

## Research Article

# On $(h, k)$ -Dichotomies for Nonautonomous Linear Difference Equations in Banach Spaces

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This paper considers two general concepts of dichotomy for noninvertible and nonautonomous linear discrete-time systems in Banach spaces. These concepts use two types of dichotomy projections sequences (invariant and strongly invariant) and generalize some well-known dichotomy concepts (uniform, nonuniform, exponential, and polynomial). In the particular case of strongly invariant dichotomy projections, we present characterizations of these sequences and connections with other dichotomy concepts existent in the literature. Some illustrative examples clarify the implications between these concepts.

## 1. Introduction

In the last few years, the theory of difference equations has witnessed an impressive development. The notion of (uniform) exponential dichotomy introduced by Perron in [1] for differential equations and by Li in [2] for difference equations plays a central role in a substantial part of the theory of differential equations and dynamical systems.

The notion of dichotomy for differential equations has gained prominence since the appearance of two fundamental monographs of Massera and Schäffer [3] and Daleckii and Krein [4]. These were followed by the important book of Coppel [5] who synthesized and improved the results that existed in the literature up to 1978. Early works in the counterpart results of difference equations appeared in the paper of Coffman and Schäffer [6] and later, in 1981, when Henry included discrete dichotomies in his book [7]. This was followed by the classical monographs due to Agarwal [8] where ordinary and exponential dichotomy properties of difference equations are studied and various applications are provided. Significant work was reported by Pötzsche in [9].

One important and useful concept of dichotomy in the study of difference equations is the so-called (uniform)  $(h, k)$ -dichotomy concept introduced by Pinto in [10]. Since then,

this concept has been extensively studied and applied; see, for example, Naulin and Pinto [11], Megan [12], Fenner [13], and Lin [14] where several examples are presented.

A new notion called nonuniform  $(\mu, \nu)$ -dichotomy is proposed by Bento and Silva in [15] for the continuous case and in [16, 17] for discrete time settings, for invertible systems, with growth rates given by increasing functions (sequences) which go to infinity. In the last years, this subject (resp., nonuniform exponential dichotomy and nonuniform polynomial dichotomy) became one of the subjects of large interest, significant results being obtained (see, e.g., [18–24]).

On the other hand, we can consider the case of noninvertible systems. From this point of view the paper of Aulbach and Kalkbrenner [25] is of interest, where the notion of exponential forward splitting is introduced, motivated by the fact that there are equations whose backward solutions are not guaranteed to exist. Indeed, this study is of interest in applications; see, for example, random dynamical systems, generated by random parabolic equations, which are not invertible (for more details, see Zhou et al. [26]).

Motivated by the above studies, in this paper, we consider two concepts of nonuniform  $(h, k)$ -dichotomies for difference equations in Banach spaces. These concepts use two concepts

of sequences of projections: invariant and strongly invariant for the respective difference systems. Thus, we offer characterizations for these concepts and present connections between them. It is worth to mention the fact that if a sequence of projections is invariant for a reversible system then it is also strongly invariant for that system, and in this way the concepts merge. In this way, our study is related to the case of noninvertible systems.

## 2. Preliminaries

Let  $X$  be a Banach space and  $\mathcal{B}(X)$  the Banach space of all bounded linear operators on  $X$ . The norms on  $X$  and on  $\mathcal{B}(X)$  will be denoted by  $\|\cdot\|$ . The identity operator on  $X$  is denoted by  $I$ . If  $T \in \mathcal{B}(X)$ , then we will denote by  $\text{Ker } T$  the kernel of  $T$ ; that is,

$$\text{Ker } T = \{x \in X \text{ with } Tx = 0\}, \quad (1)$$

$$\text{Range } T = \{Tx; \text{ with } x \in X\}, \quad (2)$$

respectively. We also denote by  $\Delta$  the set of all pairs of all natural numbers  $(m, n)$  with  $m \geq n$ ; that is,  $\Delta = \{(m, n) \in \mathbb{N}^2 \text{ with } m \geq n\}$ .

We consider the linear difference system

$$x_{n+1} = A_n x_n, \quad (\mathfrak{A})$$

where  $A : \mathbb{N} \rightarrow \mathcal{B}(X)$  is a given sequence. For  $(m, n) \in \Delta$ , we define

$$A_m^n = \begin{cases} A_{m-1} \cdots A_n, & \text{if } m > n, \\ I, & \text{if } m = n. \end{cases} \quad (3)$$

It is obvious that

$$A_m^n A_n^p = A_m^p, \quad \forall (m, n), (n, p) \in \Delta, \quad (4)$$

and every solution of  $(\mathfrak{A})$  satisfies

$$x_m = A_m^n x_n \quad \forall (m, n) \in \Delta. \quad (5)$$

If for every  $n \in \mathbb{N}$  the operator  $A_n$  is invertible, then the system  $(\mathfrak{A})$  is called *reversible*.

**Definition 1.** A sequence  $P : \mathbb{N} \rightarrow \mathcal{B}(X)$  is called a *sequence of projections* if

$$P_n^2 = P_n, \quad \text{for every } n \in \mathbb{N}. \quad (6)$$

**Remark 2.** If  $P$  is a sequence of projections, then  $Q = I - P$  is also a sequence of projections (which is called *the complementary sequence of projections of  $P$* ) with  $\text{Ker } Q_n = \text{Range } P_n$  and  $\text{Range } Q_n = \text{Ker } P_n$  for every  $n \in \mathbb{N}$ .

### 3. $(h, k)$ -Dichotomy with Invariant Projections

**Definition 3.** A projection sequence  $P$  is called *invariant* for the system  $(\mathfrak{A})$  if

$$A_n P_n = P_{n+1} A_n, \quad \forall n \in \mathbb{N}. \quad (7)$$

**Remark 4.** If  $P$  is invariant for  $(\mathfrak{A})$ , then its complementary  $Q$  is also invariant for  $(\mathfrak{A})$ . Furthermore, as a consequence of (7), we have

$$A_m^n P_n = P_m A_m^n, \quad A_m^n Q_n = Q_m A_m^n \quad (8)$$

for all  $(m, n) \in \Delta$ .

**Remark 5.** If  $P$  is invariant for  $(\mathfrak{A})$ , then

- (i)  $A_n(\text{Ker } P_n) \subset \text{Ker } P_{n+1}$  for all  $n \in \mathbb{N}$ .
- (ii)  $A_m^n(\text{Ker } P_n) \subset \text{Ker } P_m$  and  $A_m^n(\text{Range } P_n) \subset \text{Range } P_m$  for all  $(m, n) \in \Delta$ .

**Definition 6.** An nondecreasing sequence  $\varphi : \mathbb{N} \rightarrow [1, +\infty)$  is said to be a *growth rate sequence* if  $\varphi_0 = 1$  and  $\lim_{n \rightarrow +\infty} \varphi_n = +\infty$ .

Let  $h, k : \mathbb{N} \rightarrow [1, +\infty)$  be two growth rate sequences.

**Definition 7.** We say that the linear difference system  $(\mathfrak{A})$  admits a *strong  $(h, k)$ -dichotomy* if there exist a sequence of projections  $P$  invariant for  $(\mathfrak{A})$  and constants  $N \geq 1, \alpha > 0$ , and  $\beta \geq 0$  such that

$$h_m^\alpha \|A_m^n P_n x\| \leq N h_n^\alpha k_n^\beta \|P_n x\| \quad (9)$$

$$h_m^\alpha \|Q_n x\| \leq N h_n^\alpha k_n^\beta \|A_m^n Q_n x\| \quad (10)$$

for all  $(m, n, x) \in \Delta \times X$ .

As particular cases of  $(h, k)$ -dichotomy, we have

- (1) If  $h_n = k_n = e^n$ , then we say that  $(\mathfrak{A})$  admits a *strong exponential dichotomy*.
- (2) If  $h_n = e^n$  and  $\beta = 0$ , then we say that system  $(\mathfrak{A})$  admits a *strong uniform exponential dichotomy*.
- (3) If  $h_n = k_n = n + 1$ , then we say that system  $(\mathfrak{A})$  admits a *strong polynomial dichotomy*.
- (4) If  $h_n = n + 1$  and  $\beta = 0$ , then we say that system  $(\mathfrak{A})$  admits a *strong uniform polynomial dichotomy*.

**Remark 8.** If the linear difference system  $(\mathfrak{A})$  admits a strong  $(h, k)$ -dichotomy, then for all  $(m, n) \in \Delta$  the following assertion holds:

$$A_m^n Q_n x = 0 \implies Q_n x = 0. \quad (11)$$

**Definition 9.** We say that the linear difference system  $(\mathfrak{A})$  admits an  $(h, k)$  dichotomy if there exist a sequence of projections  $P$  invariant for  $(\mathfrak{A})$  and constants  $N \geq 1, \alpha > 0$ , and  $\beta \geq 0$  such that

$$h_m^\alpha \|A_m^n P_n\| \leq N h_n^\alpha k_n^\beta \quad (12)$$

$$h_m^\alpha \leq N h_n^\alpha k_n^\beta \|A_m^n Q_n\| \quad (13)$$

for all  $(m, n, x) \in \Delta \times X$ .

If the system  $(\mathfrak{A})$  admits a  $(h, k)$ -dichotomy and we take into account the particular cases from above, we obtain the notions of *exponential dichotomy*, *uniform exponential*

*dichotomy, polynomial dichotomy, and uniform polynomial dichotomy, respectively.*

From the previous definition, we have that

$$\|P_n\| \leq Nk_n^\beta, \quad \forall n \in \mathbb{N}. \tag{14}$$

*Definition 10.* If  $k : \mathbb{N} \rightarrow [1, +\infty)$  is a given growth rate sequence and  $P$  is a sequence of projections that verifies (14), then we say that  $P$  is *k-bounded*.

As particular cases of *k-boundedness*, we have the following.

- (i) If there exist  $M \geq 1$  and  $\gamma \geq 0$  such that  $\|P(n)\| \leq Me^{\gamma n}$  for all  $n \in \mathbb{N}$ , then we say that  $P$  is *exponentially bounded*.
- (ii) If there exist  $M \geq 1$  and  $\gamma \geq 0$  such that  $\|P(n)\| \leq M(n + 1)^\gamma$  for all  $n \in \mathbb{N}$ , then we say that  $P$  is *polynomially bounded*.
- (iii) If there exist  $M \geq 1$  such that  $\|P(n)\| \leq M$  for all  $n \in \mathbb{N}$ , then we say that  $P$  is *bounded*.

*Remark 11.* It is worth to mention that the concept that we name “strong  $(h, k)$ -dichotomy,” in the exponential case with bounded sequences of projections, is in fact the notion of exponential forward splitting considered in [25, Definition 2.1]. The notion of strong (uniform) exponential dichotomy has been also considered in [27]. The name of “strong dichotomy” derives from the fact that it implies the other dichotomy concept in this paper (under the *k-boundedness* assumption of the sequence of the projections). It is important to state that the “strong dichotomy” from this paper differs from the “strong dichotomy” defined in [18].

*Remark 12.* Through the following two examples we will show that for a difference system  $(\mathfrak{A})$  the concepts of strong  $(h, k)$ -dichotomy and  $(h, k)$ -dichotomy are distinct, neither of them implying the other one.

*Example 13.* Consider  $X = l^1$  the Banach space of all real valued sequences  $x = (x_n)_{n \geq 0}$  endowed with the norm  $\|x\|_1 = \sum_{j=0}^{+\infty} |x_j|$ . Fix two growth rate sequences  $h, k : \mathbb{N} \rightarrow [1, +\infty)$ . Define, for every  $n \in \mathbb{N}$ ,  $P_n : l^1 \rightarrow l^1$  by  $P_n x = (y_j^n)_{j \geq 0}$  given by

$$y_j^n = \begin{cases} x_j + k_n^{k_n} \cdot x_{j+1}, & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases} \tag{15}$$

One can see that  $P_n \in \mathcal{B}(l^1)$  for all  $n \in \mathbb{N}$ ,  $P$  is a sequence of projections on  $l^1$  and if we pick  $x \in l^1$  satisfying  $x_{2j+1} = 1/(j + 1)^2$  and  $x_{2j} = 0$ , for all  $j \in \mathbb{N}$ , a simple computation gives us that

$$\|P_n x\|_1 = k_n^{k_n} \|x\|_1, \tag{16}$$

from where we deduce that

$$\|P_n\| \geq k_n^{k_n}, \quad \forall n \in \mathbb{N}. \tag{17}$$

In addition, we have the following properties:

$$P_m P_n = P_n, \quad P_n P_m = P_m, \quad \|Q_m x\|_1 \geq \|Q_n x\|_1 \tag{18}$$

for all  $(m, n, x) \in \Delta \times l^1$ .

Consider the linear difference system

$$x_{n+1} = \frac{h_n}{h_{n+1}} P_n x_n + \frac{h_{n+1}}{h_n} Q_{n+1} x_n, \tag{19}$$

where  $Q : \mathbb{N} \rightarrow \mathcal{B}(l^1)$  is the complementary sequence of projections of  $P$ .

Some computations show us that  $A_m^n$  is given by

$$A_m^n = \frac{h_n}{h_m} P_n + \frac{h_m}{h_n} Q_m \tag{20}$$

for all  $(m, n) \in \Delta$ .

One can check that the sequence of projections is invariant for the system  $(\mathfrak{A})$ . From the fact that

$$\begin{aligned} h_m \|A_m^n P_n x\|_1 &= h_n \|P_n x\|_1 \leq h_n k_n \|P_n x\|_1, \\ h_n k_m \|A_m^n Q_n x\|_1 &= k_m h_m \|Q_m x\|_1 \geq h_m \|Q_n x\|_1 \end{aligned} \tag{21}$$

for all  $(m, n, x) \in \Delta \times l^1$ , we have that the system  $(\mathfrak{A})$  admits a strong  $(h, k)$ -dichotomy.

Indeed, assume that the system  $(\mathfrak{A})$  admits a  $(h, k)$ -dichotomy. Then, by (14) and (17), we get the inequalities

$$k_n^{k_n} \leq \|P_n\| \leq Nk_n^\beta \quad \forall n \in \mathbb{N}. \tag{22}$$

Because the sequence  $k$  is a growth rate, choose  $n_\beta$  such that

$$k_n \geq \beta + 1 \quad \forall n \geq n_\beta. \tag{23}$$

Hence, the preceding inequality implies that

$$k_n \leq k_n^{k_n - \beta} \leq N \quad \forall n \geq n_\beta, \tag{24}$$

which, by making  $n \rightarrow \infty$ , contradicts the fact that  $k$  is a growth rate. Thus, we can see that the system  $(\mathfrak{A})$  fails to admit a  $(h, k)$ -dichotomy.

*Example 14.* On  $X = \mathbb{R}^3$  consider the sequence  $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$  defined, for  $n \in \mathbb{N}$  and  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ , by

$$A_n x = \left( \frac{h_n}{h_{n+1}} x_1, \frac{h_{n+1}}{h_n} x_2, 0 \right). \tag{25}$$

By considering the linear difference system  $(\mathfrak{A})$  associated to the above-defined sequence of bounded linear operators, we obtain that for all  $(m, n, x) \in \Delta \times \mathbb{R}^3$  we have

$$A_m^n x = \left( \frac{h_n}{h_m} x_1, \frac{h_m}{h_n} x_2, 0 \right). \tag{26}$$

We define the sequence of projections  $P : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$  by

$$P_n(x_1, x_2, x_3) = (x_1, 0, 0) \tag{27}$$

for all  $n \in \mathbb{N}$  and  $(x_1, x_2, x_3) \in \mathbb{R}^3$ . One can see that  $P$  is invariant for the system  $(\mathfrak{A})$  and

$$h_m \|A_m^n P_n\| = h_n \leq h_n k_m \quad (28)$$

for all  $(m, n) \in \Delta$ . Moreover, for  $(m, n) \in \Delta$ , if we pick  $x = (0, 1, 0)$ , then we get that

$$h_n \|A_m^n Q_n x\| = h_m \|x\|, \quad (29)$$

from where we obtain that

$$h_m \leq h_n \|A_m^n Q_n\| \leq h_n k_m \|A_m^n Q_n\| \quad (30)$$

for all  $(m, n) \in \Delta$ .

By (28) and (30), we conclude that the system  $(\mathfrak{A})$  admits a  $(h, k)$ -dichotomy.

Because  $Q_0(0, 0, 1) \neq (0, 0, 0)$  and  $A_1^0(0, 0, 1) = (0, 0, 0)$ , by Remark 8, it follows that the system  $(\mathfrak{A})$  does not admit a strong  $(h, k)$ -dichotomy.

#### 4. $(h, k)$ -Dichotomy with Strongly Invariant Projections

*Definition 15.* A sequence of projections  $P$  is called *strongly invariant* for the system  $(\mathfrak{A})$  if  $P$  is invariant for  $(\mathfrak{A})$  and for all  $n \in \mathbb{N}$  the restriction of  $A_n$  to  $\text{Ker } P_n$  is an isomorphism from  $\text{Ker } P_n$  to  $\text{Ker } P_{n+1}$ .

*Remark 16.*  $P$  is strongly invariant for the system  $(\mathfrak{A})$  if and only if for all  $(m, n) \in \Delta$  the restriction of  $A_m^n$  to  $\text{Ker } P_n$  is an isomorphism from  $\text{Ker } P_n$  to  $\text{Ker } P_m$ .

*Remark 17.* In the exponential setting, under the hypotheses that  $P$  is a strongly invariant for a system  $(\mathfrak{A})$  and it is bounded, it can be easily checked that the concept of strong exponential dichotomy defined in our paper is in fact the concept of *exponential splitting* defined in [25, Definition 4.1] or the concept of *exponential dichotomy* from [28, Definition 1.1].

*Remark 18.* If the sequence of projections  $P$  is invariant for the reversible system  $(\mathfrak{A})$ , then it is also strongly invariant for  $(\mathfrak{A})$ .

Indeed, if  $P$  is invariant for the reversible system  $(\mathfrak{A})$ , then for every  $y \in \text{Ker } P_{n+1}$  we have  $y = A_n(A_n)^{-1}y = A_n x$  with  $x = (A_n)^{-1}y \in \text{Ker } P_n$ .

There are linear difference systems which admit  $(h, k)$ -dichotomy with respect to projection sequences which are invariant but are not strongly invariant for the system. This phenomenon is illustrated by the following example.

*Example 19.* On  $X = \mathbb{R}^3$ , with  $\|x\| = |x_1| + |x_2| + |x_3|$ , we consider the system generated by the sequence  $A_n$  defined by

$$A_n(x_1, x_2, x_3) = \left( \frac{x_1}{e}, ex_2, 0 \right) \quad (31)$$

for all  $n \in \mathbb{N}$  and  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ . It is immediate to see that

$$A_m^n = \begin{cases} (e^{n-m}x_1, e^{m-n}x_2, 0), & m \neq n, \\ I, & m = n. \end{cases} \quad (32)$$

Define the sequence of projections  $P : \mathbb{N} \rightarrow \mathcal{B}(X)$  by

$$P_n(x_1, x_2, x_3) = (x_1, 0, 0) \quad \forall n \in \mathbb{N}, (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (33)$$

We have that the sequence  $P$  is invariant for the system  $(\mathfrak{A})$ . Moreover, one can see that

$$A_m^n P_n x = e^{n-m} P_n x, \quad (34)$$

and hence

$$\|A_m^n P_n\| = e^{-(m-n)} \|P_n\| \quad \forall (m, n) \in \Delta. \quad (35)$$

Moreover, for  $(m, n) \in \Delta$ ,  $m \neq n$ , and  $x_0 = (0, 1, 0) \in \mathbb{R}^3$ , we have that

$$\|A_m^n Q_n x_0\| = e^{m-n} \|Q_n x_0\|. \quad (36)$$

On the other hand, it is easy to check that

$$A_n^n Q_n x_0 = e^{n-n} Q_n x_0; \quad (37)$$

hence

$$\|A_m^n Q_n\| \geq e^{m-n} \|Q_n\| = e^{m-n} \quad \forall (m, n) \in \Delta. \quad (38)$$

This shows us that we have a system  $(\mathfrak{A})$  which is (uniformly) exponentially dichotomic with the invariant projection sequence  $P$ . But  $P$  is not strongly invariant because, for example,  $A_1 Q_n(0, 0, 1) = (0, 0, 0)$ .

*Remark 20.* Our second concept of  $(h, k)$ -dichotomy introduced in this paper (see Definition 9) can be viewed in the light of the general dichotomies presented in [17], by making the choice  $a(m, n) = (h(n)/h(m))^\alpha \cdot k(n)^\beta$  and  $b(m, n) = (h(n)/h(m))^\alpha \cdot k(m)^\beta$  in the case of strongly invariant sequences of projections and by observing that the inequality (D2) from [17], namely,  $\|(A_m^n|_{\text{Ker } P_n})^{-1} Q_m\| \leq b(m, n)$ , implies the corresponding instability inequality from our dichotomy concept by observing that

$$1 \leq \|A_m^n|_{\text{Ker } P_n} (A_m^n|_{\text{Ker } P_n})^{-1} Q_m\| \leq \|A_m^n|_{\text{Ker } P_n}\| \cdot b(m, n). \quad (39)$$

Moreover, under the assumption of  $k$ -boundedness of the sequence of the projections and strong invariance, the notion of strong  $(h, k)$ -dichotomy from our paper is a particular case of the general dichotomies from [17], by making the same choice of the sequences  $a(m, n)$  and  $b(m, n)$ .

The above-mentioned concept is useful in input-output techniques, and we mention the work of B. Sasu and A. L. Sasu [29], where, in the uniform exponential setting, from a sufficient condition of admissibility of pairs of function spaces to an associated control system of  $(\mathfrak{A})$ , they deduce the property of uniform exponential dichotomy and the boundedness of the sequence of projections.

Even in the particular case of exponential dichotomy, the implication that gives (39) from the preceding remark fails to hold, showing that there exist reversible systems that verify Definition 9 and do not satisfy the dichotomic behavior pointed out by (39), even in the case of bounded sequences of projections.

*Remark 21.* If the sequence of projections  $P$  is strongly invariant for the system  $(\mathfrak{A})$ , then

- (i) for every  $n \in \mathbb{N}$  there is an isomorphism  $B_n$  from the  $\text{Ker } P_{n+1}$  to  $\text{Ker } P_n$  such that  $A_n B_n Q_{n+1} = Q_n$  and  $B_n A_n Q_n = Q_n$ ;
- (ii) for all  $(m, n) \in \Delta$  there is an isomorphism  $B_m^n$  from  $\text{Ker } P_m$  to  $\text{Ker } P_n$  with

$$A_m^n B_m^n Q_m = Q_m, \tag{40}$$

$$B_m^n A_m^n Q_n = Q_n \tag{41}$$

for all  $(m, n) \in \Delta$ .

In particular case when  $(\mathfrak{A})$  is invertible and  $P$  is invariant for  $(\mathfrak{A})$ , then  $B_n = (A_n)^{-1}$  for every  $n \in \mathbb{N}$  and  $B_m^n = (A_m^n)^{-1}$  for all  $(m, n) \in \Delta$ .

**Proposition 22.** Let  $h$  and  $k$  be growth rate sequences and  $P$  a sequence of projections that is strongly invariant for a system  $(\mathfrak{A})$ . Then the system  $(\mathfrak{A})$  admits a strong  $(h, k)$ -dichotomy if and only if there exist  $N \geq 1$ ,  $\alpha > 0$ , and  $\beta \geq 0$  such that

$$h_m^\alpha \|A_m^n P_n x\| \leq N h_n^\alpha k_n^\beta \|P_n x\|, \tag{42}$$

$$h_m^\alpha \|B_m^n Q_m x\| \leq N h_n^\alpha k_m^\beta \|Q_m x\| \tag{43}$$

for all  $(m, n, x) \in \Delta \times X$ , where  $B_m^n$  are the images of the function  $B$  defined in Remark 21.

*Proof.* We have to prove the equivalence of (10) and (43). Let  $(m, n, x) \in \Delta \times X$ . For the necessity, by (40), we have

$$h_m^\alpha \|B_m^n Q_m x\| \leq N h_n^\alpha k_m^\beta \|A_m^n B_m^n Q_m x\| = N h_n^\alpha k_m^\beta \|Q_m x\|. \tag{44}$$

For the sufficiency, by (41), we have

$$\begin{aligned} h_m^\alpha \|Q_n x\| &= h_m^\alpha \|B_m^n Q_m A_m^n Q_n x\| \\ &\leq N h_n^\alpha k_m^\beta \|Q_m A_m^n Q_n x\| = N h_n^\alpha k_m^\beta \|A_m^n Q_n x\| \end{aligned} \tag{45}$$

which concludes our proof.  $\square$

**Proposition 23.** Let  $h$  and  $k$  be growth rate sequences and  $P$  a sequence of projections that is strongly invariant for a system  $(\mathfrak{A})$ . If there exist  $N \geq 1$ ,  $\alpha > 0$ , and  $\beta \geq 0$  such that the system  $(\mathfrak{A})$  has the properties

$$h_m^\alpha \|A_m^n P_n\| \leq N h_n^\alpha k_n^\beta, \tag{46}$$

$$h_m^\alpha \|B_m^n Q_m\| \leq N h_n^\alpha k_m^\beta \tag{47}$$

for all  $(m, n) \in \Delta$ , where  $B_m^n$  are the images of the function  $B$  defined in Remark 21. Then,  $(\mathfrak{A})$  admits a  $(h, k)$ -dichotomy.

*Proof.* We have to show that (47) implies (13). Let  $(m, n) \in \Delta$ . The conclusion follows from

$$\begin{aligned} h_m^\alpha &\leq h_m^\alpha \|Q_n\| = h_m^\alpha \|B_m^n Q_m A_m^n Q_n\| \\ &\leq h_m^\alpha \|B_m^n Q_m\| \|A_m^n Q_n\| \leq N h_n^\alpha k_m^\beta \|A_m^n Q_n\|. \end{aligned} \tag{48}$$

$\square$

The converse of the preceding proposition is not generally valid, as shown in the following example.

*Example 24.* On  $X = l^1$ , consider  $(k_n)$  an arbitrary growth rate,  $h_n = e^n$  for  $n \in \mathbb{N}$ ,  $\alpha = 1$  and  $\beta = 0$ . Define the sequence  $A : \mathbb{N} \rightarrow \mathcal{B}(l^1)$  by  $A_n x = (y_j(n))_{j \geq 0}$ , where

$$\begin{aligned} y_{3j+1}(n) &= \frac{1}{e} \cdot x_{3j+1} \\ y_{3j+2}(n) &= e \cdot x_{3j+2} \\ y_{3j}(n) &= e^{e^n - e^{n+1}} \cdot x_{3j} \end{aligned} \tag{49}$$

$j \in \mathbb{N}$ .

Then, we have that for  $(m, n, x) \in \Delta \times l^1$ ,  $A_m^n x = (y_j(m, n))_{j \geq 0}$  with

$$\begin{aligned} y_{3j+1}(m, n) &= e^{n-m} \cdot x_{3j+1} \\ y_{3j+2}(m, n) &= e^{m-n} \cdot x_{3j+2} \\ y_{3j}(m, n) &= e^{e^m - e^n} \cdot x_{3j} \end{aligned} \tag{50}$$

$j \in \mathbb{N}$ .

Define, for every  $n \in \mathbb{N}$ , the projection  $P_n : l^1 \rightarrow l^1$  by  $P_n x = (a_j)_{j \geq 0}$  where

$$\begin{aligned} a_{3j+1} &= x_{3j+1} \\ a_{3j+2} &= 0 \\ a_{3j} &= 0 \end{aligned} \tag{51}$$

$j \in \mathbb{N}$ .

It is easy to check that the sequence of projections  $P$  is strongly invariant for the system  $(\mathfrak{A})$ , and for  $(m, n, x) \in \Delta \times l^1$ ,  $B_m^n Q_m x = (z_j(m, n))_{j \geq 0}$  is given by

$$\begin{aligned} z_{3j+1}(m, n) &= 0 \\ z_{3j+2}(m, n) &= e^{n-m} \cdot x_{3j+2} \\ z_{3j}(m, n) &= e^{e^m - e^n} \cdot x_{3j} \end{aligned} \tag{52}$$

$j \in \mathbb{N}$ .

A simple computation shows us that

$$e^m \|A_m^n P_n\| = e^n \tag{53}$$

for all  $(m, n) \in \Delta$  and by choosing  $x \in l^1$  with  $x_{3j+1} = x_{3j} = 0$  and  $x_{3j+2} = 1/(j+1)^2$ , it follows that

$$e^m \leq \|A_m^n Q_n\| e^n. \tag{54}$$

Hence, the system  $(\mathfrak{A})$  admits a  $(h, k)$ -dichotomy.

Assume now that there exist  $\alpha > 0$ ,  $N \geq 1$ , and  $\beta \geq 0$  with

$$h_m^\alpha \|B_m^n Q_m\| \leq N h_n^\alpha k_m^\beta \quad (55)$$

for all  $(m, n) \in \Delta$ . In our case, the preceding inequality becomes

$$e^{\alpha m} \|B_m^n Q_m\| \leq N e^{\alpha n} \quad (56)$$

for all  $(m, n) \in \Delta$ .

Consider that  $n = 0$ ,  $m \in \mathbb{N}$ , and  $x \in l^1$  given by

$$\begin{aligned} x_{3j+1} &= x_{3j+2} = 0 \\ x_{3j} &= \frac{1}{(j+1)^2} \\ j &\in \mathbb{N}. \end{aligned} \quad (57)$$

It follows that

$$\|B_m^0 Q_m x\|_1 = e^{e^m - 1} \cdot \frac{\pi}{\sqrt{6}} = e^{e^m - 1} \|x\|_1, \quad (58)$$

from where we get that

$$\|B_m^0 Q_m\| \geq e^{e^m - 1}. \quad (59)$$

Combining the above inequality with (56), we get that

$$e^{e^m + \alpha m - 1} \leq N \quad (60)$$

for all  $m \in \mathbb{N}$ , which is a contradiction.

Moreover, the converse of the implication from Proposition 23 does not hold not even in the finite dimensional case, as we can see from the following example.

*Example 25.* Consider that  $h_n = k_n = e^n$ ,  $n \in \mathbb{N}$ . Consider, on  $X = \mathbb{R}^3$ , the sequence  $A : \mathbb{N} \rightarrow \mathcal{B}(\mathbb{R}^3)$  is defined by

$$A_n(x_1, x_2, x_3) = \left( \frac{x_1}{e}, ex_2, \frac{x_3}{e^2} \right). \quad (61)$$

A simple computation shows us that

$$A_m^n(x_1, x_2, x_3) = \left( e^{n-m} x_1, e^{m-n} x_2, e^{n^2 - m^2} x_3 \right) \quad (62)$$

for all  $(m, n) \in \Delta$  and  $(x_1, x_2, x_3) \in \mathbb{R}^3$ . Define the (constant) sequence of projections  $P : \mathbb{N} \rightarrow \mathcal{B}(X)$  by

$$P_n(x_1, x_2, x_3) = (x_1, 0, 0) \quad \forall n \in \mathbb{N}, (x_1, x_2, x_3) \in \mathbb{R}^3. \quad (63)$$

We have that  $P$  is strongly invariant to the system  $(\mathfrak{A})$  with

$$(A_m^n |_{\text{Ker } P_n})^{-1}(0, x_2, x_3) = (0, e^{n-m} x_2, e^{m^2 - n^2} x_3) \quad (64)$$

for all  $(m, n) \in \Delta$  and for all  $(0, x_2, x_3) \in \text{Ker } P_n$ . A straightforward computation shows us that

$$\|A_m^n P_n\| = e^{-(m-n)}, \quad \|A_m^n Q_n\| \geq e^{m-n} \quad (65)$$

for all  $(m, n) \in \Delta$ ; hence,  $(\mathfrak{A})$  satisfies the conditions from Definition 9 (in the exponential case).

Assume by a contradiction that there exist  $N \geq 1$ ,  $\alpha > 0$ , and  $\beta \geq 0$  such that

$$\|A_m^n P_n\| \leq N e^{\beta n} e^{-\alpha(m-n)} \quad (66)$$

$$\|(A_m^n |_{\text{Ker } P_n})^{-1}\| \leq N e^{\beta m} e^{-\alpha(m-n)} \quad (67)$$

for all  $(m, n) \in \Delta$ . Choose  $n = 0$ . From the fact that

$$\|(A_m^0 |_{\text{Ker } P_0})^{-1}\| = e^{m^2} \quad \forall m \in \mathbb{N}. \quad (68)$$

From (67) and (68), we get the contradiction

$$e^{m^2} \leq N e^{(\beta-\alpha)m} \quad \forall m \in \mathbb{N}. \quad (69)$$

**Proposition 26.** *Let  $h$  and  $k$  be growth rate sequences and  $P$  a sequence of projections that is strongly invariant for a system  $(\mathfrak{A})$ . If there exist  $N \geq 1$ ,  $\alpha > 0$ , and  $\beta \geq 0$  such that the system  $(\mathfrak{A})$  verifies (46) and (47), then  $(\mathfrak{A})$  admits a strong  $(h, k)$ -dichotomy.*

*Proof.* Let  $(m, n, x) \in \Delta \times X$ . It follows that

$$h_m^\alpha \|A_m^n P_n x\| \leq h_m^\alpha \|A_m^n P_n\| \|P_n x\| \leq N h_n^\alpha k_n^\beta \|P_n x\|, \quad (70)$$

$$h_m^\alpha \|B_m^n Q_m x\| \leq h_m^\alpha \|B_m^n Q_m\| \|Q_m x\| \leq N h_n^\alpha k_m^\beta \|Q_m x\|$$

which, by Proposition 22, concludes the proof.  $\square$

**Theorem 27.** *Let  $h$  and  $k$  be growth rates and  $P$  a sequence of projections that is strongly invariant for a system  $(\mathfrak{A})$ . Then, the system  $(\mathfrak{A})$  verifies (46) and (47) if and only if  $(\mathfrak{A})$  admits a strong  $(h, k)$ -dichotomy and  $P$  is  $k$ -bounded.*

*Proof.* The necessity follows from Proposition 26 and (14).

For the sufficiency, let  $M \geq 1$  and  $\gamma \geq 0$  such that

$$\|P_n\| \leq M k_n^\gamma, \quad \forall n \in \mathbb{N}. \quad (71)$$

Let  $(m, n, x) \in \Delta \times X$ . From

$$h_m^\alpha \|A_m^n P_n x\| \leq N h_n^\alpha k_n^\beta \|P_n x\| \leq 2MN h_n^\alpha k_n^{\beta+\gamma} \|x\|, \quad (72)$$

$$h_m^\alpha \|B_m^n Q_m x\| \leq N h_n^\alpha k_m^\beta \|Q_m x\| \leq 2MN h_n^\alpha k_m^{\beta+\gamma} \|x\|,$$

the conclusion follows.  $\square$

**Corollary 28.** *Let  $P$  be a sequence of projections that is strongly invariant for a system  $(\mathfrak{A})$ .*

- (i) *If  $h_n = k_n = e^n$ , then the system  $(\mathfrak{A})$  verifies (46) and (47) if and only if it is exponentially dichotomic and  $P$  is exponentially bounded.*
- (ii) *If  $h_n = k_n = n+1$ , then the system  $(\mathfrak{A})$  verifies (46) and (47) if and only if it is polynomially dichotomic and  $P$  is polynomially bounded.*
- (iii) *If  $h_n = e^n$  (resp.,  $h_n = n+1$ ) and  $\beta = 0$ , then the system  $(\mathfrak{A})$  verifies (46) and (47) if and only if it is uniformly exponentially (polynomially) dichotomic and  $P$  is bounded.*

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