Research Article

Unified Treatment of the Krätzel Transformation for Generalized Functions

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We discuss a generalization of the Krätzel transforms on certain spaces of ultradistributions. We have proved that the Krätzel transform of an ultradifferentiable function is an ultradifferentiable function and satisfies its Parseval's inequality. We also provide a complete reading of the transform constructing two desired spaces of Boehmians. Some other properties of convergence and continuity conditions and its inverse are also discussed in some detail.

1. Introduction

Krätzel, in [1, 2], introduced a generalization of the Meijer transform by the integral:

$$\left(K_v^p f\right)(x) = \int_{\mathbf{R}_+} \mathbf{Z}_p^v(xy) f(y)\,dy, \quad x > 0, \tag{1}$$

where

$$\mathbf{Z}_p^v(xy) = \int_{\mathbf{R}_+} t^{v-1} e^{-t^p - xy/t}\,dt, \tag{2}$$

$p > 0 (\in \mathbf{N})$, $v \in \mathbb{C}$. Then this generalization is known as Krätzel transform.

Let x be in $\mathbf{R}_+$. Denote by $\mathscr{S}_+$, or $\mathscr{S}(\mathbf{R}_+)$, the space of all complex-valued smooth functions $\phi(t)$ on $\mathbf{R}_+$ such that

$$\sup_{x \in \mathbf{K}} \left|\mathscr{D}^k \phi(x)\right| < \infty, \tag{3}$$

$x \in \mathbf{R}_+$, where $\mathbf{K}$ runs through compact subsets of $\mathbf{R}_+$; see [3]. The strong dual $\mathscr{S}_+'$ of $\mathscr{S}_+$ consists of distributions of compact supports.

Later, the authors in [4] have studied the K_v^p transformation in a space of distributions of compact support inspired by known kernel method. They, also, have obtained its properties

of analyticity and boundedness and have established its inversion theorem. In the sense of classical theory, the Meijer transformation and the Laplace transformation in [5] are presented as special forms of the cited transform for $p = 1$ and $p = 1$, $v = \pm 1/2$, respectively.

It is worth mentioning in this note that a suitable motivation of the cited transform has thoroughly been discussed in [6] by the aid of a Fréchet space of constituted functions of infinitely differentiable functions over $(0, \infty)$.

This paper is a continuation of the work obtained in [4]. We are concerned with a general study of the transform in the space of ultradistributions and further discuss its extension to Boehmian spaces in some detail. We are employing the adjoint method and method of kernels for our purpose to extend the classical integral transform to generalized functions and hence ultradistributions.

2. Ultradistributions

The theory of ultradistributions is one of generalizations of the theory of Schwartz distributions; see [3, 7]. Since then, in the recent past and even earlier, it was extensively studied by many authors such as Roumieu [8, 9], Komatsu [10], Beurling [11], Carmichael et al. [12], Pathak [13, 14], and Al-Omari [15, 16].

By an ultradifferentiable function we mean an infinitely smooth function whose derivatives satisfy certain growth conditions as the order of the derivatives increases. Unlike sequences presented in [15, 16], a_i, $i = 0, 1, \ldots$, wherever it appears, denotes a sequence of positive real numbers. Such omission of constraints may ease the analysis.

Let α be a real number but fixed and $\mathscr{L}^r$ be the space of Lebesgue integrable functions on $\mathbf{R}_+$. Denote by $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$), $1 \le r \le \infty$, the subsets of $\mathscr{S}_+$ of all complex valued infinitely smooth functions on $\mathbf{R}_+$ such that, for some constant $m_1 (> 0)$,

$$\sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k \varphi(x) \right\|_{\mathscr{L}^r} \le m_1 a^\alpha a_\alpha \tag{4}$$

for all $a > 0$ (for some $a > 0$), where $\mathbf{K}$ is a compact set traverses $\mathbf{R}_+$.

The elements of the dual spaces, $\mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathscr{S}'_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$), are the Beurling-type (Roumieu-type) ultradistributions. It may be noted that $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a) \subset \mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a) \subset \mathscr{S}_+$. Thus, every distribution of compact support is an ultradistribution of Roumieu type and further, and an ultradistribution of Roumieu type is of Beurling-type. Natural topologies on $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) can be generated by the collection of seminorms:

$$\|\varphi\|_{r,a} = \sup_{\alpha, x \in \mathbf{K}} \frac{\left\| \mathscr{D}^k \varphi(x) \right\|_{\mathscr{L}^r}}{a^\alpha a_\alpha}, \quad a > 0. \tag{5}$$

A sequence $(\varphi_n) \to \varphi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ (resp., $\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) if

$$\lim_{n \to \infty} \sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k (\varphi_n(x) - \varphi(x)) \right\|_{\mathscr{L}^r} = 0, \tag{6}$$

and there is a constant $m_1 > 0$ independent of n such that

$$\lim_{n \to \infty} \sup_{\alpha, x \in \mathbf{K}} \left\| \mathscr{D}^k (\varphi_n(x) - \varphi(x)) \right\|_{\mathscr{L}^r} \le m_1 a^\alpha a_\alpha \tag{7}$$

for all $a > 0$. $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) is dense in $\mathscr{S}_+$, convergence in $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) implies convergence in $\mathscr{S}_+$, and consequently a restriction of any $f \in \mathscr{S}'_+$ to $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathscr{S}_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$) is in $\mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ ($\mathscr{S}'_+(\mathscr{L}^r, \alpha, \{a_i\}, a)$).

3. The Krätzel Transform of Tempered Ultradistributions

In this section of this paper we define the Krätzel transform of tempered ultradistributions by using both of kernel and adjoint methods. We restrict our investigation to the case of Beurling type since the other investigation for the Roumieu-type tempered ultradistributions is almost similar.

Lemma 1. *Let* $\phi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ *and then* $K_\nu^p \phi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$.

Proof. Let $\phi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ and x be fixed, and then $K_\nu^p \phi$ certainly exists. By differentiation with respect to x we get

$$\begin{aligned}
\mathscr{D}_x \left(K_\nu^p \phi \right)(x) &= \iint_{\mathbf{R}_+} \mathscr{D}_x t^{\nu-1} e^{-t^p - xy/t} dt \phi(y) \, dy \\
&= \iint_{\mathbf{R}_+} \frac{-y}{t} t^{\nu-1} e^{-t^p - xy/t} dt \phi(y) \, dy \\
&= \int_{\mathbf{R}_+} -y \int_{\mathbf{R}_+} t^{(\nu-1)-1} e^{-t^p - xy/t} dt \phi(y) \, dy \\
&= \int_{\mathbf{R}_+} -y Z_p^{\nu-1}(xy) \phi(y) \, dy.
\end{aligned} \tag{8}$$

Hence the principle of mathematical induction on the kth derivative gives

$$\mathscr{D}_x^k \left(K_\nu^p \phi \right)(x) = (-1)^k \int_{\mathbf{R}_+} Z_p^{\nu-k}(xy) \, y^k \phi(y) \, dy. \tag{9}$$

From [4], we deduce that

$$\begin{aligned}
&\left| \mathscr{D}_x^k \left(K_\nu^p \phi \right)(x) \right| \\
&\le \alpha_1 \int_{\mathbf{R}_+} \left| (y)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \right| y^k \phi(y) \, dy
\end{aligned} \tag{10}$$

for some constant α_1. The assumption that $\phi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ implies that the products under the integral sign, $\Psi = y^k \phi(y)$ and $(xy)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \Psi$, are also in $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. Moreover, $\phi \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ ensures that the integral:

$$F(x) = \alpha_1 \int_{\mathbf{R}_+} \left| (xy)^{(2\nu-p)/(2p+2)} e^{-(xy)^p/(p+1)} \right| y^k \phi(y) \, dy \tag{11}$$

belongs to $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. Hence

$$\left\| \mathscr{D}_x^k F(x) \right\|_{\mathscr{L}^r} \le m_1 a^\alpha a_\alpha \tag{12}$$

for some constant m_1. Therefore, from the above inequality we get

$$\left\| \mathscr{D}_x^k \left(K_\nu^p \phi \right)(x) \right\|_{\mathscr{L}^r} \le \left\| \mathscr{D}_x^k F(x) \right\|_{\mathscr{L}^r} \le m_1 a^\alpha a_\alpha, \tag{13}$$

for certain positive constant m_1. This proves the lemma. $\quad\square$

From Lemma 1 we deduce that the Krätzel transform is bounded and closed from $\mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ into itself. Next, we establish the Parseval's relation for the Krätzel transform.

Theorem 2. *Let f and g be absolutely integrable functions over $\mathbf{R}_+$ and then*

$$\int_{\mathbf{R}_+} f(x) \left(K_\nu^p g \right)(x) \, dx = \int_{\mathbf{R}_+} \left(K_\nu^p f \right)(x) \, g(x) \, dx, \tag{14}$$

where $K_\nu^p f$ and $K_\nu^p g$ are the Krätzel transforms of f and g, respectively.

Proof. It is clear that $K_v^p f$ and $K_v^p g$ are continuous and bounded on $\mathbf{R}_+$. Moreover, the Fubini's theorem allows us to interchange the order of integration:

$$\int_{\mathbf{R}_+} f(x)(K_v^p g)(x)\,dx$$

$$= \int_{\mathbf{R}_+} \left(\int_{\mathbf{R}_+} f(x) \, Z_p^v(xy)\,dx \right) g(y)\,dy. \tag{15}$$

Equation (15) follows since the Krätzel kernel $Z_p^v(xy)$ applies for the functions f and g, when the order of integration is interchanged. This completes the proof of the theorem. $\square$

Now, in consideration of Theorem 2, the adjoint method of extending the Krätzel transform can be read as

$$\langle K_v^p f, \phi \rangle = \langle f, K_v^p \phi \rangle, \tag{16}$$

where $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\phi \in \mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$.

Theorem 3. *Given that* $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ *then* $K_v^p f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$.

Proof. Consider a zero convergent sequence (ϕ_n) in $\mathcal{S}_+(\mathcal{L}^r, \alpha, (a_i), a)$ then certainly $(K_v^p \phi_n)$ is a zero-convergent sequence in the same space. It follows from (16) that

$$\langle K_v^p f, \phi_n \rangle = \langle f, K_v^p \phi_n \rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{17}$$

Linearity is obvious. This completes the proof. $\square$

From the above theorem we deduce that the Krätzel transform of a tempered ultradistribution is a tempered ultradistribution. Moreover, the boundedness property of $K_v^p f$, $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ follows from the following theorem.

Theorem 4. *Let* $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ *and then* $K_v^p f$ *is bounded.*

Proof. See [4, Proposition 2.3]. $\square$

It is interesting to know that the Krätzel transform can be defined in an alternative way, namely, by the kernel method. Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, and then

$$(K_v^p f)(x) = \langle f(y), Z_p^v(xy) \rangle. \tag{18}$$

In fact, (18) is a straightforward consequence of Lemma 1.

Theorem 5. *Let* $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ *then* $K_v^p f$ *is infinitely differentiable and*

$$\mathcal{D}_x^k (K_v^p f)(x) = \langle f(t), \mathcal{D}_x^k Z_p^v(xt) \rangle \tag{19}$$

for every $k \in \mathbf{N}$ *and* $x \in \mathbf{R}_+$.

Proof. See [4, Proposition 2.2]. $\square$

4. Boehmian Spaces

Boehmians were first constructed as a generalization of regular Mikusinski operators [17]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(i) a nonempty set A,

(ii) a commutative semigroup $(\mathsf{B}, *)$,

(iii) an aperation $\odot : \mathsf{A} \times \mathsf{B} \rightarrow \mathsf{A}$ such that for each $x \in \mathsf{A}$ and $s_1, s_2 \in \mathsf{B}$, $x \odot (s_1 * s_2) = (x \odot s_1) \odot s_2$,

(iv) a collection $\Delta \subset \mathsf{B}^N$ such that

(a) if $x, y \in \mathsf{A}$, $(s_n) \in \Delta$, $x \odot s_n = y \odot s_n$ for all n, then $x = y$,

(b) if $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Elements of Δ are called delta sequences. Consider

$$\mathsf{g} = \{(x_n, s_n) : x_n \in \mathsf{A}, (s_n) \in \Delta, x_n \odot s_m = x_m \odot s_n, \forall m, n \in \mathbf{N}\}. \tag{20}$$

If $(x_n, s_n), (y_n, t_n) \in \mathsf{g}$, $x_n \odot t_m = y_m \odot s_n$, for all $m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in g. The space of equivalence classes in g is denoted by β. Elements of β are called Boehmians. Between A and β there is a canonical embedding expressed as

$$x \longrightarrow \frac{x \odot s_n}{s_n}. \tag{21}$$

The operation $\odot$ can be extended to $\beta \times \mathsf{A}$ by

$$\frac{x_n}{s_n} \odot t = \frac{x_n \odot t}{s_n}. \tag{22}$$

In β, there are two types of convergence:

(δ convergence) a sequence (h_n) in β is said to be δ convergent to h in β, denoted by $h_n \xrightarrow{\delta} h$, if there exists a delta sequence (s_n) such that $(h_n \odot s_n), (h \odot s_n) \in \mathsf{A}$, for all $k, n \in \mathbf{N}$, and $(h_n \odot s_k) \rightarrow (h \odot s_k)$ as $n \rightarrow \infty$, in A, for every $k \in \mathbf{N}$,

(Δ convergence) a sequence (h_n) in β is said to be Δ convergent to h in β, denoted by $h_n \xrightarrow{\Delta} h$, if there exists a $(s_n) \in \Delta$ such that $(h_n - h) \odot s_n \in \mathsf{A}$, for all $n \in \mathbf{N}$, and $(h_n - h) \odot s_n \rightarrow 0$ as $n \rightarrow \infty$ in A. For further discussion see [17–21].

5. The Ultra-Boehmian Space β_{s_+}

Denote by D_+, or $\mathsf{D}(\mathbf{R}_+)$, the Schwartz space of $\mathbb{C}^\infty$ functions of bounded support. Let Δ_+ be the family of sequences $(s_n) \in \mathsf{D}(\mathbf{R}_+)$ such that the following holds:

(Δ_1) $\int_{\mathbf{R}_+} s_n(x)\,dx = 1$, for all $n \in \mathbf{N}$,

(Δ_2) $s_n(x) \geq 0$, for all $n \in \mathbf{N}$,

(Δ_3) supp $s_n \subset (0, \varepsilon_n)$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

It is easy to see that each (s_n) in Δ_+ forms a delta sequence.

Let $f \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $\sigma \in D(\mathbf{R}_+)$ be related by the expression:

$$(f \cdot \sigma) v = f(\sigma \circledast v), \tag{23}$$

where

$$(\sigma \circledast v)(x) = \int_{\mathbf{R}_+} \sigma(t) v(xt) \, dt \tag{24}$$

for every $v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$.

Lemma 6. *Let $f \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $\sigma \in D_+$ and then $f \cdot \sigma \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$.*

Proof. Using the weak topology of $\mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$, we write

$$\left|(f \cdot \sigma) v\right| = \left|f(\sigma \circledast v)\right| \le C \|\sigma \circledast v\|_{r,a}, \tag{25}$$

where $v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. Hence, to complete the proof, we are merely required to show that $\sigma \circledast v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$.

First, if $\sigma \in D_+$ and $v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$, then choosing a compact set $\mathbf{K}$ containing the support of σ yields

$$\frac{(\sigma \circledast v)(x + \Delta x) - (\sigma \circledast v)(x)}{\Delta x}$$
$$= \int_{\mathbf{R}_+} \sigma(t) \frac{v((x + \Delta x)t) - v(xt)}{\Delta x} \, dt \tag{26}$$

which is dominated by $\sigma(t)|\mathscr{D}_x v(x)|$. The dominated convergence theorem and the principle of mathematical induction implies

$$\mathscr{D}_x^k(\sigma \circledast v) = \sigma \circledast \mathscr{D}_x^k v. \tag{27}$$

Finally

$$\int_{\mathbf{R}_+} \left|\mathscr{D}^k(\sigma \circledast v)(x)\right|^r dx = \int_{\mathbf{R}_+} \left|(\sigma \circledast \mathscr{D}^k v)(x)\right|^r dx$$
$$\le \int_{\mathbf{R}_+} \int_{\mathbf{K}} \left|\sigma(t) \, dt \, \mathscr{D}^k v(xt)\right|^r dx$$
$$\le M \int_{\mathbf{R}_+} \left|\mathscr{D}^k v(xt)\right|^r dx. \tag{28}$$

Therefore

$$\|\sigma \circledast v\|_{r,a} \le d\|v\|_{r,a} < d a^\alpha a_\alpha \quad \text{for some constant } d. \tag{29}$$

Thus $\sigma \circledast v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$. This completes the proof of the lemma. $\qquad\square$

Lemma 7. *Let $f_1, f_2 \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $\sigma \in D_+$ and then*

  (i) $\alpha f \cdot \sigma = \alpha(f \cdot \sigma), \alpha \in \mathbb{C}$,
  (ii) $(f_1 + f_2) \cdot \sigma = f_1 \cdot \sigma_1 + f_2 \cdot \sigma_2$.

Proof of the above Lemma is obvious.

Lemma 8. *Let $f_n, f \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $\sigma \in D_+$ and then*

$$f_n \cdot \sigma \longrightarrow f \cdot \sigma. \tag{30}$$

Proof. Let $v \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and then $\sigma \circledast v \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$. Therefore

$$(f_n \cdot \sigma - f \cdot \sigma) v = ((f_n - f) \cdot \sigma) v = (f_n - f)(\sigma \circledast v). \tag{31}$$

Hence $f_n \cdot \sigma \to f \cdot \sigma$ as $n \to \infty$. $\qquad\square$

Lemma 9. *Let $f \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $(s_n) \in \Delta_+$ then*

$$f \cdot s_n \longrightarrow f \quad \text{in } \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a) \quad \text{as } n \longrightarrow \infty. \tag{32}$$

Proof. Let $v \in \mathscr{S}_+(\mathscr{L}^r, \alpha, (a_i), a)$ and $\operatorname{supp} s_n \subset (0, \varepsilon_n), n \in \mathbf{N}$ and then

$$(f \cdot s_n) v = f(s_n \circledast \sigma). \tag{33}$$

It is sufficient to establish that $s_n \circledast v \to v$ as $n \to \infty$. By using (27) and Δ_1 imply that

$$\int_{\mathbf{R}_+} \left|\mathscr{D}_x^k(s_n \circledast v - v)(x)\right|^r dx$$
$$= \int_{\mathbf{R}_+} \left|(s_n \circledast \mathscr{D}_x^k v - \mathscr{D}_x^k v)(x)\right|^r dx \tag{34}$$
$$= \int_{\mathbf{R}_+} \left|\int_0^{\varepsilon_n} s_n(t) \left(\mathscr{D}_x^k v(xt) - \mathscr{D}_x^k v(x)\right) dt\right|^r dx.$$

Hence, the mean value theorem implies

$$\int_{\mathbf{R}_+} \left|\mathscr{D}_x^k(s_n \circledast v - v)(x)\right|^r dx$$
$$\le \int_{\mathbf{R}_+} \left|\int_0^{\varepsilon_n} \xi s_n(t) \mathscr{D}_x^{k+1} v(x\xi) \, dt\right|^r dx, \tag{35}$$

$\xi \in (0, t)$. Let $A_1 = \sup_{s \in \mathbf{K}} |\mathscr{D}_x^{k+1} \psi(s)|$, where $\mathbf{K}$ is certain compact set. The calculations show that

$$\|s_n \circledast v - v\|_{r,a} \le F \varepsilon_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{36}$$

where F is certain constant. Hence Lemma 9. The Boehmian space β_{s_+} is therefore constructed. $\qquad\square$

6. β_{Z_+} and the Krätzel Transform of Ultra-Boehmians

Denote by $Z(\mathbf{R}_+)$, or Z_+, the space of functions which are Krätzel transforms of ultradistributions in $\mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$, and then convergence on Z_+ can be defined in such away that $E_n \to E_*$ in Z_+ if $f_n \to f \in \mathscr{S}'_+(\mathscr{L}^r, \alpha, (a_i), a)$ as $n \to \infty$, where $E_n = K_v^p f_n$ and $E_* = K_v^p f$. Let $E \in Z_+$ and $\sigma \in D_+$ and then it is proper to define

$$(E \odot \sigma)(u) = \int_{\mathbf{R}_+} E(ut) \sigma(t) \, dt \tag{37}$$

for each $u \in \mathbf{R}_+$.

Lemma 10. *Let $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ and $\sigma \in D_+$ and then $K^p_v(f \cdot \sigma) = K^p_v f \circledcirc \sigma$.*

Proof. Let $\mathbf{K}$ be a compact set containing the support of σ, and then from (18) it follows that

$$K^p_v(f \cdot \sigma)(u) = \left\langle (f \cdot \sigma)(y), Z^v_p(uy) \right\rangle$$
$$= \left\langle f(y), \left\langle \sigma(t), Z^v_p((ut)\,y) \right\rangle \right\rangle$$
$$= \int_{\mathbf{R}_+} \left\langle f(y), Z^v_p((ut)\,y) \right\rangle \sigma(t)\,dt = K^p_v f \circledcirc \sigma. \tag{38}$$

Hence the lemma follows. $\qquad\square$

Lemma 11. *Let $E \in Z_+$ and $\sigma \in D_+$ and then $E \circledcirc \sigma \in Z_+$.*

Proof. $E \in Z_+$ implies $K^p_v f = E$, for some $f \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$. Hence $E \circledcirc \sigma = K^p_v f \circledcirc \sigma = K^p_v(f \cdot \sigma) \in Z_+$. $\qquad\square$

The following are lemmas which can be easily proved by the aid of the corresponding lemmas from the previous section. Detailed proof is avoided. First, if $K^{-1}_{v^p}$ is the inverse Krätzel transform of K^p_v, and then

Lemma 12. *Let $E \in Z_+$ and $\sigma \in D_+$ and then*

$$K^{-1}_{v^p}(E \circledcirc \sigma) = K^{-1}_{v^p} E \cdot \sigma. \tag{39}$$

Lemma 13. *Let $E_1, E_2 \in Z_+$ and then for all $\sigma_1, \sigma_2 \in D_+$ we have*

(1) $(E_1 + E_2) \circledcirc \sigma_1 = E_1 \circledcirc \sigma_1 + E_2 \circledcirc \sigma_1$,

(2) $(aE) \circledcirc \sigma_1 = a(E \circledcirc \sigma_1)$.

Lemma 14. *Let $E_n \to E$ and $(s_n) \in \Delta_+$ and then $E_n \circledcirc s_n \to E$.*

Lemma 15. *Let $E_n \to E$ and $\sigma \in D_+$ and then $E_n \circledcirc \sigma \to E \circledcirc \sigma$.*

With the previous analysis, the Boehmian space β_{Z_+} is constructed. The sum of two Boehmians and multiplication by a scalar in β_{Z_+} is defined in a natural way $[f_n/\phi_n] + [g_n/\psi_n] = [((f_n \circledcirc \psi_n) + (g_n \circledcirc \phi_n))/(\phi_n \circledcirc \psi_n)]$ and $\alpha[f_n/\phi_n] = [\alpha(f_n/\phi_n)]$, $\alpha \in \mathbb{C}$.

The operation $\circledcirc$ and the differentiation are defined by

$$\left[\frac{f_n}{\phi_n}\right] \circledcirc \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n \circledcirc g_n}{\phi_n \circledcirc \psi_n}\right], \qquad \mathcal{D}^k \left[\frac{f_n}{\phi_n}\right] = \left[\frac{\mathcal{D}^k f_n}{\phi_n}\right]. \tag{40}$$

With the aid of Lemma 10 we define the extended Krätzel transform of a Boehmian $[f_n/s_n] \in \beta_{s_+}$ to be a Boehmian in β_{Z_+} expressed by the relation:

$$\vec{\mathbf{K}}^p_v \left[\frac{f_n}{s_n}\right] = \left[\frac{K^p_v f_n}{s_n}\right]. \tag{41}$$

Lemma 16. $\vec{\mathbf{K}}^p_v : \beta_{s_+} \to \beta_{Z_+}$ *is well defined and linear mapping.*

Proof is a straightforward conclusion of definitions.

Definition 17. Let $[E_n/s_n] \in \beta_{Z_+}$ and then the inverse of $\vec{\mathbf{K}}$ is defined as follows:

$$\vec{\mathbf{K}}^{-1}_{v^p} \left[\frac{E_n}{s_n}\right] = \left[\frac{K^{-1}_{v^p} E_n}{s_n}\right], \tag{42}$$

for each $(s_n) \in \Delta_+$.

Lemma 18. $\vec{\mathbf{K}}^p_v$ *is an isomorphism from β_{s_+} into β_{Z_+}.*

Proof. Assume $\vec{\mathbf{K}}^p_v[f_n/s_n] = \vec{\mathbf{K}}^p_v[g_n/t_n]$, then it follows from (41) and the concept of quotients of two sequences $K^p_v f_n \circledcirc t_m = K^p_v g_m \circledcirc s_n$. Therefore, Lemma 10 implies $K^p_v(f_n \cdot t_m) = K^p_v(g_m \cdot \phi_n)$. Employing properties of K^p_v implies $f_n \cdot t_m = g_m \cdot s_n$. Thus, $[f_n/s_n] = [g_n/t_n]$. Next we establish that $\vec{\mathbf{K}}^p_v$ is onto. Let $[K^p_v f_n/s_n] \in \beta_{Z_+}$ be arbitrary and then $K^p_v f_n \circledcirc s_m = K^p_v f_m \circledcirc s_n$ for every $m, n \in \mathbf{N}$. Hence $f_n, f_m \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ are such that $K^p_v(f_n \cdot s_m) = K^p_v(f_m \cdot s_n)$. Hence the Boehmian $[f_n/s_n] \in \beta_{s_+}$ satisfies the equation $\vec{\mathbf{K}}^p_v[f_n/s_n] = [K^p_v f_n/s_n]$.

This completes the proof of the lemma. $\qquad\square$

Lemma 19. *Let $[E_n/s_n] \in \beta_{Z_+}$, $E_n = K^p_v f_n$, and $\phi \in D_+$ and then*

$$\vec{\mathbf{K}}^{-1}_{v^p}\left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) = \left[\frac{E^{-1}_n}{s_n}\right] \cdot \phi,$$
$$\vec{\mathbf{K}}^p_v\left(\left[\frac{f_n}{s_n}\right] \bullet \phi\right) = \left[\frac{E_n}{s_n}\right] \circledcirc \phi. \tag{43}$$

Proof. It follows from (42) that

$$\vec{\mathbf{K}}^{-1}_{v^p}\left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) = \vec{\mathbf{K}}^{-1}_{v^p}\left(\left[\frac{E_n \circledcirc \phi}{s_n}\right]\right) = \left[\frac{K^{-1}_{v^p}(E_n \circledcirc \phi)}{s_n}\right]. \tag{44}$$

Applying Lemma 12 leads to

$$\vec{\mathbf{K}}^{-1}_{v^p}\left(\left[\frac{E_n}{s_n}\right] \circledcirc \phi\right) = \left[\frac{K^{-1}_{v^p} E_n}{s_n}\right] \cdot \phi = \left[\frac{E^{-1}_n}{s_n}\right] \cdot \phi. \tag{45}$$

Proof of the second part is similar. This completes the proof of the lemma. $\qquad\square$

Theorem 20. $\vec{\mathbf{K}}^p_v : \beta_{s_+} \to \beta_{Z_+}$ *and $\vec{\mathbf{K}}^{-1}_{v^p} : \beta_{Z_+} \to \beta_{s_+}$ are continuous with respect to δ and Δ convergences.*

Proof. First of all, we show that $\vec{\mathbf{K}}^p_v : \beta_{s_+} \to \beta_{Z_+}$ and $\vec{\mathbf{K}}^{-1}_{v^p} : \beta_{Z_+} \to \beta_{s_+}$ are continuous with respect to δ convergence. Let $\beta_n \xrightarrow{\delta} \beta$ in β_{s_+} as $n \to \infty$ and then we establish that

$\vec{\mathbf{K}}_v^p \beta_n \to \vec{\mathbf{K}}_v^p \beta$ as $n \to \infty$. In view of [10], there are $f_{n,k}$ and f_k in $\mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$ such that

$$\beta_n = \left[\frac{f_{n,k}}{s_k}\right], \qquad \beta = \left[\frac{f_k}{s_k}\right] \qquad (46)$$

such that $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbf{N}$. The continuity condition of the Krätzel transform implies $E_{n,k} \to E_k$ as $n \to \infty$, $E_{n,k} = K_v^p f_{n,k}$, and $E_k = K_v^p f_k$ in the space $\mathbf{Z}_+$. Thus, $[E_{n,k}/s_k] \to [E_k/s_k]$ as $n \to \infty$ in $\beta_{\mathbf{Z}_+}$.

To prove the second part of the lemma, let $g_n \xrightarrow{\delta} g \in \beta_{\mathbf{Z}_+}$ as $n \to \infty$. From [10], we have $g_n = [E_{n,k}/s_k]$ and $g = [E_k/s_k]$ for some $E_{n,k}, E_k \in \mathbf{Z}_+$ where $E_{n,k} \to E_k$ as $n \to \infty$. Hence $K_v^{-1^p} E_{n,k} \to K_v^{-1^p} E_k$ in β_{s_+} as $n \to \infty$. That is, $[E_{n,k}^{-1}/s_k] \to [E_k^{-1}/s_k]$ as $n \to \infty$. Using (42) we get $\vec{\mathbf{K}}_v^{-1^p}[E_{n,k}/s_k] \to \vec{\mathbf{K}}_v^{-1^p}[E_k/s_k]$ as $n \to \infty$.

Now, we establish continuity of $\vec{\mathbf{K}}_v^p$ and $\vec{\mathbf{K}}_v^{-1^p}$ with respect to Δ_+ convergence. Let $\beta_n \xrightarrow{\Delta} \beta$ in β_{s_+} as $n \to \infty$. Then, we find $f_n \in \mathcal{S}'_+(\mathcal{L}^r, \alpha, (a_i), a)$, and $(s_n) \in \Delta_+$ such that $(\beta_n - \beta) \cdot s_n = [(f_n \cdot s_k)/s_k]$ and $f_n \to 0$ as $n \to \infty$. Employing (41) we get

$$\vec{\mathbf{K}}_v^p\left((\beta_n - \beta) \cdot s_n\right) = \left[\frac{K_v^p(f_n \cdot s_k)}{s_k}\right]. \qquad (47)$$

Hence, from (41) and Lemma 19 we have $\vec{\mathbf{K}}_v^p((\beta_n - \beta) \cdot s_n) = [(E_n \circledcirc s_k)/s_k] = E_k \to 0$ as $n \to \infty$ in $\mathbf{Z}_+$. Therefore

$$\begin{aligned} &\vec{\mathbf{K}}_v^p\left((\beta_n - \beta) \cdot s_n\right) \\ &= \left(\vec{\mathbf{K}}_v^p \beta_n - \vec{\mathbf{K}}_v^p \beta\right) \circledcirc s_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \qquad (48)$$

Hence, $\vec{\mathbf{K}}_v^p \beta_n \xrightarrow{\Delta} \vec{\mathbf{K}}_v^p \beta$ as $n \to \infty$. Finally, let $g_n \xrightarrow{\Delta} g$ in $\beta_{\mathbf{Z}_+}$ as $n \to \infty$ and then we find $E_k \in \mathbf{Z}_+$ such that $(g_n - g) \circledcirc s_n = [(E_k \circledcirc s_k)/s_k]$ and $E_k \to 0$ as $n \to \infty$ for some $(s_n) \in \Delta_+$ and $E_k = K_v^p f_n$.

Next, using (42), we obtain $\vec{\mathbf{K}}_v^{-1^p}((g_n - g) \circledcirc s_n) = [K_v^{-1^p}(E_k \circledcirc s_k)/s_k]$. Lemma 19 implies that

$$\begin{aligned} &\vec{\mathbf{K}}_v^{-1^p}\left((g_n - g) \circledcirc s_n\right) \\ &= \left[\frac{f_n \cdot s_k}{s_k}\right] = f_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \text{ in } \mathcal{S}'_+\left(\mathcal{L}^r, \alpha, (a_i), a\right). \end{aligned} \qquad (49)$$

Thus $\vec{\mathbf{K}}_v^{-1^p}((g_n - g) \circledcirc s_n) = (\vec{\mathbf{K}}_v^{-1^p} g_n - \vec{\mathbf{K}}_v^{-1^p} g) \cdot s_n \to 0$ as $n \to \infty$. Hence, we have $\vec{\mathbf{K}}_v^{-1^p} g_n \xrightarrow{\Delta} \vec{\mathbf{K}}_v^{-1^p} g$ as $n \to \infty$ in β_{s_+}.

This completes the proof of the theorem. □

Acknowledgment

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