f : c)$ if and only if (18) holds and there are $\alpha, \alpha_k \in \mathbb{C}$ such that*

$$\lim_{n \to \infty} a_{nk} = \alpha_k \quad \forall k \in \mathbb{N}, \tag{19}$$

$$\lim_{n \to \infty} \sum_k a_{nk} = \alpha, \tag{20}$$

$$\lim_{n \to \infty} \sum_k |\Delta(a_{nk} - \alpha_k)| = 0. \tag{21}$$

**Theorem 10.** *Define the sets $t_1^r$ and $t_2^r$ by*

$$t_1^r = \left\{ a = (a_k) \in \omega : \sum_k \left| \Delta\left( \frac{a_k}{1+r^k} \right)(k+1) \right| < \infty \right\},$$

$$t_2^r = \left\{ a = (a_k) \in \omega : \sup_k \left| \frac{a_k(k+1)}{1+r^k} \right| < \infty \right\}, \tag{22}$$

*where $\Delta(a_k/(1+r^k)) = a_k/(1+r^k) - a_{k+1}/(1+r^{k+1})$ for all $k \in \mathbb{N}$. Then $(a_f^r)^\gamma = t_1^r \cap t_2^r$.*

*Proof.* Take any sequence $a = (a_k) \in \omega$, and consider the following equality:

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left[ \sum_{i=k-1}^{k} (-1)^{k-j} \frac{i+1}{1+r^i} y_i \right]$$

$$= \sum_{k=0}^{n-1} \Delta\left( \frac{a_k}{1+r^k} \right)(k+1) y_k \tag{23}$$

$$+ \frac{n+1}{1+r^n} a_n y_n$$

$$= (Ty)_n,$$

where $T = \{t_{nk}^r\}$ is

$$t_{nk}^r = \begin{cases} \Delta\left( \dfrac{a_k}{1+r^k} \right)(k+1) & (0 \leqslant k \leqslant n-1) \\ \dfrac{n+1}{1+r^n} a_n & (k=n) \\ 0 & (k>n) \end{cases} \tag{24}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from (23) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in a_f^r$ if and only if $Ty \in \ell_\infty$ whenever $y = (y_k) \in f$ where $T = \{t_{nk}^r\}$ is defined in (24). Therefore, with the help of Lemma 8, $(a_f^r)^\gamma = t_1^r \cap t_2^r$. ∎

**Theorem 11.** *The $\beta$-dual of the space $a_f^r$ is the intersection of the sets*

$$t_3^r = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_k |\Delta(t_{nk}^r - \alpha_k)| = 0 \right\},$$

$$t_4^r = \left\{ a = (a_k) \in \omega : \left( \frac{k+1}{1+r^k} a_k \right) \in cs \right\}, \tag{25}$$

*where $\lim_{n \to \infty} t_{nk}^r = \alpha_k$ for all $k \in \mathbb{N}$. Then, $(a_f^r)^\beta = t_3^r \cap t_4^r$.*

*Proof.* Let us take any sequence $a \in \omega$. By (23), $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in a_f^r$ if and only if $Ty \in c$ whenever $y = (y_k) \in f$. It is obvious that the columns of that matrix $T$ in $c$ where $T = \{t_{nk}^r\}$ defined in (24), we derive the consequence by Lemma 9 that $(a_f^r)^\beta = t_3^r \cap t_4^r$. $\square$

**Theorem 12.** *The $\gamma$-dual of the space $a_{fs}^r$ is the intersection of the sets*

$$c_1^r = \left\{ a = (a_k) \in \omega : \sum_k \left| \Delta \left[ \Delta \left( \frac{a_k}{1 + r^k} \right) (k + 1) \right. \right. \right.$$

$$\left. \left. \left. + \frac{a_k}{1 + r^k} (k + 1) \right] \right| < \infty \right\},$$

$$c_2^r = \left\{ a = (a_k) \in \omega : \left( \frac{a_k (k + 1)}{1 + r^k} \right) \in c_0 \right\}.$$

$$(26)$$

*In other words, we have $(a_{fs}^r)^\gamma = c_1^r \cap c_2^r$.*

*Proof.* We obtain from (23) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in a_{fs}^r$ if and only if $Ty \in \ell_\infty$ whenever $y = (y_k) \in fs$, where $T = \{t_{nk}^r\}$ is defined in (24). Therefore, by Lemma 19 (viii), $(a_{fs}^r)^\gamma = c_1^r \cap c_2^r$. $\square$

**Theorem 13.** *Define the set $c_3^r$ by*

$$c_3^r = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_k \left| \Delta^2 (t_{nk}^r) \right| \text{ exists} \right\}. \quad (27)$$

*Then, $(a_{fs}^r)^\beta = c_1^r \cap c_2^r \cap c_3^r$.*

*Proof.* This may be obtained in the same way as mentioned in the proof of Theorem 12 with Lemma 19(viii) instead of Lemma 19(vii). So we omit details. $\square$

For the sake of brevity, the following notations will be used:

$$a(n, k, m) = \frac{1}{m + 1} \sum_{i=0}^m a_{n+i,k}, \qquad a(n, k) = \sum_{i=0}^n a_{ik},$$

$$\widehat{a}_{nk} = \Delta \left( \frac{a_{nk}}{1 + r^k} \right) (k + 1) = \left( \frac{a_{nk}}{1 + r^k} - \frac{a_{n,k+1}}{1 + r^{k+1}} \right) (k + 1),$$

$$\Delta a_{nk} = a_{n,k} - a_{n,k+1}, \qquad \widetilde{a}_{nk} = \sum_{j=0}^n \frac{(1 + r^j) e_{jk}}{n + 1},$$

$$(28)$$

for all $k, n \in \mathbb{N}$. Assume that the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with relation (10) to the sequences $u = (u_n)$ and $v = (v_n)$, respectively; that is,

$$u_n = (Ax)_n = \sum_k a_{nk} x_k \quad \forall n \in \mathbb{N}, \quad (29)$$

$$v_n = (By)_n = \sum_k b_{nk} y_k \quad \forall n \in \mathbb{N}. \quad (30)$$

One can easily conclude here that the method $A$ is directly applied to the terms of the sequence $x = (x_k)$, while the method $B$ is applied to the $A^r$-transform of the sequence $x = (x_k)$. So the methods $A$ and $B$ are essentially different.

Now, suppose that the matrix product $BA^r$ exists which is a much weaker assumption than the conditions on the matrix $B$ belonging to any matrix class, in general. It is not difficult to see that the sequence in (30) reduces to the sequence in (29) under the application of formal summation by parts. This leads us to the fact that $BA^r$ exists and is equal to $A$ and $(BA^r)x = B(A^r x)$ formally holds if one side exists. This statement is equivalent to the following relation between the entries of the matrices $A = (a_{nk})$ and $B = (b_{nk})$ which are connected with the relation

$$\widehat{a}_{nk} = b_{nk} = \Delta \left( \frac{a_{nk}}{1 + r^k} \right) (k + 1) \text{ or}$$

$$a_{nk} = (1 + r^k) \sum_{j=k}^\infty \frac{b_{nj}}{1 + j} \quad (31)$$

$$\forall k, n \in \mathbb{N}.$$

Note that the methods $A$ and $B$ are not necessarily equivalent since the order of summation may not be reversed. We now give the following fundamental theorem connected with the matrix mappings on/into the almost convergent spaces $a_f^r$ and $a_{fs}^r$.

**Theorem 14.** *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with relation (31) for all $k, n \in \mathbb{N}$, and let $\lambda$ be any given sequence space. Then, $A \in (a_f^r : \lambda)$ if and only if*

$$B \in (f : \lambda),$$

$$\left\{ \frac{n + 1}{1 + r^k} a_{nk} \right\}_{k \in \mathbb{N}} \in c_0. \quad (32)$$

*Proof.* Suppose that $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation (31), and let $\lambda$ be any given sequence space, and keep in mind that the spaces $a_f^r$ and $f$ are norm isomorphic.

Let $A \in (a_f^r : \lambda)$, and take any sequence $x \in a_f^r$, and keep in mind that $y = A^r x$. Then, $(a_{nk})_{k \in \mathbb{N}} \in (a_f^r)^\beta$; that is, (32) holds for all $n \in \mathbb{N}$ and $BA^r$ exists which implies that $(b_{nk})_{k \in \mathbb{N}} \in \ell_1 = f^\beta$ for each $n \in \mathbb{N}$. Thus, $By$ exists for all $y \in f$, and thus, we have $m \to \infty$ in the equality

$$\sum_{k=0}^m b_{nk} y_k = \sum_{k=0}^m \sum_{j=k}^m (1 + r^k) \frac{b_{nj}}{1 + j} x_k \quad (33)$$

for all $m, n \in \mathbb{N}$, and we have (31) $By = Ax$ which means that $B \in (f : \lambda)$. On the other hand, assume that (32) holds and $B \in (f : \lambda)$. Then, we have $(b_{nk})_{k \in \mathbb{N}} \in \ell_1$ for all $n \in \mathbb{N}$ which

gives together with $(v_{nk})_{k\in\mathbb{N}} \in (a_f^r)^\beta$ for each $n \in \mathbb{N}$ that $Ax$ exists. Then, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \Delta \left( \frac{a_{nk}}{1+r^k} \right) (k+1) y_k + \frac{m+1}{1+r^m} a_{nm} y_m$$

$$= \sum_{k=0}^{m} b_{nk} y_k \qquad (34)$$

for all $m, n \in \mathbb{N}$, as $m \to \infty$, that $Ax = By$, and this shows that $A \in (a_f^r : \lambda)$. $\square$

**Theorem 15.** *Suppose that the entries of the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$ are connected with the relation*

$$f_{nk} = \tilde{a}_{nk} \qquad (35)$$

*for all $m, n \in \mathbb{N}$ and $\lambda$ is any given sequence space. Then, $E \in (\lambda : a_f^r)$ if and only if $F \in (\lambda : f)$.*

*Proof.* Let $x = (x_k) \in \lambda$, and consider the following equality:

$$\sum_{j=0}^{n} \sum_{k=0}^{m} \frac{1+r^j}{n+1} e_{jk} x_k = \sum_{k=0}^{m} f_{nk} x_k \qquad (36)$$

for all $k, m, n \in \mathbb{N}$, which yields as $m \to \infty$ that $Ex \in a_f^r$ whenever $x \in \lambda$ if and only if $Fx \in f$ whenever $x \in \lambda$. This step completes the proof. $\square$

**Theorem 16.** *Let $\lambda$ be any given sequence space, and the matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation (31). Then, $A \in (a_{fs}^r : \lambda)$ if and only if $B \in (fs : \lambda)$ and $(a_{nk})_{k\in\mathbb{N}} \in (a_{fs}^r)^\beta$ for all $n \in \mathbb{N}$.*

*Proof.* The proof is based on the proof of Theorem 14. $\square$

**Theorem 17.** *Let $\lambda$ be any given sequence space, and the elements of the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$ are connected with relation (35). Then, $E = (e_{nk}) \in (\lambda : a_{fs}^r)$ if and only if $F \in (\lambda : fs)$.*

*Proof.* The proof is based on the proof of Theorem 15. $\square$

By Theorems 14, 15, 16, and 17, we have quite a few outcomes depending on the choice of the space $\lambda$ to characterize certain matrix mappings. Hence, by the help of these theorems, the necessary and sufficient conditions for the classes $(a_f^r : \lambda)$, $(\lambda : a_f^r)$, $(a_{fs}^r : \lambda)$ and $(\lambda : a_{fs}^r)$ may be derived by replacing the entries of $A$ and $B$ by those of $B = A(A^r)^{-1}$, and $F = A^r E$, respectively, where the necessary and sufficient conditions on the matrices $E$ and $F$ are read from the concerning results in the existing literature

**Lemma 18.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) $A \in (c_0(p) : f)$ *if and only if*

$$\exists N > 1 \ni \sup_{m\in\mathbb{N}} \sum_k |a(n,k,m)| N^{1/p_k} < \infty, \quad \forall n \in \mathbb{N},$$

$$\exists \alpha_k \in \mathbb{C} \quad \forall k \in \mathbb{N} \ni \lim_{m\to\infty} a(n,k,m) = \alpha_k \quad \textit{uniformly in } n; \qquad (37)$$

(ii) $A \in (c(p) : f)$ *if and only if* (37) *and*

$$\exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_k a(n,k,m) = \alpha \quad \textit{uniformly in } n; \qquad (38)$$

(iii) $A \in (\ell_\infty(p) : f)$ *if and only if* (37) *and*

$$\exists N > 1 \ni \lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| N^{1/p_k} = 0 \qquad (39)$$

$$\textit{uniformly in } n.$$

**Lemma 19.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) *(Duran, [19]) $A \in (\ell_\infty : f)$ if and only if (18) holds and*

$$f - \lim a_{nk} = \alpha_k \textit{ exists for each fixed } k, \qquad (40)$$

$$\lim_{m\to\infty} \sum_k |a(n,k,m) - \alpha_k| = 0 \quad \textit{uniformly in } n; \qquad (41)$$

(ii) *(King, [20]) $A \in (c : f)$ if and only if (18), (40) hold and*

$$f - \lim \sum_k a_{nk} = \alpha; \qquad (42)$$

(iii) *(Başar and Çolak, [21]) $A \in (cs : f)$ if and only if (40) holds and*

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a_{nk}| < \infty; \qquad (43)$$

(iv) *(Başar and Çolak, [21]) $A \in (bs : f)$ if and only if (40), (43) hold and*

$$\lim_k a_{nk} = 0 \quad \textit{exists for each fixed } n, \qquad (44)$$

$$\lim_{q\to\infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} |\Delta [a(n+i,k) - \alpha_k]| = 0 \quad \textit{uniformly in } n; \qquad (45)$$

(v) *(Duran, [19]) $A \in (f : f)$ if and only if (18), (40), and (42) hold and*

$$\lim_{m\to\infty} \sum_k |\Delta [a(n,k,m) - \alpha_k]| = 0 \quad \textit{uniformly in } n; \qquad (46)$$

(vi) *(Başar, [22]) $A \in (fs : f)$ if and only if (40), (44), (46), and (45) hold;*

(vii) *(Öztürk, [23]) $A \in (fs : c)$ if and only if (19), (43), and (44) hold and*

$$\lim_{n \to \infty} \sum_k \left| \Delta^2 a_{nk} \right| = \alpha;$$

(47)

(viii) *$A \in (fs : \ell_\infty)$ if and only if (43) and (44) hold;*

(ix) *(Başar and Solak, [24]) $A \in (bs : fs)$ if and only if (44), (45) hold and*

$$\sup_{n \in \mathbb{N}} \sum_k |\Delta a(n,k)| < \infty,$$

(48)

*$f - \lim a(n,k) = \alpha_k$ exists for each fixed $k$;*

(x) *(Başar, [22]) $A \in (fs : fs)$ if and only if (45), (48) hold and*

$$\lim_{q \to \infty} \sum_k \frac{1}{q+1} \sum_{i=0}^{q} \left| \Delta^2 \left[ a(n+i,k) - \alpha_k \right] \right| = 0 \quad \text{uniformly in } n;$$

(49)

(xi) *(Başar and Çolak, [21]) $A \in (cs : fs)$ if and only if (48) holds;*

(xii) *(Başar, [25]) $A \in (f : cs)$ if and only if*

$$\sup_{n \in \mathbb{N}} \sum_k |a(n,k)| < \infty,$$

(50)

$$\sum_n a_{nk} = \alpha_k \quad \text{exists for each fixed } k,$$

(51)

$$\sum_n \sum_k a_{nk} = \alpha,$$

(52)

$$\lim_{m \to \infty} \sum_k \left| \Delta \left[ a(n,k) - \alpha_k \right] \right| = 0.$$

(53)

Now we give our main results which are related to matrix mappings on/into the spaces of almost convergent series $a_{fs}^r$ and sequences $a_f^r$.

**Corollary 20.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *$A \in (a_{fs}^r : f)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (40), (44) hold with $\hat{a}_{nk}$ instead of $a_{nk}$, (46) holds with $\hat{a}(n,k,m)$ instead of $a(n,k,m)$, and (45) holds with $\hat{a}(n,k)$ instead of $a(n,k)$.*

(ii) *$A \in (a_{fs}^r : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (19), (43), (44), and (47) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.*

(iii) *$A \in (a_{fs}^r : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (43) and (44) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.*

(iv) *$A \in (a_{fs}^r : fs)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_{fs}^r)^\beta$ for all $n \in \mathbb{N}$ and (45), (48), and (49) hold with $\hat{a}(n,k)$ instead of $a(n,k)$.*

(v) *$A \in (cs : a_{fs}^r)$ if and only if (48) holds with $\tilde{a}(n,k)$ instead of $a(n,k)$.*

(vi) *$A \in (bs : a_{fs}^r)$ if and only if (44) holds with $\tilde{a}_{nk}$ instead of $a_{nk}$ and (45), (48) hold with $\tilde{a}(n,k)$ instead of $a(n,k)$.*

(vii) *$A \in (fs : a_{fs}^r)$ if and only if (45), (48), and (49) hold with $\tilde{a}(n,k)$ instead of $a(n,k)$.*

**Corollary 21.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *$A \in (c(p) : a_{fs}^r)$ if and only if (37) and (38) hold with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$.*

(ii) *$A \in (c_0(p) : a_{fs}^r)$ if and only if (37) holds with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$.*

(iii) *$A \in (\ell_\infty(p) : a_{fs}^r)$ if and only if (37) and (39) hold with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$.*

**Corollary 22.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *$A \in (a_f^r : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (18) holds with $\hat{a}_{nk}$ instead of $a_{nk}$.*

(ii) *$A \in (a_f^r : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (18), (19), (20), and (21) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.*

(iii) *$A \in (a_f^r : cs)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in (a_f^r)^\beta$ for all $n \in \mathbb{N}$ and (50),(53) hold with $\hat{a}(n,k)$ instead of $a(n,k)$ and (51),(52) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.*

**Corollary 23.** *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold.*

(i) *$A \in (\ell_\infty : a_f^r)$ if and only if (18), (40) hold with $\tilde{a}_{nk}$ instead of $a_{nk}$ and (41) holds with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$.*

(ii) *$A \in (f : a_f^r)$ if and only if (18), (40), and (46) hold with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$ and (42) holds with $\tilde{a}_{nk}$ instead of $a_{nk}$.*

(iii) *$A \in (c : a_f^r)$ if and only if (18), (40), and (42) hold with $\tilde{a}_{nk}$ instead of $a_{nk}$.*

(iv) *$A \in (bs : a_f^r)$ if and only if (40), (43), and (44) hold with $\tilde{a}_{nk}$ instead of $a_{nk}$ and (45) holds with $\tilde{a}(n,k)$ instead of $a(n,k)$.*

(v) *$A \in (fs : a_f^r)$ if and only if (40), (44) hold with $\tilde{a}_{nk}$ instead of $a_{nk}$, (46) holds with $\tilde{a}(n,k,m)$ instead of $a(n,k,m)$, and (45) holds with $\tilde{a}(n,k)$ instead of $a(n,k)$.*

(vi) *$A \in (cs : a_f^r)$ if and only if (40) and (43) hold with $\tilde{a}_{nk}$ instead of $a_{nk}$.*

*Remark 24.* Characterization of the classes $(a_f^r : f_\infty)$, $(f_\infty : a_f^r)$, $(a_{fs}^r : f_\infty)$, and $(f_\infty : a_{fs}^r)$ is redundant since the spaces of almost bounded sequences $f_\infty$ and $\ell_\infty$ are equal.

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