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Research Article

On Bilipschitz Extensions in Real Banach Spaces

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Suppose that E and E' denote real Banach spaces with dimension at least 2, that $D \neq E$ and $D' \neq E'$ are bounded domains with connected boundaries, that $f : D \to D'$ is an M-QH homeomorphism, and that D' is uniform. The main aim of this paper is to prove that f extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D}'$ and $\overline{f}|_{\partial D}$ is bilipschitz if and only if f is bilipschitz in $\overline{D}$. The answer to some open problems of Väisälä is affirmative under a natural additional condition.

1. Introduction and Main Results

During the past three decades, the quasihyperbolic metric has become an important tool in geometric function theory and in its generalizations to metric spaces and Banach spaces [1]. Yet, some basic questions of the quasihyperbolic geometry in Banach spaces are open. For instance, only recently the convexity of quasihyperbolic balls has been studied in [2, 3] in the setup of Banach spaces.

Our study is motivated by Väisälä's theory of freely quasiconformal maps and other related maps in the setup of Banach spaces [1, 4, 5]. Our goal is to study some of the open problems formulated by him. We begin with some basic definitions and the statements of our results. The proofs and necessary supplementary notation terminology will be given thereafter.

Throughout the paper, we always assume that E and E' denote real Banach spaces with dimension at least 2. The norm of a vector z in E is written as $|z|$, and for every pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. We begin with the following concepts following closely the notation and terminology of [4–8] or [9].

We first recall some definitions.

Definition 1. A domain D in E is called *c-uniform* in the norm metric, provided there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc α in D satisfying

(1) $\min_{j=1,2} \ell(\alpha[z_j, z]) \le c d_D(z)$ for all $z \in \alpha$, and

(2) $\ell(\alpha) \le c|z_1 - z_2|$,

where $\ell(\alpha)$ denotes the length of α, $\alpha[z_j, z]$ the part of α between z_j and z, and $d_D(z)$ the distance from z to the boundary ∂D of D.

Definition 2. Suppose $G \subsetneq E$, $G' \subsetneq E'$, and $M \ge 1$. We say that a homeomorphism $f : G \to G'$ is *M-bilipschitz* if

$$\frac{1}{M}|x - y| \le |f(x) - f(y)| \le M|x - y| \tag{1}$$

for all x, $y \in G$, and *M-QH* if

$$\frac{1}{M} k_G(x, y) \le k_{G'}(f(x), f(y)) \le M k_G(x, y) \tag{2}$$

for all x, $y \in G$.

As for the extension of bilipschitz maps in $\mathbb{R}^2$, Ahlfors [10] proved that if a planar curve through ∞ admits a quasiconformal reflection, it also admits a bilipschitz reflection. Furthermore, Gehring gave generalizations of Ahlfors' result in the plane.

Theorem A (see [11, Theorem 7]). *Suppose that D is a K-quasidisk in $\mathbb{R}^2$, that D' is a Jordan domain in $\mathbb{R}^2$, and that $\phi : \partial D \to \partial D'$ is L_1-bilipschitz. Then there exist L-bilipschitz $f : \overline{D} \to \overline{D'}$ and $f^\star : \overline{D^\star} \to \overline{D'^\star}$ such that $f = f^\star = \phi$ on ∂D and L depends only on K and L_1, where $D^\star = \overline{\mathbb{R}}^2 \setminus \overline{D}$ and $D'^\star = \overline{\mathbb{R}}^2 \setminus \overline{D'}$.*

Tukia and Väisälä [12] dealt with the curious phenomenon that sometimes a quasiconformal property implies the corresponding bilipschitz property.

Theorem B (see [12, Theorem 2.12]). *Suppose that X is a closed set in $\mathbb{R}^n$, $n \neq 4$, and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-QC map such that $f|_X$ is L-bilipschitz. Then there is an L_1-bilipschitz map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) *$g|_X = f|_X$;*

(2) *$g(D) = f(D)$ for each component D of $\mathbb{R}^n \setminus X$;*

(3) *L_1 depends only on K, L, and n.*

In [13], Gehring raised the following two related problems.

Open Problem 1. Suppose that D is a Jordan domain in $\overline{\mathbb{R}}^2$ and that $f|_{\partial D}$ is M-bilipschitz. Characterize mappings f having M'-bilipschitz extension to D with $M' = M'(c, M)$.

Open Problem 2. Suppose that D is a Jordan domain in $\overline{\mathbb{R}}^2$. For which domains D does each M-bilipschitz f in the ∂D have M'-bilipschitz extension to D with $M' = M'(c, M)$?

Gehring himself discussed these two problems and got the following two results.

Theorem C (see [13, Theorem 2.11]). *Suppose that D and D' are Jordan domains in $\overline{\mathbb{R}}^2$ and that $\infty \in D'$ if and only if $\infty \in D$. Suppose also that $f : D \to D'$ is a K-quasiconformal mapping and that f extends to a homeomorphism $f : \overline{D} \to \overline{D'}$ such that $f|_{\partial D}$ is M-bilipschitz. Then there exists an M-bilipschitz map $g : \overline{D} \to \overline{D'}$ with $g|_{\partial D} = f|_{\partial D}$, where $M' = M'(M, K)$.*

Theorem D (see [13, Theorem 4.9]). *Suppose that D and D' are Jordan domains in $\overline{\mathbb{R}}^2$. Then each M-bilipschitz f in ∂D has an M'-bilipschitz extension $g : D \to D'$ with $g|_{\partial D} = f|_{\partial D}$ if and only if D is a K-quasidisk, where $M' = M'(M, K)$ and $K = K(M)$.*

We remark that Theorem C is a partial answer to Open Problem 1 and Theorem D is an affirmative answer to Open Problem 2. In the proof of Theorem C, the modulus of a path family, which is an important tool in the quasiconformal theory in $\mathbb{R}^n$, was applied. In general, this tool is no longer applicable in the context of Banach spaces (see [4]). A natural problem is whether Theorem C is true or false in Banach spaces. In fact, this problem was raised by Väisälä in [1] in the following form.

Open Problem 3. Suppose that D and D' are bounded domains with connected boundaries in E and E'. Suppose also that $f : D \to D'$ is M-QH and that f extends to a homeomorphism $f : \overline{D} \to \overline{D'}$ such that $f|_{\partial D}$ is M-bilipschitz. Is it true that f M'-bilipschitz with $M' = M'(c, M)$?

Our result is as follows.

Theorem 3. *Suppose that D and D' are bounded domains with connected boundaries in E and E', respectively. Suppose also that $f : D \to D'$ is M-QH and that f extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$ such that $f|_{\partial D}$ is M-bilipschitz. If D' is a c-uniform domain, then f is M'-bilipschitz with $M' = M'(c, M)$.*

We see from Theorem 3 that the answer to Open Problem 3 is positive by replacing the hypothesis "D' being bounded" in Open Problem 3 with the one "D' being bounded and uniform."

The organization of this paper is as follows. The proof of Theorem 3 will be given in Section 3.1. In Section 2, some preliminaries are introduced.

2. Preliminaries

The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number (cf. [14, 15])

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)}. \tag{3}$$

For each pair of points z_1, z_2 in D, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha), \tag{4}$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D. For all z_1, z_2 in D, we have (cf. [15])

$$\begin{aligned} & k_D(z_1, z_2) \\ & \quad \geq \inf \left\{ \log \left(1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}} \right) \right\} \\ & \quad \geq \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|, \end{aligned} \tag{5}$$

where the infimum is taken over all rectifiable curves α in D connecting z_1 and z_2.

In [5], Väisälä characterized uniform domains by the quasihyperbolic metric.

Theorem E (see [5, Theorem 6.16]). *For a domain D, the following are quantitatively equivalent:*

(1) *D is a c-uniform domain;*

(2) *$k_D(z_1, z_2) \leq c' \log(1 + |z_1 - z_2| / \min\{d_D(z_1), d_D(z_2)\})$ for all $z_1, z_2 \in D$;*

(3) $k_D(z_1, z_2) \le c_1' \log(1+|z_1-z_2|/\min\{d_D(z_1), d_D(z_2)\})+d$ *for all* $z_1, z_2 \in D$.

Gehring and Palka [14] introduced the quasihyperbolic metric of a domain in $\mathbb{R}^n$, and it has been recently used by many authors in the study of quasiconformal mappings and related questions [16]. In the case of domains in $\mathbb{R}^n$, the equivalence of items (1) and (3) in Theorem E is due to Gehring and Osgood [17] and the equivalence of items (2) and (3) is due to Vuorinen [18]. Many of the basic properties of this metric may be found in [4, 5, 17].

Recall that an arc α from z_1 to z_2 is a *quasihyperbolic geodesic* if $\ell_k(\alpha) = k_D(z_1, z_2)$. Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in E exists if the dimension of E is finite, see [17, Lemma 1]. This is not true in arbitrary spaces (cf. [19, Example 2.9]). In order to remedy this shortage, Väisälä introduced the following concepts [5].

Definition 4. Let α be an arc in E. The arc may be closed, open, or half open. Let $\overline{x} = (x_0, \ldots, x_n)$, $n \ge 1$, be a finite sequence of successive points of α. For $h \ge 0$, we say that $\overline{x}$ is h-coarse if $k_D(x_{j-1}, x_j) \ge h$ for all $1 \le j \le n$. Let $\Phi_k(\alpha, h)$ be the family of all h-coarse sequences of α. Set

$$s_k(\overline{x}) = \sum_{j=1}^{n} k_D(x_{j-1}, x_j),$$
$$\ell_{k_D}(\alpha, h) = \sup\{s_k(\overline{x}) : \overline{x} \in \Phi_k(\alpha, h)\} \tag{6}$$

with the agreement that $\ell_k(\alpha, h) = 0$ if $\Phi_k(\alpha, h) = \emptyset$. Then the number $\ell_k(\alpha, h)$ is the *h-coarse quasihyperbolic length* of α.

In this paper, we will use this concept in the case where D is a domain equipped with the quasihyperbolic metric k_D. We always use $\ell_k(\alpha, h)$ to denote the h-coarse quasihyperbolic length of α.

Definition 5. Let D be a domain in E. An arc $\alpha \subset D$ is (ν, h)-solid with $\nu \ge 1$ and $h \ge 0$ if

$$\ell_k(\alpha[x, y], h) \le \nu k_D(x, y) \tag{7}$$

for all $x, y \in \alpha$. A $(\nu, 0)$-*solid arc* is said to be a ν-*neargeodesic*, that is, an arc $\alpha \subset D$ is a ν- neargeodesic if and only if $\ell_k(\alpha[x, y]) \le \nu k_D(x, y)$ for all $x, y \in \alpha$.

Obviously, a ν-neargeodesic is a quasihyperbolic geodesic if and only if $\nu = 1$.

In [19], Väisälä got the following property concerning the existence of neargeodesic in E.

Theorem F (see [19, Theorem 3.3]). *Let* $\{z_1, z_2\} \subset D$ *and* $\nu > 1$. *Then there is a* ν*-neargeodesic in* D *joining* z_1 *and* z_2.

The following result due to Väisälä is from [5].

Theorem G (see [5, Theorem 4.15]). *For domains* $D \ne E$ *and* $D' \ne E'$, *suppose that* $f : D \to D'$ *is* M*-QH. If* γ *is a* c*-neargeodesic in* D*, then the arc* γ' *is* c_1*-neargeodesic in* D' *with* c_1 *depending only on* c *and* M.

Let $G \ne E$ *and* $G' \ne E'$ *be metric spaces, and let* $\varphi : [0, \infty) \to [0, \infty)$ *be a growth function, that is, a homeomorphism with* $\varphi(t) \ge t$. *We say that a homeomorphism* $f : G \to G'$ *is* φ*-semisolid if*

$$k_{G'}(f(x), f(y)) \le \varphi(k_G(x, y)) \tag{8}$$

for all $x, y \in G$*, and* φ*-solid if both* f *and* f^{-1} *satisfy this condition.*

We say that f *is fully* φ*-semisolid (resp. fully* φ*-solid) if* f *is* φ*-semisolid (resp.* φ*-solid) on every subdomain of* G*. In particular, when* $G = E$*, corresponding subdomains are taken to be proper ones. Fully* φ*-solid mapsare also called freely* φ*-quasiconformal maps, or briefly* φ*-FQC maps.*

For convenience, in the following, we always assume that $x, y, z, \ldots$ denote points in D and $x', y', z', \ldots$ the images in D' of $x, y, z, \ldots$ under f, respectively. Also we assume that α, $\beta, \gamma, \ldots$ denote curves in D and $\alpha', \beta', \gamma', \ldots$ the images in D' of $\alpha, \beta, \gamma, \ldots$ under f, respectively.

3. Bilipschitz Mappings

First we introduce the following Theorems.

Theorem H (see [5, Theorem 7.18]). *Let* D *and* D' *be domains in* E *and* E'*, respectively. Suppose that* D *is a* c*-uniform domain and that* $f : D \to D'$ *is* φ*-FQC (see Section 2 for the definition). Then the following conditions are quantitatively equivalent:*

(1) D' *is a* c_1*-uniform domain;*

(2) f *is* η*-quasimöbius.*

Theorem I (see [20, Theorem 1.1]). *Suppose that* D *is a* c*-uniform domain and that* $f : D \to D'$ *is* (M, C)*-CQH, where* $D \subsetneq E$ *and* $D' \subsetneq E'$*. Then the following conditions are quantitatively equivalent:*

(1) D' *is a* c_1*-uniform domain;*

(2) f *extends to a homeomorphism* $\overline{f} : \overline{D} \to \overline{D}'$ *and* $\overline{f}$ *is* η*-QM rel* ∂D.

The following theorem easily follows from Theorems H and I.

Theorem 6. *Suppose that* $D \subsetneq E$ *and* $D' \subsetneq E'$*, that* D *is a* c*-uniform domain, and that* $f : D \to D'$ *is* φ*-FQC. Then the following conditions are quantitatively equivalent:*

(1) D' *is a* c_1*-uniform domain;*

(2) f *is* θ*-quasimöbius;*

(3) f *extends to a homeomorphism* $\overline{f} : \overline{D} \to \overline{D}'$ *and* $\overline{f}$ *is* θ_1*-QM rel* ∂D.

Let us recall the following three theorems which are useful in the proof of Theorem 3.

Theorem J (see [1, Theorem 2.44]). *Suppose that $G \subsetneq E$ and $G' \subsetneq E'$ is a c-uniform domain, and that $f : G \to G'$ is M-QH. If $D \subset G$ is a c-uniform domain, then $D' = f(D)$ is a c'-uniform domain with $c' = c'(c, M)$.*

Theorem K (see [5, Theorem 6.19]). *Suppose that $D \subsetneq E$ is a c-uniform domain and that γ is a c_1-neargeodesic in D with endpoints z_1 and z_2. Then there is a constant $b = b(c, c_1) \geq 1$ such that*

(1) $\min_{j=1,2} \ell(\gamma[z_j, z]) \leq b d_D(z)$ *for all $z \in \alpha$, and*

(2) $\ell(\gamma) \leq b|z_1 - z_2|$.

Theorem L (see [21, Theorem 1.2]). *Suppose that D_1 and D_2 are convex domains in E, where D_1 is bounded and D_2 is c-uniform for some $c > 1$, and that there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathbb{B}(z_0, r) \subset D_1 \cap D_2$. If there exist constants $R_1 > 0$ and $c_0 > 1$ such that $R_1 \leq c_0 r$ and $D_1 \subset \overline{\mathbb{B}}(z_0, R_1)$, then $D_1 \cup D_2$ is a c'-uniform domain with $c' = (c + 1)(2c_0 + 1) + c$.*

Basic Assumption A. In this paper, we always assume that D and D' are bounded domains with connected boundaries in E and E', respectively, that $f : D \to D'$ is M-QH, that f extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$ such that $\overline{f}|_{\partial D}$ is M-bilipschitz, and that D' is a c-uniform domain.

Before the proof of Theorem 3, we prove a series of lemmas.

Lemma 7. *There is a constant $M_0 = M_0(M) > M$ such that if the points $z_1, z_2 \in D$ satisfies $\operatorname{dist}(z_1, \partial D) \leq \varepsilon$ and $\operatorname{dist}(z_2, \partial D) \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, then*

$$\frac{1}{M_0} |z_1 - z_2| \leq |z_1' - z_2'| \leq M_0 |z_1 - z_2|. \tag{9}$$

Proof. Let $x_1, x_2 \in \partial D$ be such that $|z_1 - x_1| = (4/3)\operatorname{dist}(z_1, \partial D)$, $|z_2 - x_2| \leq (4/3)\operatorname{dist}(z_2, \partial D)$ and $|x_1 - x_2| \leq \max\{|z_1 - x_1|, |z_2 - x_2|\} < 3|x_1 - x_2|$ for sufficiently small $\varepsilon > 0$. It follows from "f being M-QH in D and homeomorphic in $\overline{D}$" that $H(x, f) \leq K$ (cf. [1]) for each $x \in D$, where K depends only on M. Hence,

$$|z_1' - x_1'| < \frac{3}{2} K |x_1' - x_2'|, \qquad |z_2' - x_2'| < \frac{3}{2} K |x_1' - x_2'|. \tag{10}$$

If $|z_1 - z_2| \leq (1/4K^2 M) \max\{|z_1 - x_1|, |z_2 - x_2|\}$, then for each $z \in [z_1, z_2]$,

$$d_D(z) \geq \frac{3K^2 M - 1}{4K^2 M} \max\{|z_1 - x_1|, |z_2 - x_2|\}, \tag{11}$$

and so we have

$$\begin{aligned}
\frac{2|z_1' - z_2'|}{\min\{d_{D'}(z_1'), d_{D'}(z_2')\}} &\leq \log\left(1 + \frac{|z_1' - z_2'|}{\min\{d_{D'}(z_1'), d_{D'}(z_2')\}}\right) \\
&\leq k_{D'}(z_1', z_2') \leq M k_D(z_1, z_2) \\
&\leq M \int_{[z_1, z_2]} \frac{|dz|}{d_D(z)} \\
&\leq \frac{4K^2 M^2 |z_1 - z_2|}{(3K^2 M - 1) \max\{|z_1 - x_1|, |z_2 - x_2|\}},
\end{aligned} \tag{12}$$

which shows that

$$|z_1' - z_2'| \leq \frac{12K^3 M^3}{3K^2 M - 1} |z_1 - z_2|. \tag{13}$$

If $|z_1 - z_2| > (1/4K^2 M) \max\{|z_1 - x_1|, |z_2 - x_2|\}$, then by the assumption "f being M-bilipschitz in ∂D,"

$$\begin{aligned}
|z_1' - z_2'| &\leq |z_1' - x_1'| + |z_2' - x_2'| + |x_1' - x_2'| \\
&\leq (3K + 1) |x_1' - x_2'| \\
&\leq (3K + 1) M |x_1 - x_2| \\
&\leq (12K + 4) K^2 M^2 |z_1 - z_2|.
\end{aligned} \tag{14}$$

The same discussion as the above shows that

$$|z_1 - z_2| \leq (12K + 4) K^2 M^2 |z_1' - z_2'|. \tag{15}$$

□

Lemma 8. *There is a constant $M_1 = M_1(c, M)$ such that if the points $x \in D$ and $z \in \mathbb{S}(x, d_D(x)) \cap \overline{D}$ satisfies $\operatorname{dist}(z, \partial D) \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, then*

$$|z' - x'| \leq M_1 d_D(x). \tag{16}$$

Proof. Let $x_0 \in \mathbb{S}(x, d_D(x)) \cap \overline{D}$ such that $\operatorname{dist}(x_0, \partial D) \leq \varepsilon$ for sufficiently small $\varepsilon > 0$, and let x_2 be the intersection point of $\mathbb{S}(x_0, (1/2)d_D(x))$ with $[x_0, x]$. Then we have

$$\begin{aligned}
k_D(x_2, x) &\leq \log\left(1 + \frac{|x - x_2|}{d_D(x) - |x - x_2|}\right) \\
&\leq \log \frac{d_D(x)}{d_D(x_2)} = \log 2,
\end{aligned} \tag{17}$$

which implies that

$$\log \frac{|x_2' - x'|}{|x_2' - x_0'|} \leq k_{D'}(x_2', x') \leq M k_D(x_2, x) = M \log 2. \tag{18}$$

Hence,

$$\left|x_2' - x'\right| \le 2^M \left|x_2' - x_0'\right|, \tag{19}$$

and so

$$\begin{aligned}\left|x' - x_0'\right| &\le \left|x' - x_2'\right| + \left|x_2' - x_0'\right| \\ &\le \left(2^M + 1\right)\left|x_2' - x_0'\right|.\end{aligned} \tag{20}$$

Let T be a 2-dimensional linear subspace of E which contains x_0 and x_2, and we use τ to denote the circle $T \cap \mathbb{S}(x_0, (1/2)d_D(x))$. Take $w_1 \in \tau \cap \partial D$ such that $\tau(x_2, w_1) \subset D$ and $\ell(\tau[x_2, w_1]) \le 2d_D(x)$. Let $x_1 \in \mathbb{S}(x, d_D(x)) \cap \tau[x_2, w_1] \cap \overline{D}$ and denote $\tau(x_1, w_1)$ by τ_1.

Claim 1. There must exist a 2^{32}-uniform domain D_1 in D and $x_3 \in \partial D_1 \cap \overline{D}$ satisfying $\mathrm{dist}(x_3, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$ such that $x_0, x \in \overline{D}_1$ and $(1/12)d_D(x) \le |x_3 - x_0| \le (11/12)d_D(x)$.

If $d_D(x_1) = 0$, then we take $D_1 = \mathbb{B}(x, d_D(x))$ and $x_3 = x_1$. Obviously, $|x_3 - x_0| = (1/2)d_D(x)$. Hence Claim 1 holds true in this case.

If $d_D(x_1) > 0$, we divide the proof of Claim 1 into two parts.

Case 1. ($d_D(x_1) \le (5/12)d_D(x)$). Then we take $D_1 = \mathbb{B}(x, d_D(x)) \cup \mathbb{B}(x_1, d_D(x_1))$ and $x_3 \in \mathbb{S}(x_1, d_D(x_1)) \cap \overline{D}$ such that $\mathrm{dist}(x_3, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. It follows from Theorem L that D_1 is a 29-uniform domain and

$$\begin{aligned}\frac{1}{12}d_D(x) &\le \left|x_1 - x_0\right| - \left|x_1 - x_3\right| \le \left|x_3 - x_0\right| \\ &\le \left|x_1 - x_0\right| + \left|x_1 - x_3\right| \le \frac{11}{12}d_D(x),\end{aligned} \tag{21}$$

from which we see that Claim 1 is true.

Case 2. ($d_D(x_1) > (5/12)d_D(x)$). Obviously, $d_D(x_1) > (5/6)|x_1 - x_0|$. We let $w_2 \in \tau_1$ be the first point along the direction from x_1 to w_1 such that

$$d_D(w_2) = \frac{5}{12}d_D(x). \tag{22}$$

If $|w_2 - x_1| \le (1/3)d_D(x)$, then we take $D_1 = \mathbb{B}(x, d_D(x)) \cup \mathbb{B}(w_2, d_D(w_2))$,and let $x_3 \in \mathbb{S}(w_2, d_D(w_2)) \cap \overline{D}$ such that $\mathrm{dist}(x_3, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Then

$$\begin{aligned}&d_D(w_2) + d_D(x) - \left|w_2 - x\right| \\ &\quad \ge d_D(w_2) - \left|w_2 - x_1\right| \ge \frac{1}{12}d_D(x), \\ &\frac{1}{12}d_D(x) \le \left|x_3 - x_0\right| \\ &\quad \le \left|w_2 - x_0\right| + \left|w_2 - x_3\right| \le \frac{11}{12}d_D(x).\end{aligned} \tag{23}$$

It follows from Theorem L that D_1 is a 677-uniform domain, which shows that Claim 1 is true.

If $|w_2 - x_1| > (1/3)d_D(x)$, then we first prove the following subclaim.

Subclaim 1. There exists a simply connected domain $D_1 = \cup_{i=0}^{t} B_i$ in D, where $t = 1$ or 2, such that

(1) $x_0, x \in \overline{D}_1$;

(2) for each $i \in \{0, \dots, t\}$, $(5/12)d_D(x) \le r_i \le d_D(x)$;

(3) if $t = 2$, then $|x - w_2| - r_0 - r_2 \ge (1/144)d_D(x)$;

(4) $r_i + r_{i+1} - |v_i - v_{i+1}| \ge (1/144)d_D(x)$, where $i \in \{0, 1\}$ if $t = 2$ or $i = 0$ if $t = 1$.

Here $B_i = \mathbb{B}(v_i, r_i)$, $v_i \in \tau[x_2, w_2]$, $v_1 \notin B_0$, and $v_2 \notin \tau[x_2, v_1]$.

To prove this subclaim, we let $y_2 \in \tau_1$ be such that $|x_1 - y_2| = (1/3)d_D(x)$ and let $C_0 = \mathbb{B}(x, d_D(x))$ and $C_1 = \mathbb{B}(y_2, d_D(y_2))$. Since $d_D(y_2) > (5/12)d_D(x)$, we have

$$d_D(y_2) + d_D(x) - \left|y_2 - x\right| \ge \frac{1}{12}d_D(x). \tag{24}$$

Next, we construct a ball denoted by C_2.

If $w_2 \in \overline{C}_1$, then we let $C_2 = \mathbb{B}(w_2, d_D(w_2))$.

If $w_2 \notin \overline{C}_1$, then we let y_3 be the intersection of $\mathbb{S}(y_2, d_D(y_2))$ with $\tau_1[y_2, w_1]$. Since $\ell(\tau_1) \le 2d_D(x)$ and $d_D(z) \ge (5/12)d_D(x)$ for all $z \in \tau_1(x_1, x_2)$, we have

$$\begin{aligned}&\left|w_1 - w_2\right| + \left|w_2 - y_3\right| + \left|y_3 - y_2\right| + \left|y_2 - x_1\right| + \left|x_2 - x_1\right| \\ &\quad \le \ell(\tau_1) \le 2d_D(x),\end{aligned} \tag{25}$$

which implies that

$$\left|w_2 - y_3\right| \le \frac{1}{3}d_D(x). \tag{26}$$

We take $C_2 = \mathbb{B}(w_2, d_D(w_2))$. Then (26) implies

$$\begin{aligned}&d_D(w_2) + d_D(x_2) - \left|x_2 - w_2\right| \\ &\quad \ge d_D(w_2) - \left|w_2 - y_3\right| \ge \frac{1}{12}d_D(x).\end{aligned} \tag{27}$$

Now we are ready to construct the needed domain D_1.

If $d_D(w_2) + d_D(x) - |w_2 - x| \ge (1/48)d_D(x)$, then we take $B_0 = C_0$, $B_1 = C_2$, and $D_1 = B_0 \cup B_1$ with $v_0 = x$, $v_1 = w_2$, $r_0 = d_D(x)$, and $r_1 = d_D(w_2)$. Obviously, D_1 satisfies all the conditions in Subclaim 1. In this case, $t = 1$.

If $d_D(w_2) + d_D(x) - |w_2 - x| < (1/48)d_D(x)$, then we take $B_0 = \mathbb{B}(x, (35/36)d_D(x))$ with $r_0 = (35/36)d_D(x)$ and $v_0 = x$, $B_1 = C_1$ with $r_1 = d_D(y_2)$ and $v_1 = y_2$, and $B_2 = C_2$ with $r_2 = d_D(w_2)$ and $v_2 = w_2$. Then Inequalities (24) and (27) show that $D_1 = \bigcup_{i=0}^{2} B_i$ satisfies all the conditions in Subclaim 1. In this case, $t = 2$.

Hence, the proof of Subclaim 1 is complete.

The following follows from a similar argument as in the proof of [22, Theorem 1.1].

Corollary 9. *The domain D_1 constructed in Subclaim 1 is a 2^{32}-uniform domain.*

Let $x_3 \in \mathbb{S}(w_2, d_D(w_2)) \cap \overline{D}$ such that $\operatorname{dist}(x_3, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Then

$$\frac{1}{12} d_D(x) \le |x_3 - x_0| \le \frac{11}{12} d_D(x). \tag{28}$$

Then the proof of Claim 1 easily follows from (28), Subclaim 1, and Corollary 9.

We come back to the proof of Lemma 8. It follows from (28) and Lemma 7 that

$$\begin{aligned} |x - x_3| &\le |x - x_0| + |x_0 - x_3| \le \frac{23}{12} d_D(x), \\ \frac{1}{12M_0} d_D(x) &\le \frac{1}{M_0} |x_3 - x_0| \le |x_3' - x_0'| \le M_0 |x_3 - x_0| \\ &\le \frac{11M_0}{12} d_D(x). \end{aligned} \tag{29}$$

Then it follows from Theorem J that D_1' is an M'-uniform domain, where $M' = M'(c, M)$. Hence, we know from Theorem 6 that f^{-1} is a θ-Quasimöbius in $\overline{D}_1$, where $\theta = \theta(c, M)$, and so (19), (20), (28), and (29) imply that

$$\begin{aligned} \frac{1}{23} \le \frac{|x_3 - x_0|}{|x_2 - x_0|} \cdot \frac{|x_2 - x|}{|x - x_3|} &\le \theta\left(\frac{|x_3' - x_0'|}{|x_2' - x_0'|} \cdot \frac{|x_2' - x'|}{|x' - x_3'|}\right) \\ &\le \theta\left(\frac{M_0 2^{M+1} d_D(x)}{|x' - x_3'|}\right), \end{aligned} \tag{30}$$

which, together with (20), shows

$$\begin{aligned} |x' - x_0'| &\le |x' - x_3'| + |x_3' - x_0'| \\ &\le \left(\frac{2^{M+1}}{\theta^{-1}(1/23)} + \frac{11M_0}{12}\right) d_D(x) \\ &< \frac{2^{M_0+2}}{\theta^{-1}(1/23)} d_D(x). \end{aligned} \tag{31}$$

Thus, the proof of Lemma 8 is complete. □

Lemma 10. *For all $x \in D$, if $z \in \mathbb{S}(x, d_D(x)) \cap \overline{D}$ such that $\operatorname{dist}(x, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$, then $|z' - x'| \ge (1/e^{4M_0M_1^2}) d_D(x)$, where $M_1 = M_1(c, M)$.*

Proof. Suppose on the contrary that there exist points $x_1 \in D$ and $y_1 \in \mathbb{S}(x_1, d_D(x_1)) \cap \overline{D}$ with $\operatorname{dist}(y_1, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$ such that

$$|x_1' - y_1'| < \frac{1}{e^{4M_0M_1^2}} |x_1 - y_1|. \tag{32}$$

We take $y_2 \in \mathbb{S}(y_1, d_D(x_1)) \cap \overline{D}$ such that $\operatorname{dist}(y_2, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. From Lemma 7 we know that

$$|y_1' - y_2'| \ge \frac{1}{M_0} |y_1 - y_2| = \frac{1}{M_0} d_D(x_1). \tag{33}$$

Let T_1 be a 2-dimensional linear subspace of E determined by x_1, y_1 and y_2, and ω the circle $T_1 \cap \mathbb{S}(y_1, d_D(x_1))$. We take $y_3 \in \omega \cap \partial D$ which satisfies $\omega(x_1, y_3) \subset D$ and $\ell(\omega[x_1, y_3]) \le 4d_D(x_1)$. Let $\omega_1 = \omega(x_1, y_3)$ and w_1 be the first point along the direction from x_1 to y_3 such that

$$d_D(w_1) = \frac{1}{4M_0M_1} d_D(x_1). \tag{34}$$

Let $v_1 \in \mathbb{S}(w_1, d_D(w_1)) \cap \overline{D}$ such that $\operatorname{dist}(w_1, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Then it follows from Lemma 8 that

$$d_{D'}(w_1') \le |w_1' - v_1'| \le M_1 d_D(w_1) = \frac{1}{4M_0} d_D(x_1), \tag{35}$$

which, together with Lemmas 7 and 8 and (32), implies that

$$\begin{aligned} |x_1' - w_1'| &\ge |y_1' - v_1'| - |x_1' - y_1'| - |v_1' - w_1'| \\ &\ge \frac{1}{M_0} |y_1 - v_1| \\ &\quad - \frac{1}{e^{4M_0M_1^2}} |x_1 - y_1| - M_1 |v_1 - w_1| \\ &\ge \frac{1}{M_0} (d_D(x_1) - d_D(w_1)) \\ &\quad - \frac{1}{e^{4M_0M_1^2}} |x_1 - y_1| - M_1 |v_1 - w_1| \\ &> \frac{1}{2M_0} d_D(x_1). \end{aligned} \tag{36}$$

Hence, we infer from (32) that

$$k_{D'}(x_1', w_1') \ge \log\left(1 + \frac{|x_1' - w_1'|}{d_{D'}(x_1')}\right) > M_1^2. \tag{37}$$

Since $\ell(\omega_1) \le 4d_D(x_1)$, by the choice of w_1, one has

$$k_D(x_1, w_1) \le \int_{\omega_1[x_1, w_1]} \frac{|dx|}{d_D(x)} \le 16M_0M_1, \tag{38}$$

whence

$$k_{D'}(x_1', w_1') \le M k_D(x_1, w_1) \le 16MM_0M_1, \tag{39}$$

which contradicts with (37). The proof of Lemma 10 is complete. □

Lemma 11. *For $x_1 \in D$ and $x_2 \in \partial D$, we have*

$$|x_1' - x_2'| \le M_2 |x_1 - x_2|, \tag{40}$$

where $M_2 = 2M_0 + M_1$.

Proof. For $x_1 \in D$, we let $y_1 \in \mathbb{S}(x_1, d_D(x_1)) \cap \overline{D}$ such that $\operatorname{dist}(y_1, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Then it follows from Lemma 8 that

$$|x_1' - y_1'| \le M_1 |x_1 - y_1|. \tag{41}$$

For $x_2 \in \partial D$, if $|y_1 - x_2| \le 2|x_1 - y_1|$, then by Lemma 7, we have

$$\begin{aligned} |x_1' - x_2'| &\le |x_1' - y_1'| + |y_1' - x_2'| \\ &\le M_1 |x_1 - y_1| + M_0 |y_1 - x_2| \\ &\le (2M_0 + M_1) |x_1 - y_1| \\ &\le (2M_0 + M_1) |x_1 - x_2|. \end{aligned} \tag{42}$$

If $|y_1 - x_2| > 2|y_1 - x_1|$, then we have

$$|x_1 - x_2| > |y_1 - x_2| - |x_1 - y_1| > \frac{1}{2} |y_1 - x_2|. \tag{43}$$

Hence, by Lemma 7 and (41),

$$\begin{aligned} |x_1' - x_2'| &\le |x_1' - y_1'| + |y_1' - x_2'| \\ &\le M_1 |x_1 - y_1| + M_0 |y_1 - x_2| \\ &\le (2M_0 + M_1) |x_1 - x_2|, \end{aligned} \tag{44}$$

from which the proof follows. □

Lemma 12. *For $x_1 \in D$ and $x_2 \in \partial D$, one has*

$$|x_1' - x_2'| \ge \frac{1}{M_3} |x_1 - x_2|, \tag{45}$$

where $M_3 = 2M_0M_1e^{(5MM_0+8M_0)M_1^2}$.

Proof. We begin with a claim.

Claim 2. For all $z \in D$, we have $d_{D'}(z') \ge (1/e^{(5MM_0+8M_0)M_1^2})d_D(z)$.

To prove this claim, we let $w_2 \in [z, y_1]$ be such that $|w_2 - y_1| = (1/2M_1e^{4M_0M_1^2})d_D(z)$. It follows from [18] that

$$k_D(w_2, z) \le \log\left(1 + \frac{|w_2 - z|}{d_D(z) - |w_2 - z|}\right) < 5M_0M_1^2. \tag{46}$$

By Lemma 8, we have

$$|w_2' - y_1'| \le M_1 |w_2 - y_1| = \frac{1}{2e^{4M_0M_1^2}} d_D(z). \tag{47}$$

Hence, Lemma 10 implies $|w_2' - z'| \ge (1/2e^{4M_0M_1^2})d_D(z)$, whence

$$\log \frac{|w_2' - z'|}{d_{D'}(z')} \le k_{D'}(w_2', z') \le Mk_D(w_2, z) \le 5MM_0M_1^2, \tag{48}$$

which shows that Claim 2 is true.

Now we are ready to finish the proof of Lemma 12. For $x_1 \in D$ and $x_2 \in \partial D$, if $|x_1 - x_2| \le 2M_0M_1d_D(x_1)$, then by Claim 2,

$$\begin{aligned} |x_1' - x_2'| \ge d_{D'}(x_1') &\ge \frac{1}{e^{(5MM_0+8M_0)M_1^2}} d_D(x_1) \\ &\ge \frac{1}{2M_0M_1e^{(5MM_0+8M_0)M_1^2}} |x_1 - x_2|. \end{aligned} \tag{49}$$

If $|x_1 - x_2| > 2M_0M_1d_D(x_1)$, then we take $w_3 \in \mathbb{S}(x_1, d_D(x_1)) \cap \overline{D}$ such that $\operatorname{dist}(w_3, \partial D) \le \varepsilon$ for sufficiently small $\varepsilon > 0$, and so

$$\begin{aligned} |w_3 - x_2| &\ge |x_1 - x_2| - |x_1 - w_3| \\ &\ge \left(1 - \frac{1}{2M_0M_1}\right) |x_1 - x_2|, \\ |w_3 - x_2| &\ge |x_1 - x_2| - |x_1 - w_3| \\ &\ge (2M_0M_1 - 1) |x_1 - w_3|, \end{aligned} \tag{50}$$

whence Lemmas 7 and 8 imply

$$\begin{aligned} |x_1' - x_2'| &\ge |w_3' - x_2'| - |x_1' - w_3'| \\ &\ge \frac{1}{M_0} |w_3 - x_2| - M_1 |x_1 - w_3| \\ &\ge \left(\frac{1}{M_0} - \frac{M_1}{2M_0M_1 - 1}\right) |w_3 - x_2| \\ &\ge \frac{1}{3M_0} |x_1 - x_2|, \end{aligned} \tag{51}$$

from which the proof is complete. □

By the previous lemmas, we get the following result.

Lemma 13. *D is a c_1-uniform domain, where $c_1 = c_1(c, M)$.*

Proof. We first prove that f^{-1} is θ_1-Quasimöbius rel $\partial D'$, where $\theta_1(t) = (M_2M_3)^2t$, M_2 and M_3 are the same as in Lemmas 11 and 12, respectively. By definition, it is necessary to prove that for $x_1', x_2', x_3', x_4' \in \overline{D'}$,

$$\begin{aligned} &\frac{|x_4 - x_1|}{|x_4 - x_2|} \cdot \frac{|x_2 - x_3|}{|x_1 - x_3|} \\ &\quad \le (M_2M_3)^2 \frac{|x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{|x_2' - x_3'|}{|x_1' - x_3'|}, \end{aligned} \tag{52}$$

where $x_1, x_2 \in \partial D'$. Obviously, to prove Inequality (52), we only need to consider the following three cases.

Case 3 ($x_1', x_2', x_3', x_4' \in \partial D'$). Since f is M-bilipschitz in ∂D, we have

$$\frac{|x_4 - x_1|}{|x_4 - x_2|} \cdot \frac{|x_2 - x_3|}{|x_1 - x_3|} \le M^4 \frac{|x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{|x_2' - x_3'|}{|x_1' - x_3'|}. \tag{53}$$

Case 4 (x_1', x_2', $x_3' \in \partial D'$ and $x_4' \in D'$). It follows from Lemmas 11 and 12 that

$$\begin{aligned} \frac{|x_4 - x_1|}{|x_4 - x_2|} \cdot \frac{|x_2 - x_3|}{|x_1 - x_3|} &\le \frac{M_2M_3 |x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{M^2 |x_2' - x_3'|}{|x_1' - x_3'|} \\ &\le M^2M_2M_3 \frac{|x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{|x_2' - x_3'|}{|x_1' - x_3'|}. \end{aligned} \tag{54}$$

Case 5 ($x_1', x_2' \in \partial D'$ and $x_3', x_4' \in D'$). We obtain from Lemmas 11 and 12 that

$$\frac{|x_4 - x_1|}{|x_4 - x_2|} \cdot \frac{|x_2 - x_3|}{|x_1 - x_3|} \le \frac{M_2 M_3 |x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{M_2 M_3 |x_2' - x_3'|}{|x_1' - x_3'|} \le (M_2 M_3)^2 \frac{|x_4' - x_1'|}{|x_4' - x_2'|} \cdot \frac{|x_2' - x_3'|}{|x_1' - x_3'|}. \tag{55}$$

The combination of Cases 3 ~ 5 shows that Inequality (52) holds, which implies that f^{-1} is a θ_1-Quasimöbius rel $\partial D'$. Hence, Theorem 6 shows that D is a c_1-uniform domain, where c_1 depends only on c and M. □

3.1. The Proof of Theorem 3. For any $z_1, z_2 \in \overline{D}$, it suffices to prove that

$$\frac{1}{M'} |z_1 - z_2| \le |z_1' - z_2'| \le M' |z_1 - z_2|, \tag{56}$$

where M' depends only on c and M.

It follows from the hypothesis "f being M-bilipschitz in ∂D," Lemmas 11 and 12 that we only need to consider the case $z_1, z_2 \in D$.

If $|z_1 - z_2| \le (1/2)\max\{d_D(z_1), d_D(z_2)\}$, then

$$k_D(z_1, z_2) \le \int_{[z_1,z_2]} \frac{|dx|}{d_D(x)} \le \frac{2|z_1 - z_2|}{\max\{d_D(z_1), d_D(z_2)\}} \le 1, \tag{57}$$

which shows that

$$\log\left(1 + \frac{|z_1' - z_2'|}{\min\{d_{D'}(z_1'), d_{D'}(z_2')\}}\right) \le k_{D'}(z_1', z_2') \le M k_D(z_1, z_2) \le M, \tag{58}$$

and so

$$\frac{|z_1' - z_2'|}{e^M \min\{d_{D'}(z_1'), d_{D'}(z_2')\}} \le \log\left(1 + \frac{|z_1' - z_2'|}{\min\{d_{D'}(z_1'), d_{D'}(z_2')\}}\right) \le \frac{2M|z_1 - z_2|}{\max\{d_D(z_1), d_D(z_2)\}}. \tag{59}$$

We see from Lemma 8 that

$$\min\{d_{D'}(z_1'), d_{D'}(z_2')\} \le M_1 \max\{d_D(z_1), d_D(z_2)\}. \tag{60}$$

Then (59) implies that

$$|z_1' - z_2'| \le 2MM_1 e^M |z_1 - z_2|. \tag{61}$$

For the other case $|z_1 - z_2| > (1/2)\max\{d_D(z_1), d_D(z_2)\}$, we let β be a 2-neargeodesic joining z_1 and z_2 in D. It follows from Theorem G that β' is a c_2-neargeodesic, where c_2 depends only on M. Let $z' \in \beta'$ such that

$$|z_1' - z'| = \frac{1}{2}|z_1' - z_2'|. \tag{62}$$

Then we know from $|z_2' - z'| \ge (1/2)|z_1' - z_2'|$ and Theorem K that

$$\begin{aligned} |z_1' - z_2'| &\le 2\min\{|z_1' - z'|, |z_2' - z'|\} \\ &\le 2\min\{\operatorname{diam}(z_1', z'), \operatorname{diam}(z_2', z')\} \\ &\le 2\mu d_{D'}(z'), \end{aligned} \tag{63}$$

where μ depends only on c and M.

We claim that

$$d_D(z) \le 3\ell(\beta). \tag{64}$$

Otherwise,

$$\begin{aligned} \max\{d_D(z_1), d_D(z_2)\} &\ge d_D(z) - \max\{|z_1 - z|, |z_2 - z|\} > 2\ell(\beta) \\ &\ge 2|z_1 - z_2|. \end{aligned} \tag{65}$$

This is the desired contradiction.

By Theorem K and Lemma 13, we have

$$d_D(z) \le 3\ell(\beta) \le 3b|z_1 - z_2|, \tag{66}$$

where $b = b(c_1)$. Hence, Lemma 8 and (63) show that

$$|z_1' - z_2'| \le 2\mu d_{D'}(z') \le 6bM_1\mu|z_1 - z_2|. \tag{67}$$

By Lemma 13, we see that D is a c_1-uniform domain. Hence a similar argument as in the proofs of Inequalities (61) and (67) yields that

$$|z_1 - z_2| \le M_4 |z_1' - z_2'|, \tag{68}$$

where $M_4 = M_4(c, M)$.

Obviously, the inequalities (61), (67), and (68) show that (56) holds, and thus the proof of the theorem is complete.

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