Research Article

On the Dirichlet Problem for the Stokes System in Multiply Connected Domains

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The Dirichlet problem for the Stokes system in a multiply connected domain of $\mathbb{R}^n$ ($n \geq 2$) is considered in the present paper. We give the necessary and sufficient conditions for the representability of the solution by means of a simple layer hydrodynamic potential, instead of the classical double layer hydrodynamic potential.

1. Introduction

Potential theory methods have been employed for a long time in the study of boundary value problems. In particular they were widely used in BVPs for the Stokes system, starting from [1, 2].

Recently some papers have used the integral representations of solutions for studying some BVPs for the Stokes system also in multiply connected domains [3–8]. All these papers concern the double layer hydrodynamic potential approach for the Dirichlet problem and the simple layer hydrodynamic potential approach for the traction problem.

The aim of the present paper is to investigate a different integral representation for the Dirichlet problem for the Stokes system in a multiply connected bounded domain of $\mathbb{R}^n$ ($n \geq 2$). Namely, we consider the simple layer potential approach for the Dirichlet problem in a domain

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j, \tag{1}$$

where Ω_j ($j = 0, \ldots, m$) are suitable domains with connected boundaries in $C^{1,\lambda}$, $\lambda \in (0,1]$.

We use a new method which hinges on a singular integral system in which the unknown is a usual vector valued function, while the data is a vector whose components are differential forms.

The paper is organized as follows. In Section 2 we give an outlook of the method with a brief description of some previous results.

After the preliminary Section 3, in Section 4 we study in detail the case $n = 2$, where some particular phenomena appear.

Section 5 is devoted to determine the eigenspace of a certain singular integral system in which the unknowns are differential forms of degree $n - 2$ on $\partial\Omega$. In the same section, we recall some known results concerning the eigenspaces of some classical integral systems.

In Section 6 we construct a left reduction for the singular integral system under study. Such a singular integral system is equivalent in a precise sense to the Fredholm system obtained through the reduction.

Finally, in the last section, we find the solution of the Dirichlet problem for the Stokes system in a multiply connected domain by means of a simple layer hydrodynamic potential.

The main result is that, given $f \in [W^{1,p}(\partial\Omega)]^n$, we can represent the solution of the Dirichlet problem

$$\mu\Delta v = \nabla r \quad \text{in } \Omega,$$
$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{2}$$
$$v = f \quad \text{on } \partial\Omega,$$

by means of a simple layer hydrodynamic potential if, and only if, the conditions

$$\int_{\partial\Omega_j} f \cdot v d\sigma = 0, \quad j = 0, 1, \ldots, m \quad (3)$$

are satisfied (v being the outwards unit normal on $\partial\Omega$). Moreover, if the data f satisfies only the condition

$$\int_{\partial\Omega} f \cdot v d\sigma = 0 \quad (4)$$

(which is necessary for the existence of a solution of the Dirichlet problem (2)) we show how to modify the integral representation of the solution (see Theorem 23).

2. Sketch of the Method

The aim of this section is to give a better understanding of the method we are going to use in the present paper.

We will do that by considering the Dirichlet problem for Laplace equation in a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, whose boundary we denote by Σ as follows:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Sigma. \end{aligned} \quad (5)$$

Suppose that $g \in W^{1,p}(\Sigma)$, $1 < p < \infty$. If we want to find the solution in the form of a simple layer potential whose density belongs to $L^p(\Sigma)$, we have to solve an integral equation of the first kind on Σ as follows:

$$\int_\Sigma \varphi(y) s(x, y) d\sigma_y = g(x), \quad x \in \Sigma, \quad (6)$$

where $s(x, y)$ is the fundamental solution of Laplace equation

$$s(x, y) = \begin{cases} -\dfrac{1}{2\pi} \log \dfrac{1}{|x - y|}, & \text{if } n = 2, \\ -\dfrac{1}{\omega_n (n - 2)} \dfrac{1}{|x - y|^{n-2}}, & \text{if } n \geq 3. \end{cases} \quad (7)$$

In [9] a new method for discussing such an equation was proposed. Namely, the first step is to consider the differential (in the sense of the theory of differential forms) of both sides in (6). In this way we obtain the equation

$$\int_\Sigma \varphi(y) d_x [s(x, y)] d\sigma_y = dg(x), \quad x \in \Sigma, \quad (8)$$

in which we look for a solution $\varphi \in L^p(\Sigma)$.

The integral on the left hand side is a singular integral and it can be considered as a linear and continuous operator from $L^p(\Sigma)$ to $L_1^p(\Sigma)$ (we denote by $L_h^p(\Sigma)$ the space of the differential forms of degree h whose coefficients belong to $L^p(\Sigma)$ in every local coordinate system).

It must be remarked that, if $n \geq 3$, the space in which we look for the solution of (8) and the space in which the data is given are different.

We recall that, if B and B' are two Banach spaces and $S : B \to B'$ is a continuous linear operator, S can be reduced on the left if there exists a continuous linear operator $S' : B' \to B$ such that $S'S = I + T$, where I stands for the identity operator on B, and $T : B \to B$ is compact. Analogously, one can define an operator S reducible on the right. One of the main properties of such operators is that the equation $S\alpha = \beta$ has a solution if, and only if, $\langle \gamma, \beta \rangle = 0$ for any γ such that $S^*\gamma = 0$, S^* being the adjoint of S (for more details see, e.g., [10, 11]).

Let us denote by $S\varphi$ the left hand side of (8). In [9] a reducing operator S' was explicitly constructed. This implies that there exists a solution of (8) if, and only if, the compatibility conditions

$$\int_\Sigma dg \wedge h = 0 \quad (9)$$

are satisfied for any $h \in L_{n-2}^q(\Sigma)$ ($q = p/(p - 1)$) such that $S^*h = 0$. Moreover one can show that $S^*h = 0$ if, and only if, h is a weakly closed form. Therefore the compatibility conditions (9) are satisfied, and there exists a solution $\varphi \in L^p(\Sigma)$ of (8).

A left reduction is said to be *equivalent* if $N(S') = \{0\}$, where $N(S')$ denotes the kernel of S' (see, e.g., [11, page 19-20]). Obviously this means that $Sx = y$ if, and only if, $S'Sx = S'y$. In [12] it was remarked that if $N(S'S) = N(S)$, we still have a kind of equivalence. Indeed the coincidence of these two kernels implies the following fact: if y is such that the equation $Sx = y$ is solvable, then this equation is satisfied if, and only if, $S'Sx = S'y$.

Since $N(S'S) = N(S)$, then we have (8) equivalent to the Fredholm equation $S'S\varphi = S'(dg)$. These results lead to a simple layer potential theory for the Dirichlet problem (5).

As a consequence one can obtain also a double layer representation for the Neumann problem for Laplace equation [12].

A characteristic of this method is that it uses neither the theory of pseudodifferential operators nor the concept of hypersingular integrals.

This method has been used also for studying other BVPs. In particular in [13] it was used to study the Dirichlet and the Neumann problems in multiply connected domains. Among other things, an interesting by-product of these results was obtained as follows (see [13, Theorem 6.1]).

Let u be a harmonic function of class $C^1(\overline{\Omega})$, where Ω is the multiple connected domain (1). *There exists a 2-form v conjugate to u in Ω if, and only if,*

$$\int_{\partial\Omega_j} \frac{\partial u}{\partial v} d\sigma = 0, \quad j = 0, 1, \ldots, m. \quad (10)$$

An explicit integral expression for v was also given. We recall that the 2-form v is conjugate to u if $du = \delta v$, $dv = 0$.

The method has been applied to different BVPs for several PDEs (see [12–19]).

3. Preliminaries

In this paper Ω denotes an $(m + 1)$-*connected domain* of $\mathbb{R}^n$ ($n \geq 2$), that is an open-connected set of the form (1), where each Ω_j ($j = 0, \ldots, m$) is a bounded domain of $\mathbb{R}^n$ with

connected boundaries $\Sigma_j \in C^{1,\lambda}$ ($\lambda \in (0, 1]$), and such that $\overline{\Omega}_j \subset \Omega_0$ and $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$, $j, k = 1, \dots, m$, $j \neq k$. Let ν be the outwards unit normal on the boundary $\Sigma = \partial\Omega$.

We consider the classical Stokes system for the incompressible viscous fluid

$$\begin{aligned} \mu\Delta u &= \nabla p, \\ \operatorname{div} u &= 0, \end{aligned} \quad \text{in } \Omega, \tag{11}$$

where the unknowns $u = (u_1, \dots, u_n)$ and $p = p(x)$ are the velocity and pressure of the fluid flow, respectively, and the constant $\mu > 0$ is the kinematic viscosity of the fluid. A fundamental solution for this system is given by the pair of fundamental velocity tensor and its associated pressure vector

$$\gamma_{ij}(x, y) = \begin{cases} -\dfrac{1}{4\pi\mu} \left[\delta_{ij} \log \dfrac{1}{|x - y|} \right. \\ \left. \qquad + \dfrac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} \right], & \text{if } n = 2, \\[2ex] -\dfrac{1}{2\omega_n\mu} \left[\dfrac{\delta_{ij}}{n - 2} \dfrac{1}{|x - y|^{n-2}} \right. \\ \left. \qquad + \dfrac{(x_i - y_i)(x_j - y_j)}{|x - y|^n} \right], & \text{if } n \geq 3, \end{cases} \tag{12}$$

$$\varepsilon_j(x, y) = -\frac{1}{\omega_n} \frac{x_j - y_j}{|x - y|^n}, \tag{13}$$

$(i, j = 1, \dots, n)$, ω_n being the hypersurface measure of the unit sphere in $\mathbb{R}^n$. For a solution (u, p) of (11) we consider the following classical boundary operators:

$$\begin{aligned} T_j u &= \left[-\delta_{ij}p + \mu\left(\partial_j u_i + \partial_i u_j\right) \right] \nu_i, \\ T'_j u &= \left[\delta_{ij}p + \mu\left(\partial_j u_i + \partial_i u_j\right) \right] \nu_i, \end{aligned} \quad j = 1, \dots, n. \tag{14}$$

Through this paper, p indicates a real number such that $1 < p < +\infty$. We denote by $[L^p(\Sigma)]^n$ the space of all measurable vector-valued functions $u = (u_1, \dots, u_n)$ such that $|u_j|^p$ is integrable over Σ ($j = 1, \dots, n$). If h is any nonnegative integer, $L^p_h(\Sigma)$ is the vector space of all differential forms of degree h (briefly h-forms) defined on Σ such that their components are integrable functions belonging to $L^p(\Sigma)$ in a coordinate system of class C^1 and consequently in every coordinate system of class C^1. The space $[L^p_h(\Sigma)]^n$ is constituted by the vectors $(v_1, \dots, v_n)$ such that v_j is a differential form of $L^p_h(\Sigma)$ ($j = 1, \dots, n$). $[W^{1,p}(\Sigma)]^n$ is the vector space of all measurable vector-valued functions $u = (u_1, \dots, u_n)$ such that u_j belongs to the Sobolev space $W^{1,p}(\Sigma)$ ($j = 1, \dots, n$).

The pair (v, r) with components

$$v_i(x) = -\int_\Sigma \gamma_{ij}(x, y)\, \varphi_j(y)\, d\sigma_y, \quad i = 1, \dots, n, \ x \in \mathbb{R}^n, \tag{15}$$

$$r(x) = -\int_\Sigma \varepsilon_j(x, y)\, \varphi_j(y)\, d\sigma_y, \quad x \in \mathbb{R}^n \tag{16}$$

is the simple layer hydrodynamic potential with density φ.

The pair (w, q) with components

$$w_i(x) = \int_\Sigma T'_{j,y}\left[\gamma^i(x, y)\right] \psi_j(y)\, d\sigma_y, \quad i = 1, \dots, n, \ x \in \mathbb{R}^n, \tag{17}$$

$$q(x) = 2\mu \int_\Sigma \frac{\partial}{\partial\nu_y}\left[\varepsilon_j(x, y)\right] \psi_j(y)\, d\sigma_y, \quad x \in \mathbb{R}^n \tag{18}$$

is the double layer hydrodynamic potential with density ψ.

4. On the Bidimensional Case

It is wellknown that there are some exceptional plane domains in which no every harmonic function can be represented by a simple layer potential. The simplest example of this kind is given by the unit disk, for which one has

$$\int_{|y|=1} \log|x - y|\, ds_y = 0, \quad |x| < 1. \tag{19}$$

It is also known that such domains do not occur in higher dimensions. For similar questions for the Laplace equation and the elasticity system, see [13, Section 3] and [16, Section 4], respectively.

In this section we show that also for the Stokes system there are similar domains. We say that the boundary of the domain Ω is *exceptional* if there exists some constant vector which cannot be represented in Ω by a simple layer potential.

Denoting by Σ_R the circle of radius R centered at the origin, we have the following lemma.

Lemma 1. *The circle Σ_R with $R = \exp(1/2)$ is exceptional for the Stokes system.*

Proof. Keeping in mind that (see, e.g., [16, Section 4])

$$\int_{\Sigma_R} \log|x - y|\, ds_y = 2\pi R \log R,$$

$$\int_{\Sigma_R} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2}\, ds_y = \delta_{ij}\pi R, \quad |x| < R, \tag{20}$$

we find

$$\int_{\Sigma_R} \gamma_{ij}(x, y)\, ds_y = \frac{R}{4\mu}\delta_{ij}\left(2\log R - 1\right), \quad |x| < R. \tag{21}$$

Taking $R = \exp(1/2)$ we obtain the result. $\qquad\square$

Let us consider now the exceptional boundaries of not simply connected domains.

Proposition 2. *Let $\Omega \subset \mathbb{R}^2$ be an $(m + 1)$-connected domain. Denote by $\mathscr{P}$ the eigenspace in $[L^p(\Sigma)]^2$ of the singular integral system*

$$\int_\Sigma \varphi_j(y)\, \frac{\partial}{\partial s_x} \gamma_{ij}(x, y)\, ds_y = 0, \quad a.e. \ x \in \Sigma, \ i = 1, 2. \tag{22}$$

Then $\dim \mathscr{P} = 2(m + 1)$.

Proof. As in the proof of [16, Lemma 12], one can show that

$$\frac{\partial}{\partial s_x} \gamma_{ij}(x, y) = \frac{1}{4\pi\mu} \delta_{ij} \frac{\partial}{\partial s_x} \log |x - y| + \mathcal{O}\left(|y - x|^{h-1}\right),$$
(23)

deduce that system (22) can be regularized to a Fredholm one, and see that its index is zero. Since the vectors $e_i \chi_{\Sigma_j}$ (by χ_X we denote the characteristic function of the set X) ($i = 1, 2$, $j = 0, 1, \ldots, m$) are the only eigensolutions of the adjoint system

$$\int_\Sigma \varphi_j(y) \frac{\partial}{\partial s_y} \gamma_{ij}(x, y) \, ds_y = 0, \quad \text{a.e. } x \in \Sigma, \ i = 1, 2, \quad (24)$$

we have $\dim \mathcal{P} = 2(m + 1)$. $\square$

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be an $(m+1)$-connected domain. The following conditions are equivalent*

(1) *There exists a Hölder continuous vector function $\varphi \not\equiv 0$ such that*

$$\int_\Sigma \gamma(x, y) \varphi(y) \, ds_y = 0, \quad x \in \Sigma.$$
(25)

(2) *There exists a constant vector which cannot be represented in Ω by a simple layer potential;*

(3) *Σ_0 is exceptional.*

(4) *Let $\varphi_1, \ldots, \varphi_{2m+2}$ be linearly independent vectors of $\mathcal{P}$ (see Proposition 2), and let $c_{jk} = (\alpha_{jk}, \beta_{jk}) \in \mathbb{R}^2$ be given by*

$$\int_\Sigma \gamma(x, y) \varphi_j(y) \, ds_y = c_{jk},$$
(26)

$$x \in \Sigma_k, \ j = 1, \ldots, 2m + 2, \ k = 0, 1, \ldots, m.$$

Then $\det \mathscr{C} = 0$, where

$$\mathscr{C} = \begin{pmatrix} \alpha_{1,0} & \cdots & \alpha_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \alpha_{1,m} & \cdots & \alpha_{2m+2,m} \\ \beta_{1,0} & \cdots & \beta_{2m+2,0} \\ \cdots & \cdots & \cdots \\ \beta_{1,m} & \cdots & \beta_{2m+2,m} \end{pmatrix}.$$
(27)

Proof. The proof runs as in [16, Theorem 1] with obvious modifications. We omit the details. $\square$

5. Some Eigenspaces

We determine the structure of the kernel of a particular singular integral system. Namely, let us denote by $\mathcal{N}_p$ the space of $\psi \in [L_{n-2}^p(\Sigma)]^n$ such that

$$\int_\Sigma \psi_j(y) \wedge d_y \left[\gamma_{ij}(x, y)\right] = 0, \quad \text{a.e. on } \Sigma, \ i = 1, \ldots, n.$$
(28)

We begin by proving the following result.

Lemma 4. *Let $u \in [C_0^\infty(\mathbb{R}^n)]^n$. Then, for any $x \in \mathbb{R}^n$,*

$$u_i(x) = \mu \int_{\mathbb{R}^n} \Delta u_j(y) \gamma_{ij}(x, y) \, dy + \int_{\mathbb{R}^n} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} s(x, y) \, dy,$$
(29)

where $\gamma(x, y)$ and $s(x, y)$ are given by (12) and (7), respectively.

Proof. By the well-known Stokes identity we have

$$u_i(x) = \int_{\mathbb{R}^n} \Delta u_i(y) s(x, y) \, dy = \int_{\mathbb{R}^n} \Delta u_j(y) \delta_{ij} s(x, y) \, dy.$$
(30)

Since, for every $n \neq 2, 4$,

$$\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^n} = \frac{1}{(4 - n)(2 - n)} \frac{\partial^2}{\partial y_i \partial y_j} |x - y|^{4-n} - \delta_{ij} \omega_n s(x, y),$$

$$\gamma_{ij}(x, y) = \frac{\delta_{ij}}{2\mu} s(x, y) - \frac{1}{2\mu\omega_n} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^n}, \quad \forall n \geq 2,$$
(31)

we can rewrite

$$\gamma_{ij}(x, y) = \frac{\delta_{ij}}{\mu} s(x, y) - \frac{1}{2\mu\omega_n} \frac{1}{(4 - n)(2 - n)} \frac{\partial^2}{\partial y_i \partial y_j} \quad (32)$$
$$\times |x - y|^{4-n}.$$

Then

$$\delta_{ij} s(x, y) = \mu \gamma_{ij}(x, y) + \frac{1}{2\omega_n (4-n)(2-n)} \frac{\partial^2}{\partial y_i \partial y_j} |x - y|^{4-n},$$

$$u_i(x) = \mu \int_{\mathbb{R}^n} \Delta u_j(y) \gamma_{ij}(x, y) \, dy$$
$$+ \frac{1}{2\omega_n (4-n)(2-n)} \int_{\mathbb{R}^n} \Delta u_i(y) \frac{\partial^2}{\partial y_i \partial y_j} |x - y|^{4-n} dy.$$
(33)

Integrating by parts, it follows that the last integral is equal to

$$\frac{1}{2\omega_n (4 - n)(2 - n)} \int_{\mathbb{R}^n} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} \Delta_y |x - y|^{4-n} \, dy$$
(34)
$$= \int_{\mathbb{R}^n} \frac{\partial^2 u_j(y)}{\partial y_i \partial y_j} s(x, y) \, dy,$$

since $\Delta_y |x - y|^{4-n} = 2(4 - n)|x - y|^{2-n}$. Then the claim holds for $n \neq 2, 4$.

In the same manner it is possible to show formula (29) for $n = 2$ and $n = 4$ after observing that, if $n = 2$, we have

$$\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^2} = \frac{1}{2} \frac{\partial^2}{\partial y_i \partial y_j} |x - y|^2 \log |x - y| \quad (35)$$

$$- \delta_{ij} \log |x - y| - \frac{1}{2} \delta_{ij},$$

$\Delta_y |x - y|^2 \log |x - y| = 4(\log |x - y| + 1)$, while, for $n = 4$,

$$\frac{(x_i - y_i)(x_j - y_j)}{|x - y|^4} = \frac{\delta_{ij}}{2|x - y|^2} + \frac{1}{2} \frac{\partial^2}{\partial y_i \partial y_j} \log |x - y|,$$

(36)

$$\Delta_y \log |x - y| = 2/|x - y|^2. \qquad \square$$

Lemma 5. *Let $\zeta_1, \ldots, \zeta_n$ be differential forms in $L^p_{n-2}(\Sigma)$ such that $d\zeta_j = (-1)^{n-1} v_j d\sigma$ on Σ. One has $\psi \in \mathcal{N}_p$ if, and only if,*

$$\psi_j = \sum_{h=0}^m c_h \chi_{\Sigma_h} \zeta_j + \eta_j, \quad j = 1, \ldots, n,$$

(37)

where $c_0, \ldots, c_m \in \mathbb{R}$ and $\eta_1 \ldots, \eta_n$ are weakly closed forms belonging to $L^p_{n-2}(\Sigma)$.

Proof. It is easy to construct the differential forms $\zeta_1, \ldots, \zeta_n$. For example, one can take the restriction on Σ of the following forms: $\zeta_1 = (-1)^{n-1} x_2 dx^3 \cdots dx^n$, $\zeta_j = (-1)^{n-j} x_1 dx^2 \cdots \widehat{j} \cdots dx^n$ $(j = 2, \ldots, n)$. We remark that (37) holds if, and only if, the weak differentials $d\psi_j$ exist and

$$d\psi_j = (-1)^{n-1} \sum_{h=0}^m c_h \chi_{\Sigma_h} v_j d\sigma, \quad j = 1, \ldots, n,$$

(38)

that is,

$$\int_\Sigma \psi_j \wedge du_j = \sum_{h=0}^m c_h \int_{\Sigma_h} u \cdot v d\sigma, \quad \forall u \in [C_0^\infty(\mathbb{R}^n)]^n. \quad (39)$$

Let us prove that (39) holds if, and only if,

$$\int_{\Sigma_k} \psi_j \wedge du_j = c_k \int_{\Sigma_k} u \cdot v d\sigma,$$

$$\forall u \in [C_0^\infty(\mathbb{R}^n)]^n, \quad k = 0, \ldots, m.$$

(40)

It is obvious that (40) implies (39).

Conversely, suppose that (39) is true. Define $U_k^\varepsilon = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Sigma_k) < \varepsilon\}$, where $0 < \varepsilon < \min_{0 \le h < k \le m} \text{dist}(\Sigma_h, \Sigma_k)$. Let $v_k \in C_0^\infty(U_k^\varepsilon)$ be such that $v_k = 1$ in $U_k^{\varepsilon/2}$. Since $v_k u \in [C_0^\infty(\mathbb{R}^n)]^n$, we may write

$$\int_\Sigma \psi_j \wedge d(v_k u_j) = \sum_{h=0}^m c_h \int_{\Sigma_h} v_k u \cdot v d\sigma,$$

(41)

and (40) follows immediately.

Suppose now that (39) is true. From (40) it follows that

$$\int_{\Sigma_k} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = c_k \int_{\Sigma_k} \gamma_{ij}(x, y) v_j(y) d\sigma_y,$$

$$\forall x \notin \Sigma_k.$$

(42)

An integration by parts shows that

$$\int_{\Sigma_k} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0, \quad \forall x \notin \overline{\Omega}_k.$$

(43)

Taking the exterior angular boundary value (for the definition of internal (external) angular boundary values see, e.g., [20, page 53] or [21, page 293]), we have

$$\int_{\Sigma_k} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0$$

(44)

a.e. on Σ_k. Arguing as in [9, pages 189-190], this implies that

$$\int_{\Sigma_k} \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0$$

(45)

also in Ω_k. Summing over k we find

$$\int_\Sigma \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0,$$

(46)

for every $x \in \mathbb{R}^n \setminus \Sigma$ and a.e. on Σ. In particular ψ is the solution of the singular integral system (28).

Conversely, suppose (28) holds. Arguing again as in [9, pages 189-190], from (28) it follows that

$$\int_\Sigma \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)] = 0, \quad x \notin \Sigma.$$

(47)

Since $\varepsilon_j(x, y) = -\partial_{x_j} s(x, y)$, system (11) implies that $\Delta_x [\gamma_{ij}(x, y)] = -(1/\mu)(\partial^2/\partial x_i \partial x_j) s(x, y)$. Hence,

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_\Sigma \psi_j(y) \wedge d_y [s(x, y)] = 0, \quad x \notin \Sigma.$$

(48)

Therefore, there exist some constants $a_0, a_1, \ldots, a_m$ such that

$$\partial_j \Psi_j(x) = \begin{cases} -a_h & x \in \Omega_h, \ h = 1, \ldots, m, \\ -a_0 & x \in \Omega, \\ 0 & x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

(49)

where

$$\Psi_j(x) = \int_\Sigma \psi_j(y) \wedge d_y [s(x, y)].$$

(50)

Then, on account of Lemma 4, for every $u \in [C_0^\infty(\mathbb{R}^n)]^n$,

$$\int_\Sigma \psi_j \wedge du_j = \mu \int_{\mathbb{R}^n} \Delta u_j(x) dx \int_\Sigma \psi_j(y) \wedge d_y [\gamma_{ij}(x, y)]$$

$$+ \int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) dx \int_\Sigma \psi_j(y) \wedge d_y [s(x, y)].$$

(51)

The first term of the right hand side vanishes because of (47). As far as the second one is concerned, integrating by parts we get

$$\int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) \Psi_j(x) \, dx$$

$$= \sum_{h=1}^m \int_{\Omega_h} \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) \Psi_j(x) \, dx$$

$$+ \int_\Omega \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) \Psi_j(x) \, dx$$

$$+ \int_{\mathbb{R}^n \setminus \overline{\Omega}_0} \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) \Psi_j(x) \, dx$$

$$= -\sum_{h=1}^m \int_{\Sigma_h} \partial_i u_i \Psi_j \nu_j d\sigma + \sum_{h=1}^m \int_{\Omega_h} \partial_i u_i \partial_j \Psi_j dx \quad (52)$$

$$+ \int_\Sigma \partial_i u_i \Psi_j \nu_j d\sigma - \int_\Omega \partial_i u_i \partial_j \Psi_j dx$$

$$- \int_{\Sigma_0} \partial_i u_i \Psi_j \nu_j d\sigma + \int_{\mathbb{R}^n \setminus \overline{\Omega}_0} \partial_i u_i \partial_j \Psi_j dx$$

$$= \sum_{h=1}^m \int_{\Omega_h} \partial_i u_i \partial_j \Psi_j dx - \int_\Omega \partial_i u_i \partial_j \Psi_j dx.$$

Hence, by (49),

$$\int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j} u_i(x) \Psi_j(x) \, dx$$

$$= -\sum_{h=1}^m a_h \int_{\Omega_h} \partial_i u_i dx + a_0 \int_\Omega \partial_i u_i dx$$

$$(53)$$

$$= \sum_{h=1}^m a_h \int_{\Sigma_h} u \cdot \nu d\sigma + a_0 \int_\Sigma u \cdot \nu d\sigma$$

$$= a_0 \int_{\Sigma_0} u \cdot \nu d\sigma + \sum_{h=1}^m (a_0 + a_h) \int_{\Sigma_h} u \cdot \nu d\sigma.$$

By setting $c_0 = a_0$ and $c_h = a_0 + a_h$ ($h = 1, \ldots, m$) we get the claim. $\qquad\square$

Remark 6. Lemma 5 shows that the dimension of the kernel $\mathcal{N}_p$ is infinite. However, if we consider the quotient space $\mathcal{N}_p / \Xi_p$, Ξ_p being the space of weakly closed differential forms in $L^p_{n-2}(\Sigma)$, we have $\dim(\mathcal{N}_p / \Xi_p) = m + 1$.

We conclude this section by recalling some properties concerning the following eigenspaces:

$$\mathscr{V}_\pm = \left\{ \varphi_k \in L^p(\Sigma) : \pm \frac{1}{2} \varphi_k(x) \right.$$

$$\left. + \int_\Sigma F_{ki}(x, y) \varphi_i(y) d\sigma_y = 0, \ k = 1, \ldots, n \right\},$$

$$\mathscr{W}_\pm = \left\{ \varphi_k \in L^p(\Sigma) : \mp \frac{1}{2} \varphi_k(x) \right.$$

$$\left. + \int_\Sigma F_{ik}(y, x) \varphi_i(y) d\sigma_y = 0, \ k = 1, \ldots, n \right\},$$

$$(54)$$

where (see, e.g., [22])

$$F_{ki}(x, y) := T'_{i,y} \left[\gamma^k(x, y) \right]$$

$$= -\frac{n}{\omega_n} \frac{(x_k - y_k)(x_i - y_i)(x_j - y_j)}{|x - y|^{n+2}} \nu_j(y). \quad (55)$$

For the proofs of the following two results see [7, Lemma 3.3] and [8, Theorem 3.2], respectively.

Proposition 7. *The sets $\mathscr{V}_+$ and $\mathscr{W}_-$ are linear subspaces of $L^1(\Sigma)$ and*

$$\dim(\mathscr{V}_+) = \dim(\mathscr{W}_-) = 1 + \frac{n(n+1)m}{2}. \quad (56)$$

A basis of $\mathscr{W}_-$ is expressed by the fields $\{\psi_{ih}, \nu : i = 1, \ldots, n(n+1)/2, h = 1, \ldots, m\}$. The simple layer potentials v_{ih} whose densities are ψ_{ik} such that: $v_{ih}|_{\overline{\Omega}_k} = \delta_{hk} \rho_i$, $i = 1, \ldots, n(n+1)/2$, $h, k = 1, \ldots, m$, where ρ_i are rigid displacement in $\mathbb{R}^n$, specifically $\rho_i(x) = e_i$, $i = 1, \ldots, n$, and, for $i = n+1, \ldots, n(n+1)/2$, $\rho_i(x) = (e_h \wedge e_k)x$, $h = 1, \ldots, n-1$, $k = h+1, \ldots, n$, $h[n - (h+1)/2] + k = i$.

In addition, every $\psi \in \mathscr{W}_-$ has the property that $v|_{\Sigma_0} = 0$, where v is the simple layer potential with density ψ.

Proposition 8. *The sets $\mathscr{V}_-$ and $\mathscr{W}_+$ are linear subspaces of $L^1(\Sigma)$ and*

$$\dim(\mathscr{V}_-) = \dim(\mathscr{W}_+) = \frac{n(n+1)}{2} + m. \quad (57)$$

A basis for $\mathscr{W}_+$ is expressed by the fields $\{\psi_i, \nu \chi_{\Sigma_h} : i = 1, \ldots, n(n+1)/2, h = 1, \ldots, m\}$, where ψ_i, $i = 1, \ldots, n(n+1)/2$ are zero on $\Sigma \setminus \Sigma_0$, and such that the simple layer potentials with density ψ_i are $n(n+1)/2$ rigid displacement in Ω_0 (linearly independent for $n \geq 3$).

Finally, every function φ which is the restriction to Σ of a rigid displacement belongs to $\mathscr{V}_-$.

One recalls that if $\varphi \in [L^1(\Sigma)]^n$ belongs to one of the eigenspaces $\mathscr{V}_\pm, \mathscr{W}_\pm$, then $\varphi \in [C^\lambda(\Sigma)]^n$. This follows from general results about integral equations (see [8, Lemma 31] and [7, page 81]).

Remark 9. We can make the statement of Proposition 8 slightly more precise, saying that *the simple layer potentials with density ψ_i are $n(n + 1)/2$ rigid displacement in Ω_0 linear independent for any $n \geq 2$, unless $n = 2$ and Σ_0 is exceptional.* Indeed, let us show that if $n = 2$ and Σ_0 is not exceptional, such rigid displacements are linearly independent. Let c_i be such that

$$\sum_{i=1}^{3} c_i \int_{\Sigma_0} \psi_i (y) \gamma (x, y) d\sigma_y = 0, \quad \text{in } \Omega_0. \tag{58}$$

We have also

$$\int_{\Sigma_0} \sum_{i=1}^{3} c_i \psi_i (y) \gamma (x, y) d\sigma_y = 0, \quad \text{a.e. } x \in \Sigma_0. \tag{59}$$

Let $\varphi = \sum_{i=1}^{3} c_i \psi_i$. In view of the equivalence between (1) and (3) of Theorem 3, φ has to vanish. Therefore $c_i = 0$ ($i = 1, 2, 3$) because of the linearly independence of ψ_i. On the other hand, if $n = 2$ and Σ_0 is exceptional, Theorem 3 shows that the potentials with densities $\{\psi_i\}_{i=1,2,3}$ are linearly dependent.

6. Reduction of a Certain Singular Integral Operator

For every $\psi \in L_1^p(\Sigma)$, let Θ_h be the operator defined by

$$\Theta_h (\psi) (x) = * \left(\int_{\Sigma} d_x \left[s_{n-2} (x, y) \right] \wedge \psi (y) \wedge dx^h \right), \tag{60}$$
$$x \in \Omega,$$

where $*$ and d denote the Hodge star operator and the exterior derivative, respectively, and $s_h(x, y)$ is the double h-form introduced by Hodge in [23] as follows:

$$s_h (x, y) = \sum_{j_1 < \cdots < j_h} s (x, y) dx^{j_1} \cdots dx^{j_h} dy^{j_1} \cdots dy^{j_h}. \tag{61}$$

Note that the operator Θ_h satisfies the equation

$$\partial_h \int_{\Sigma} u (y) \frac{\partial}{\partial v_y} s (x, y) d\sigma_y = -\Theta_h (du), \quad x \in \Omega, \tag{62}$$

for each $u \in W^{1,p}(\Sigma)$, since (see [9, page 187])

$$* d \int_{\Sigma} u (y) \frac{\partial}{\partial v_y} s (x, y) d\sigma_y$$
$$\tag{63}$$
$$= d_x \int_{\Sigma} du (y) \wedge s_{n-2} (x, y), \quad x \in \Omega.$$

Moreover we introduce the operators $\mathcal{H}_{jh}$ defined as

$$\mathcal{H}_{jh} (\psi) (x) = \Theta_h \left(\psi_j \right) (x) - \frac{\delta^{123\cdots n}_{lij_3\cdots j_n}}{(n-2)!}$$
$$\times \int_{\Sigma} \partial_{x_h} H_{lj} (x, y) \wedge \psi_i (y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}, \tag{64}$$

for every $\psi \in [L_1^p(\Sigma)]^n$, where

$$H_{lj} (x, y) = \frac{1}{\omega_n} \frac{(y_l - x_l) (y_j - x_j)}{|y - x|^n}. \tag{65}$$

In the sequel du denotes the vector $(du_1, \ldots, du_n)$ whose elements are 1-forms, and $\psi = (\psi_1, \ldots, \psi_n) \in [L_1^p(\Sigma)]^n$.

Lemma 10. *Let (w, q) be the double layer hydrodynamic potential of (17)-(18) with density $u \in [W^{1,p}(\Sigma)]^n$. Then, for $x \notin \Sigma$,*

$$\partial_h w_j (x) = \mathcal{H}_{jh} (du) (x), \tag{66}$$

$$q (x) = 2\mu \Theta_h (du_h) (x), \tag{67}$$

where $\mathcal{H}_{jh}$ and Θ_h are given by (60) and (64), respectively.

Proof. Note that, even if one could prove (66)-(67) directly, it seems easier to deduce them from the similar results we have already obtained for the elasticity system (see [16, Section 3]). For $k > (n - 2)/n$, let $\overset{(k)}{w}$ be the double layer elastic potential with density u, that is,

$$\overset{(k)}{w}_j (x) = \int_{\Sigma} u_i (y) \overset{(k)}{L}_{i,y} \left[\overset{(k)}{\Gamma^j} (x, y) \right] d\sigma_y, \tag{68}$$

where $\overset{(k)}{L}$ and $\overset{(k)}{\Gamma}$ are the stress operator and the Kelvin's matrix associated to the Lamé system $-\Delta u - k\nabla \operatorname{div} u = 0$, respectively.

Thanks to [16, Lemma 1], we know that

$$\partial_h \overset{(k)}{w}_j (x) = \overset{(k)}{\mathcal{H}}_{jh} (du) (x), \tag{69}$$

where

$$\overset{(k)}{\mathcal{H}}_{jh} (\psi) (x) = \Theta_h \left(\psi_j \right) (x) - \frac{\delta^{123\cdots n}_{lij_3\cdots j_n}}{(n-2)!}$$
$$\times \int_{\Sigma} \partial_{x_h} \overset{(k)}{H}_{lj} (x, y) \wedge \psi_i (y) \wedge dy^{j_3} \cdots dy^{j_n},$$

$$\overset{(k)}{H}_{lj} (x, y) = \frac{k}{\omega_n (k+1)} \frac{(y_l - x_l) (y_j - x_j)}{|y - x|^n}$$
$$- \frac{1}{k+1} \delta_{lj} s (x, y), \tag{70}$$

and Θ_h is given by (60).

From [16, formula (5)] (where we set $\xi = 1$), letting $k \to +\infty$, we get

$$\partial_{x_h} \left\{ \overset{(k)}{L}_{i,y} \left[\overset{(k)}{\Gamma^j} (x, y) \right] \right\}$$
$$\longrightarrow -\frac{n}{\omega_n} \partial_{x_h} \left\{ \frac{(y_i - x_i) (y_j - x_j) (y_k - x_k)}{|x - y|^{n+2}} v_k (y) \right\} \tag{71}$$
$$= \partial_{x_h} T'_{i,y} \left[\gamma^j (x, y) \right],$$

$x \notin \Sigma$, from which $\partial_h \overset{(k)}{w} \to \partial_h w$ as $k \to +\infty$. Therefore we obtain formula (66) by letting $k \to +\infty$ in (69). Formula (67) is an immediate consequence of (62) because $\varepsilon_j(x, y) = -\partial_{x_j} s(x, y)$. □

For the next lemma it is convenient to recall here two jump formulas proved in [16, Lemmas 2 and 3].

Let $f \in L^1(\Sigma)$. If $\eta \in \Sigma$ is a Lebesgue point for f, we get

$$\lim_{x \to \eta} \int_\Sigma f(y) \partial_{x_s} \frac{(y_l - x_l)(y_j - x_j)}{|x - y|^n} d\sigma_y$$

$$= \frac{\omega_n}{2} \left(\delta_{lj} - 2\nu_j(\eta) \nu_l(\eta) \right) \nu_s(\eta) f(\eta) \qquad (72)$$

$$+ \int_\Sigma f(y) \partial_{x_s} \frac{(y_l - \eta_l)(y_j - \eta_j)}{|x - y|^n} d\sigma_y,$$

where the limit has to be understood as an internal angular boundary value, and the integral in the right hand side is a singular integral.

Further, let $\psi \in L_1^p(\Sigma)$ and write ψ as $\psi = \psi_h dx^h$ with

$$\nu_h \psi_h = 0. \qquad (73)$$

Assumption (73) is not restrictive, because, given the 1-form ψ on Σ, there exist scalar functions ψ_h defined on Σ such that $\psi = \psi_h dx^h$ and (73) holds (see [24, page 41]). Then, for almost every $\eta \in \Sigma$,

$$\lim_{x \to \eta} \Theta_h(\psi)(x) = -\frac{1}{2} \psi_h(\eta) + \Theta_h(\psi)(\eta), \qquad (74)$$

where Θ_h is given by (60), and the limit has to be understood again as an internal angular boundary value.

Lemma 11. Let $\psi \in L_1^p(\Sigma)$. Let one write ψ as $\psi = \psi_h dx^h$ and suppose that (73) holds. Then, for almost every $\eta \in \Sigma$,

$$\lim_{x \to \eta} \frac{1}{(n - 2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(x, y) \wedge \psi(y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}$$

$$= -\frac{1}{2} \left[\nu_j(\eta) \psi_i(\eta) + \nu_i(\eta) \psi_j(\eta) \right] \nu_s(\eta)$$

$$+ \frac{1}{(n-2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(\eta, y) \wedge \psi(y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}, \qquad (75)$$

where H_{lj} is defined by (65), and the limit has to be understood as an internal angular boundary value.

Proof. We have

$$\frac{1}{(n - 2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(x, y) \wedge \psi(y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}$$

$$= \frac{1}{(n - 2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \delta_{rhj_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(x, y) \psi_h(y) \nu_r(y) d\sigma_y$$

$$= \delta_{rh}^{li} \int_\Sigma \partial_{x_s} H_{lj}(x, y) \psi_h(y) \nu_r(y) d\sigma_y. \qquad (76)$$

Hence, by (65) and (72),

$$\lim_{x \to \eta} \frac{1}{(n - 2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(x, y) \wedge \psi(y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}$$

$$= \frac{\delta_{rh}^{li}}{2} \left(\delta_{lj} - 2\nu_j(\eta) \nu_l(\eta) \right) \nu_s(\eta) \nu_r(\eta) \psi_h(\eta)$$

$$+ \frac{1}{(n-2)!} \delta_{lij_3 \cdots j_n}^{123 \cdots n} \int_\Sigma \partial_{x_s} H_{lj}(\eta, y) \wedge \psi(y) \wedge dy^{j_3} \cdots \wedge dy^{j_n}. \qquad (77)$$

Keeping in mind (73), we find

$$\frac{\delta_{rh}^{li}}{2} \left(\delta_{lj} - 2\nu_j \nu_l \right) \nu_s \nu_r \psi_h = \left(\frac{1}{2} \delta_{lj} \nu_s - \nu_j \nu_l \nu_s \right) \left(\nu_l \psi_i - \nu_i \psi_l \right)$$

$$= -\frac{1}{2} \nu_s \nu_j \psi_i - \frac{1}{2} \nu_s \nu_i \psi_j, \qquad (78)$$

and the result follows. □

Lemma 12. Let $\psi = (\psi_1, \ldots, \psi_n) \in [L_1^p(\Sigma)]^n$. Then, for almost every $\eta \in \Sigma$,

$$\lim_{x \to \eta} \mu \left[2\delta_{ij} \Theta_h(\psi_h)(x) + \mathcal{H}_{ij}(\psi)(x) + \mathcal{H}_{ji}(\psi)(x) \right] \nu_i(x)$$

$$= \mu \left[2\delta_{ij} \Theta_h(\psi_h)(\eta) + \mathcal{H}_{ij}(\psi)(\eta) + \mathcal{H}_{ji}(\psi)(\eta) \right] \nu_i(\eta), \qquad (79)$$

Θ_h and $\mathcal{H}$ being as in (60) and (64), respectively, and the limit has to be understood as an internal angular boundary value.

Proof. Let us write ψ_i as $\psi_i = \psi_{ih} dx^h$ with

$$\nu_h \psi_{ih} = 0, \quad i = 1, \ldots, n. \qquad (80)$$

On account of (72) and (74), we infer

$$\lim_{x \to \eta} \mu \left[2\delta_{ij} \Theta_h(\psi_h)(x) + \mathcal{H}_{ij}(\psi)(x) + \mathcal{H}_{ji}(\psi)(x) \right] \nu_i(x)$$

$$= \mu \Psi_{ij}(\psi)(\eta) \nu_i(\eta)$$

$$+ \mu \left[2\delta_{ij} \Theta_h(\psi_h)(\eta) \right.$$

$$\left. + \mathcal{H}_{ij}(\psi)(\eta) + \mathcal{H}_{ji}(\psi)(\eta) \right] \nu_i(\eta), \qquad (81)$$

where

$$\Psi_{ij}(\psi) = -\delta_{ij} \psi_{hh} - \frac{1}{2} \psi_{ij} + \frac{1}{2} \left(\nu_i \psi_{ss} + \nu_s \psi_{si} \right) \nu_j$$

$$- \frac{1}{2} \psi_{ji} + \frac{1}{2} \left(\nu_j \psi_{ss} + \nu_s \psi_{sj} \right) \nu_i. \qquad (82)$$

By (80) we get $\Psi_{ij}(\psi)\nu_i = -\psi_{hh}\nu_j - \psi_{ij}\nu_i/2 + \psi_{ss}\nu_j + \nu_s\psi_{sj}/2 = 0$. □

Remark 13. Whenever we consider external boundary values, we have just to change the sign in the first term on the right hand sides in (72), (74), and (75), while (79) remains unchanged.

Lemma 14. *Let w be the double layer potential* (17) *with density* $u \in [W^{1,p}(\Sigma)]^n$. *Then* $T_{+,j}w = T_{-,j}w = \mu[2\delta_{ij}\Theta_h(du_h) + \mathscr{H}_{ij}(du) + \mathscr{H}_{ji}(du)]v_i$ *a.e. on* Σ, *where* T_+w *and* T_-w *denote the internal and the external angular boundary limits of* Tw, *respectively, and* Θ_h *is given by* (60) *and* $\mathscr{H}$ *by* (64).

Proof. It is an immediate consequence of (66), (67), (79), and Remark 13. □

Proposition 15. *Let* $R : [L^p(\Sigma)]^n \rightarrow [L_1^p(\Sigma)]^n$ *be the following singular integral operator*

$$R\varphi(x) = -\int_\Sigma d_x\left[\gamma(x, y)\right]\varphi(y)\, d\sigma_y. \tag{83}$$

Let one define $R' : [L_1^p(\Sigma)]^n \rightarrow [L^p(\Sigma)]^n$ *to be the singular integral operator*

$$R'_j(\psi)(x) = \mu\left[2\delta_{ij}\Theta_h(\psi_h)(x) + \mathscr{H}_{ij}(\psi)(x) + \mathscr{H}_{ji}(\psi)(x)\right]v_i(x). \tag{84}$$

Then

$$R'R\varphi = \frac{1}{4}\varphi - K^2\varphi, \tag{85}$$

where

$$K\varphi(x) = -\int_\Sigma T_x\left[\gamma(x, y)\right]\varphi(y)\, d\sigma_y. \tag{86}$$

Proof. Let v be the simple layer potential (15) with density $\varphi \in [L^p(\Sigma)]^n$. In view of Lemma 14, we have a.e. on Σ

$$R'_j(R\varphi) = \mu\left[2\delta_{ij}\Theta_h(dv_h) + \mathscr{H}_{ij}(dv) + \mathscr{H}_{ji}(dv)\right]v_i = T_jw, \tag{87}$$

where w is the double layer potential (17) with density v. Moreover, if $x \in \Omega$,

$$w_k(x) = \int_\Sigma v_i(y)\, T'_{i,y}\left[\gamma^k(x, y)\right]d\sigma_y$$
$$= v_k(x) + \int_\Sigma \gamma_{ik}(x, y)\, T_i\left[v(y)\right]d\sigma_y, \tag{88}$$

and then, on account of (86),

$$Tw = \frac{1}{2}Tv - K(Tv) = \frac{1}{2}\left(\frac{1}{2}\varphi + K\varphi\right) - K\left(\frac{1}{2}\varphi + K\varphi\right)$$
$$= \frac{1}{4}\varphi - K^2\varphi. \tag{89}$$
□

7. The Dirichlet Problem

Let us consider the Dirichlet problem for the Stokes system

$$\mu\Delta v = \nabla r \quad \text{in } \Omega,$$
$$\text{div } v = 0 \quad \text{in } \Omega, \tag{90}$$
$$v = f \quad \text{on } \Sigma,$$

where the given data $f \in [W^{1,p}(\Sigma)]^n$ satisfies the compatibility condition (4).

The aim of the present section is to study the representability of the solution of this problem by means of a simple layer hydrodynamic potential (15)-(16).

By the symbol $\mathscr{S}^p$ we mean the class of the simple layer hydrodynamic potentials (15)-(16) with density in $[L^p(\Sigma)]^n$. Whenever $n = 2$ and Σ_0 is exceptional (see Section 4), we say that (v, r) belongs to $\mathscr{S}^p$ if, and only if,

$$v(x) = -\int_\Sigma \gamma(x, y)\, \varphi(y)\, d\sigma_y + c, \quad x \in \Omega,$$
$$r(x) = -\int_\Sigma \varepsilon_j(x, y)\, \varphi_j(y)\, d\sigma_y, \quad x \in \Omega, \tag{91}$$

where $\varphi \in [L^p(\Sigma)]^2$ and $c \in \mathbb{R}^2$.

We will see that condition (4) is not sufficient to prove the existence of the solution in the class $\mathscr{S}^p$, but it must be satisfied on each Σ_j, $j = 0, 1, \ldots, m$.

We begin by proving the following result.

Theorem 16. *Given* $\omega \in [L_1^p(\Sigma)]^n$, *there exists a solution* $\varphi \in [L^p(\Sigma)]^n$ *of the singular integral system*

$$-\int_\Sigma d_x\left[\gamma(x, y)\right]\varphi(y)\, d\sigma_y = \omega(x), \quad \text{a.e. } x \in \Sigma, \tag{92}$$

if, and only if,

$$\int_\Sigma \psi_i \wedge \omega_i = 0, \tag{93}$$

for every $\psi = (\psi_1, \ldots, \psi_n) \in [L_{n-2}^q(\Sigma)]^n$ $(q = p/(p-1))$ *such that the weak differentials* $d\psi_j$ *exist and* (38) *holds for some real constants* $c_0, \ldots, c_m$.

Proof. Consider the adjoint of R (see (83)), $R^* : [L_{n-2}^q(\Sigma)]^n \rightarrow [L^q(\Sigma)]^n$, that is, the operator whose components are given by

$$R_i^*\psi(x) = -\int_\Sigma \psi_i(y) \wedge d_y\left[\gamma_{ij}(x, y)\right]. \tag{94}$$

Proposition 15 implies that the integral system (92) has a solution $\varphi \in [L^p(\Sigma)]^n$ if, and only if,

$$\int_\Sigma \psi_i \wedge \omega_i = 0, \tag{95}$$

for each $\psi = (\psi_1, \ldots, \psi_n) \in [L_{n-2}^q(\Sigma)]^n$ such that $R^*\psi = 0$. The result follows from Lemma 5. □

Proposition 17. *Given* $f \in [W^{1,p}(\Sigma)]^n$, *there exists a solution of the BVP*

$$(\widetilde{v}, \widetilde{r}) \in \mathscr{S}^p,$$
$$\mu\Delta\widetilde{v} = \nabla\widetilde{r} \quad \text{in } \Omega,$$
$$\text{div } \widetilde{v} = 0 \quad \text{in } \Omega, \tag{96}$$
$$d\widetilde{v} = df \quad \text{on } \Sigma,$$

if, and only if, conditions (3) *are satisfied. The density* φ *of the pair* $(\widetilde{v}, \widetilde{r})$ *(see* (15)-(16)*) solves the singular integral system* $R\varphi = df$, *where* R *is given by* (83).

Proof. Clearly, there exists a solution of this BVP if, and only if, there exists a solution $\varphi \in [L^p(\Sigma)]^n$ of the singular integral system

$$-\int_\Sigma d_x \left[\gamma(x,y)\right] \varphi(y)\, d\sigma_y = df(x), \quad \text{a.e. } x \in \Sigma. \quad (97)$$

In view of Theorem 16, there exists a solution φ of this system if, and only if,

$$\int_\Sigma \psi_i \wedge df_i = 0, \quad (98)$$

for every $\psi = (\psi_1, \ldots, \psi_n) \in [L^q_{n-2}(\Sigma)]^n$ satisfying $R^*\psi = 0$, that is, such that the weak differentials $d\psi_j$ exist and (39) holds for some real constants $c_0, \ldots, c_m$. Equation (39) being true for any $u \in [W^{1,p}(\Sigma)]^n$, we can write

$$\int_\Sigma \psi_i \wedge df_i = \sum_{h=0}^m c_h \int_{\Sigma_h} f \cdot v d\sigma, \quad (99)$$

because of a density argument. In view of the arbitrariness of $c_0, \ldots, c_m$, (98) is satisfied if, and only if, (3) holds. □

Proposition 18. *Let* $a_h \in \mathbb{R}^n$ ($h = 0, \ldots, m$). *Let* ψ_{ik}, $i = 1, \ldots, n$, $k = 1, \ldots, m$, *be the elements of the basis of* $\mathcal{W}_-$ *given by Proposition 7. The pair*

$$v_0(x) = \sum_{k=1}^m \sum_{i=1}^n \left(a_k^i - a_0^i\right) \int_\Sigma \gamma(x,y)\, \psi_{ik}(y)\, d\sigma_y + a_0, \quad x \in \Omega,$$

$$r_0(x) = \sum_{k=1}^m \sum_{i=1}^n \left(a_k^i - a_0^i\right) \int_\Sigma \varepsilon(x,y)\, \psi_{ik}(y)\, d\sigma_y, \quad x \in \Omega,$$
$$(100)$$

is the solution of the BVP

$$(v_0, r_0) \in \mathcal{S}^p,$$
$$\mu \Delta v_0 = \nabla r_0 \quad in \ \Omega,$$
$$\text{div } v_0 = 0 \quad in \ \Omega,$$
$$v_0 = a_h \quad on \ \Sigma_h, \ h = 0, \ldots, m.$$
$$(101)$$

Proof. The pair (v_0, r_0) belongs to $\mathcal{S}^p$ (for $n = 2$, see Remark 9). Obviously it satisfies the Stokes system, and it satisfies the boundary conditions since, thanks to Proposition 7,

$$v_0|_{\Sigma_0} = \sum_{k=1}^m \sum_{i=1}^n \left(a_k^i - a_0^i\right) v_{ik}|_{\Sigma_0} + a_0 = a_0,$$

$$v_0|_{\Sigma_h} = \sum_{k=1}^m \sum_{i=1}^n \left(a_k^i - a_0^i\right) v_{ik}|_{\Sigma_h} + a_0 \quad (102)$$

$$= \sum_{k=1}^m \sum_{i=1}^n \left(a_k^i - a_0^i\right) \delta_{hk} e_i + a_0 = a_h,$$

for any $h = 1, \ldots, m$. □

Theorem 19. *Given* $f \in [W^{1,p}(\Sigma)]^n$, *the Dirichlet problem*

$$(v, r) \in \mathcal{S}^p,$$
$$\mu \Delta v = \nabla r \quad in \ \Omega,$$
$$\text{div } v = 0 \quad in \ \Omega,$$
$$v = f \quad on \ \Sigma,$$
$$(103)$$

is solvable if, and only if, conditions (3) *are satisfied. Moreover the solution* (v, r) *is unique* (r *is unique up to an additive constant).*

Proof. Suppose conditions (3) are satisfied. Let $(\tilde{v}, \tilde{r})$ be a solution of the problem (96). Since $d\tilde{v} = df$ on Σ, $\tilde{v} = f + a_h$ on Σ_h ($h = 0, \ldots, m$) for some $a_h \in \mathbb{R}^n$. The pair $(v, r) = (\tilde{v}, \tilde{r}) - (v_0, r_0)$, where v_0 and r_0 are given by (100), solves the problem (103).

Conversely, if there exists a solution (v, r) of (103), the compatibility condition (4) has to be satisfied. Moreover, for any $j = 1, \ldots, m$, (v, r) is the solution of the Stokes system also in Ω_j. Therefore conditions (3) are satisfied for $j = 1, \ldots, m$. These, together with (4), imply (3) also for $j = 0$. The uniqueness is known [7, Theorem 5.5]. □

Remark 20. The density (φ, ε) of (v, r) can be written as $(\varphi, \varepsilon) = (\varphi_0 + \lambda_0, \varepsilon)$, where φ_0 solves the singular integral system (97), and (λ_0, ε) is the density of a simple layer potential which is constant on every connected component of Σ.

Remark 21. If $n \geq 3$ or $n = 2$ and Σ_0 is not exceptional, denoting by φ the density of the simple layer potential (15)-(16) obtained in Theorem 19, we have φ that solves the integral system of the first kind

$$-\int_\Sigma \gamma(x,y) \varphi(y)\, d\sigma_y = f(x) \quad (104)$$

on Σ. Therefore, Theorem 19 can be seen as an existence theorem for the integral system of the first kind (104) in $L^p(\Sigma)$.

If $n = 2$ and Σ_0 is exceptional, we have the existence of a solution $(\varphi, c) \in [L^p(\Sigma)]^2 \times \mathbb{R}^2$ of the integral equation

$$-\int_\Sigma \gamma(x,y) \varphi(y)\, d\sigma_y = f(x) + c \quad on \ \Sigma. \quad (105)$$

Remark 22. Observe that the solvability of the Dirichlet problem (90) by means of a simple layer potential hinges on the singular integral system (97). Thanks to Proposition 15, the operator R' provides a left reduction for such a system. This reduction is not an equivalent one, but, as in [25, pages 253-254], one can show that R' is a weakly equivalent reduction (see definition in Section 3). Since the system $R\varphi = df$ is solvable, we have $R\varphi = df$ if, and only if, φ is solution of the Fredholm system $R'R\varphi = R'df$. In this sense, such Fredholm system is equivalent to the problem (103).

In order to obtain a similar integral representation for the solution of the Dirichlet problem (90) when f satisfies the only condition (4), we need to modify the representation of the solution by adding an extra term.

By $\widetilde{\mathcal{S}}^p$ we denote the space of all pairs (v, r) written as

$$
v_i(x) = - \int_\Sigma \gamma_{ij}(x, y) \varphi_j(y) \, d\sigma_y
$$

$$
+ \int_\Sigma T'_{j,y} \left[\gamma^i(x, y) \right] \psi_j(y) \, d\sigma_y, \quad i = 1, \ldots, n, \ x \in \Omega,
$$

$$
r(x) = - \int_\Sigma \varepsilon_j(x, y) \varphi_j(y) \, d\sigma_y
$$

$$
+ 2\mu \int_\Sigma \frac{\partial}{\partial \nu_y} \left[\varepsilon_j(x, y) \right] \psi_j(y) \, d\sigma_y, \quad x \in \Omega,
$$

(106)

where φ and ψ belong to $[L^p(\Sigma)]^n$.

Theorem 23. *Given* $f \in [W^{1,p}(\Sigma)]^n$ *satisfying (4), the Dirichlet problem*

$$
(v, r) \in \widetilde{\mathcal{S}}^p,
$$

$$
\mu \Delta v = \nabla r \quad in \ \Omega,
$$

$$
\operatorname{div} v = 0 \quad in \ \Omega,
$$

$$
v = f \quad on \ \Sigma,
$$

(107)

has one, and only one, solution (v, r) *given by*

$$
v_i(x) = - \int_\Sigma \gamma_{ij}(x, y) \varphi_j(y) \, d\sigma_y
$$

$$
+ \int_\Sigma T'_{j,y} \left[\gamma^i(x, y) \right] f_j(y) \, d\sigma_y, \quad x \in \Omega,
$$

(108)

$$
r(x) = - \int_\Sigma \varepsilon_j(x, y) \varphi_j(y) \, d\sigma_y
$$

$$
+ 2\mu \int_\Sigma \frac{\partial}{\partial \nu_y} \left[\varepsilon_j(x, y) \right] f_j(y) \, d\sigma_y, \quad x \in \Omega,
$$

(109)

where $\varphi \in [L^p(\Sigma)]^n$ *is solution of the integral system of the first kind*

$$
- \int_\Sigma \gamma_{ij}(x, y) \varphi_j(y) \, d\sigma_y
$$

$$
= \frac{1}{2} f_i(x) - \int_\Sigma T'_{j,y} \left[\gamma^i(x, y) \right] f_j(y) \, d\sigma_y, \quad a.e. \ on \ \Sigma.
$$

(110)

Proof. Let v be given by (108); imposing the boundary condition, we get (the symbol w_+ (w_-) stands for the interior (exterior) value of the double layer potential (17) on Σ)

$$
- \int_\Sigma \gamma_{ij}(x, y) \varphi_j(y) \, d\sigma_y = f_i(x) - w_{+,i}(x), \quad a.e. \ x \in \Sigma.
$$

(111)

In view of Remark 21 such a system is solvable if, and only if,

$$
\int_{\Sigma_h} (f_i - w_{+,i}) \nu_i d\sigma = 0, \quad h = 0, \ldots, m.
$$

(112)

On the other hand, because of the jump formulas, we have

$$
f_i(x) - w_{+,i}(x)
$$

$$
= f_i(x) - \left(\frac{1}{2} f_i(x) + \int_\Sigma T'_{j,y} \left[\gamma^i(x, y) \right] f_j(y) \, d\sigma_y \right)
$$

$$
:= \frac{1}{2} f_i(x) - w_i(x) = -w_{-,i}(x), \quad a.e. \ x \in \Sigma.
$$

(113)

Therefore, conditions (112) become $- \int_{\Sigma_h} w_- \cdot \nu d\sigma = 0$, $h = 0, \ldots, m$. Since w_- can be considered as the datum of the interior Dirichlet problem in Ω_h, for $h = 1, \ldots, m$, we have

$$
\int_{\Sigma_h} w_- \cdot \nu d\sigma = 0, \quad h = 1, \ldots, m.
$$

(114)

As far as $h = 0$ is concerned, first we remark that (4) implies $\int_\Sigma w \cdot \nu d\sigma = 0$, because

$$
0 = \int_\Sigma w_+ \cdot \nu d\sigma = \int_\Sigma \left(\frac{1}{2} f + w \right) \cdot \nu d\sigma = \int_\Sigma w \cdot \nu d\sigma.
$$

(115)

Keeping in mind (4) and (114), this leads to

$$
\int_{\Sigma_0} \left(\frac{1}{2} f - w \right) \cdot \nu d\sigma
$$

$$
= \frac{1}{2} \int_{\Sigma_0} f \cdot \nu d\sigma - \int_{\Sigma_0} w \cdot \nu d\sigma
$$

$$
= -\frac{1}{2} \sum_{j=1}^m \int_{\Sigma_j} f \cdot \nu d\sigma + \sum_{j=1}^m \int_{\Sigma_j} w \cdot \nu d\sigma
$$

$$
= \sum_{j=1}^m \int_{\Sigma_j} w_- \cdot \nu d\sigma = 0.
$$

(116)

Finally, assume that (v, r) is the solution of (107) with the data $f = 0$. The integral representation (108) shows that $(v, r) \in \mathcal{S}^p$, and then the uniqueness follows from Theorem 19. □

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