Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 730825, 7 pages
http://dx.doi.org/10.1155/2013/730825

Research Article

Fixed Point Results for α-ψ_λ-Contractions on Gauge Spaces and Applications

Mohamed Jleli,[1] Erdal Karapınar,[2] and Bessem Samet[1]

[1] *Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia*
[2] *Department of Mathematics, Atilim University, Incek, 06836 Ankara, Turkey*

Correspondence should be addressed to Erdal Karapınar; erdalkarapinar@yahoo.com

Received 21 July 2013; Accepted 25 August 2013

Academic Editor: Calogero Vetro

We extend the concept of α-ψ-contractive mappings introduced recently by Samet et al. (2012) to the setting of gauge spaces. New fixed point results are established on such spaces, and some applications to nonlinear integral equations on the half-line are presented.

1. Introduction and Preliminaries

Fixed point theory plays an important role in nonlinear analysis. This is because many practical problems in applied science, economics, physics, and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings. The Banach contraction principle [1] is one of the fundamental results in fixed point theory. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to approximate those fixed points.

Theorem 1 (see [1]). *Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction self-mapping on X; that is, there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y), \tag{1}$$

for all $(x, y) \in X \times X$. Then the map T admits a unique fixed point. Moreover, for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to this fixed point.

During the last few decades, several extensions of this famous principle have been established. In 1961, Edelstein [2] established the following result.

Theorem 2 (see [2]). *Let (X, d) be complete and ε-chainable for some $\varepsilon > 0$. Let $T : X \to X$ be such that*

$$x, y \in X, \quad d(x, y) < \varepsilon \Longrightarrow d(Tx, Ty) \leq kd(x, y), \tag{2}$$

where $k \in (0, 1)$ is a constant. Then T has a unique fixed point.

Kirk et al. [3] introduced the concept of cyclic mappings and proved the following fixed point theorem.

Theorem 3 (see [3]). *Let A and B be two nonempty closed subsets of a complete metric space (X, d). Let $T : X \to X$ be a self-mapping such that*

$$x, y \in X, \quad (x, y) \in A \times B \Longrightarrow d(Tx, Ty) \leq kd(x, y), \tag{3}$$

where $k \in (0, 1)$ is a constant. Suppose also that $T(A) \subseteq B$ and $T(B) \subset A$. Then T has a unique fixed point in $A \cap B$.

Ran and Reurings [4] extended the Banach contraction principle to a metric space endowed with a partial order. They established the following result.

Theorem 4 (see [4]). *Let (X, d) be a complete metric space endowed with a partial order $\preceq$. Let $T : X \to X$ be a continuous mapping such that*

$$x, y \in X, \quad x \preceq y \Longrightarrow d(Tx, Ty) \leq kd(x, y), \tag{4}$$

where $k \in (0, 1)$ is a constant. Suppose also that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then T has a fixed point.

Many extensions of the previous result exist in the literature; for more details, we refer the reader to [5–11] and the references therein.

Observe that all the contractive conditions (2), (3), and (4) can be written as

$$d(Tx,Ty) \le kd(x,y), \quad \forall (x,y) \in H, \tag{5}$$

where H is a subset of $X \times X$. In Theorem 2, we have

$$H := \{(x,y) \in X \times X : d(x,y) < \varepsilon\}. \tag{6}$$

In Theorem 3, we have

$$H := (A \times B) \cup (B \times A). \tag{7}$$

In Theorem 4, we have

$$H := \{(x,y) \in X \times X : (x \preceq y) \wedge (y \preceq x)\}. \tag{8}$$

The contractive condition (5) is said to be a partial contraction, that is, a contraction satisfied only on a subset of $H \subseteq X \times X$.

Very recently, Samet et al. [12] observed that a partial contraction can be considered as a total contraction, that is, a contraction satisfied for every pair $(x,y) \in X \times X$. More precisely, if we define the function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) := \begin{cases} 1, & \text{if } (x,y) \in H, \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

we show that (5) is equivalent to

$$\alpha(x,y)\, d(Tx,Ty) \le kd(x,y), \quad \forall (x,y) \in X \times X. \tag{10}$$

In [12], the authors considered a more general inequality; that is,

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(d(x,y)), \quad \forall (x,y) \in X \times X, \tag{11}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a function satisfying some conditions. The above inequality is called an α-ψ-contraction. In [12], some fixed point results were established under this contractive condition. For other works in this direction, we refer the reader to [13–16].

The aim of this work is to extend, generalize, and improve the obtained results in [12]. More precisely, the concept of α-ψ-contractive mappings is extended to the setting of gauge spaces. New fixed point results are established on such spaces, and some applications to nonlinear integral equations on the half-line are presented.

Through this paper, $\mathbb{E}$ will denote a gauge space endowed with a separating gauge structure $\mathscr{D} = \{d_\lambda\}_{\lambda \in \Lambda}$, where Λ is a directed set.

A sequence $\{x_n\} \subset \mathbb{E}$ is said to be convergent if there exists an $x \in \mathbb{E}$ such that for every $\varepsilon > 0$ and $\lambda \in \Lambda$, there is an $N \in \mathbb{N}$ with $d_\lambda(x_n, x) < \varepsilon$, for all $n \ge N$.

A sequence $\{x_n\} \subset \mathbb{E}$ is said to be Cauchy, if for every $\varepsilon > 0$ and $\lambda \in \Lambda$, there is an $N \in \mathbb{N}$ with $d_\lambda(x_n, x_{n+p}) < \varepsilon$, for all $n \ge N$ and $p \in \mathbb{N}$.

A gauge space is called complete if any Cauchy sequence is convergent.

A subset of $\mathbb{E}$ is said to be closed if it contains the limit of any convergent sequence of its elements.

For more details on guage spaces, we refer the reader to Dugundji [17].

We denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(C1) ψ is nondecreasing;

(C2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, for all $t > 0$, where ψ^n is the nth iterate of ψ;

(C3) $\psi(a) + \psi(b) \le \psi(a+b)$, for all $a, b \ge 0$.

It is easy to show that under the conditions (C1) and (C2), we have $\psi(t) < t$, for all $t > 0$. Moreover, under the conditions (C1) and (C3), we have

$$\psi^n(a) + \psi^n(b) \le \psi^n(a+b), \tag{12}$$

for all $n \in \mathbb{N}$ and $a, b \ge 0$.

Example 5. Let $\psi : [0,\infty) \to [0,\infty)$ be the function defined by

$$\psi(t) := \begin{cases} \dfrac{t^2}{4}, & \text{if } t \in [0,1), \\ \dfrac{t}{2}, & \text{if } t \ge 1. \end{cases} \tag{13}$$

Then $\psi \in \Psi$.

Definition 6. Let $\alpha : \mathbb{E} \times \mathbb{E} \to [0,\infty)$ be a given function. Let $N \in \mathbb{N}$ and $x, y \in \mathbb{E}$. We say that $(x^i)_{i=0}^N \subset \mathbb{E}$ is an N-α-path from x to y if

$$\begin{gathered} x^0 = x, \qquad x^N = y, \\ \alpha(x^{i-1}, x^i) \ge 1, \quad \forall i = 1, \dots, N. \end{gathered} \tag{14}$$

We denote

$$x[N,\alpha] := \{y \in \mathbb{E} : \text{there is an } N\text{-}\alpha\text{-path from } x \text{ to } y\}. \tag{15}$$

Let $N \in \mathbb{N}$ and $x \in \mathbb{E}$. For $\lambda \in \Lambda$ and $y \in x[N,\alpha]$, and let

$$\begin{aligned} p_\lambda^N(x,y) := \inf \Bigg\{ & \sum_{i=1}^{N} d_\lambda(x^{i-1}, x^i) : (x^i)_{i=0}^N \subset X \\ & \text{is an } N\text{-}\alpha\text{-path from } x \text{ to } y \Bigg\}. \end{aligned} \tag{16}$$

2. Fixed Point Results for α-ψ_λ-Contractions

Definition 7. Let $T : \mathbb{E} \to \mathbb{E}$ be a given self-mapping. Let $\alpha : \mathbb{E} \times \mathbb{E} \to [0,\infty)$ be a given function, and let $\{\psi_\lambda\}_{\lambda \in \Lambda} \subset \Psi$. We say that T is an α-ψ_λ-contraction if

$$\alpha(x,y)\, d_\lambda(Tx,Ty) \le \psi_\lambda(d_\lambda(x,y)), \tag{17}$$

for all $\lambda \in \Lambda$ and $x, y \in \mathbb{E}$.

Definition 8. Let $T : \mathbb{E} \to \mathbb{E}$ be a given self-mapping. Let $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ be a given function. We say that T is α-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad \forall x, y \in \mathbb{E}. \tag{18}$$

The following lemma will be useful to establish our fixed point results.

Lemma 9. *Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping. Suppose that there exist $\alpha : \mathbb{E}\times\mathbb{E} \to [0, \infty)$ and $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$ such that the following conditions hold:*

(i) *T is an α-ψ_λ-contraction;*

(ii) *T is α-admissible.*

Let $\varepsilon > 0$ and $N \in \mathbb{N}$. Then, for every $\lambda \in \Lambda$, $x \in X$, and $y \in x[N, \alpha]$, one has

(I) *$Ty \in Tx[N, \alpha]$ with*

$$p_\lambda^N(Tx, Ty) \leq \psi_\lambda\left(p_\lambda^N(x, y) + \varepsilon\right); \tag{19}$$

(II) *for all $k \in \mathbb{N} \cup \{0\}$, one has $T^k y \in T^k x[N, \alpha]$ with*

$$p_\lambda^N\left(T^k x, T^k y\right) \leq \psi_\lambda^k\left(p_\lambda^N(x, y) + \varepsilon\right). \tag{20}$$

Proof. Let $\lambda \in \Lambda$, $x \in X$, and $y \in x[N, \alpha]$. Let $\{x^i\}_{i=0}^N$ be an N-α-path from x to y such that

$$\sum_{i=1}^N d_\lambda\left(x^{i-1}, x^i\right) \leq p_\lambda^N(x, y) + \varepsilon. \tag{21}$$

Since $\alpha(x, x^1) \geq 1$, we have

$$d_\lambda\left(Tx, Tx^1\right) \leq \alpha\left(x, x^1\right) d_\lambda\left(Tx, Tx^1\right) \leq \psi_\lambda\left(d_\lambda\left(x, x^1\right)\right). \tag{22}$$

Since $\alpha(x^1, x^2) \geq 1$, we have

$$\begin{aligned} d_\lambda\left(Tx^1, Tx^2\right) &\leq \alpha\left(x^1, x^2\right) d_\lambda\left(Tx^1, Tx^2\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(x^1, x^2\right)\right). \end{aligned} \tag{23}$$

Recursively from $i = 3$ to N, since $\alpha(x^{i-1}, x^i) \geq 1$, we have

$$\begin{aligned} d_\lambda\left(Tx^{i-1}, Tx^i\right) &\leq \alpha\left(x^{i-1}, x^i\right) d_\lambda\left(Tx^{i-1}, Tx^i\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(x^{i-1}, x^i\right)\right). \end{aligned} \tag{24}$$

On the other hand, since T is α-admissible, $\{Tx^i\}_{i=0}^N$ is an N-α-path from Tx to Ty. So, we have

$$\begin{aligned} p_\lambda^N(Tx, Ty) &\leq \sum_{i=1}^N d_\lambda\left(Tx^{i-1}, Tx^i\right) \\ &\leq \sum_{i=1}^N \psi_\lambda\left(d_\lambda\left(x^{i-1}, x^i\right)\right) \\ &\leq \psi_\lambda\left(\sum_{i=1}^N d_\lambda\left(x^{i-1}, x^i\right)\right) \\ &\leq \psi_\lambda\left(p_\lambda^N(x, y) + \varepsilon\right). \end{aligned} \tag{25}$$

Thus, we proved (I).

Again, since $\alpha(Tx, Tx^1) \geq 1$, we have

$$\begin{aligned} d_\lambda\left(T^2x, T^2x^1\right) &\leq \alpha\left(Tx, Tx^1\right) d_\lambda\left(T^2x, T^2x^1\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(Tx, Tx^1\right)\right) \leq \psi_\lambda^2\left(d_\lambda\left(x, x^1\right)\right). \end{aligned} \tag{26}$$

Since $\alpha(Tx^1, Tx^2) \geq 1$, we have

$$\begin{aligned} d_\lambda\left(T^2x^1, T^2x^2\right) &\leq \alpha\left(Tx^1, Tx^2\right) d_\lambda\left(T^2x^1, T^2x^2\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(Tx^1, Tx^2\right)\right) \leq \psi_\lambda^2\left(d_\lambda\left(x^1, x^2\right)\right). \end{aligned} \tag{27}$$

Recursively from $i = 3$ to N, since $\alpha(Tx^{i-1}, Tx^i) \geq 1$, we have

$$\begin{aligned} d_\lambda\left(T^2x^{i-1}, T^2x^i\right) &\leq \alpha\left(Tx^{i-1}, Tx^i\right) d_\lambda\left(T^2x^{i-1}, T^2x^i\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(Tx^{i-1}, Tx^i\right)\right) \\ &\leq \psi_\lambda^2\left(d_\lambda\left(x^{i-1}, x^i\right)\right). \end{aligned} \tag{28}$$

On the other hand, since T is α-admissible, $\{T^2x^i\}_{i=0}^N$ is an N-α-path from T^2x to T^2y. So, we have

$$\begin{aligned} p_\lambda^N\left(T^2x, T^2y\right) &\leq \sum_{i=1}^N d_\lambda\left(T^2x^{i-1}, T^2x^i\right) \\ &\leq \sum_{i=1}^N \psi_\lambda^2\left(d_\lambda\left(x^{i-1}, x^i\right)\right) \\ &\leq \psi_\lambda^2\left(\sum_{i=1}^N d_\lambda\left(x^{i-1}, x^i\right)\right) \\ &\leq \psi_\lambda^2\left(p_\lambda^N(x, y) + \varepsilon\right). \end{aligned} \tag{29}$$

Continuing this process, by induction, we get (II). □

Definition 10. Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping, and let $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ be a given function. For $N \in \mathbb{N}$, we say that a sequence $\{x_n\} \subset \mathbb{E}$ is an N-α-Picard trajectory from x_0 if $x_n = Tx_{n-1} \in x_{n-1}[N, \alpha]$ for all $n \in \mathbb{N}$. We denote by $\mathcal{T}_N(T, \alpha, x_0)$, the set of all N-α-Picard trajectories from x_0.

Definition 11. Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping, and let $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ be a given function. For $N \in \mathbb{N}$, we say that T is N-α-Picard continuous from $x_0 \in \mathbb{E}$ if the limit of any convergent sequence $\{x_n\} \in \mathcal{T}_N(T, \alpha, x_0)$ is a fixed point of T.

We have the following fixed point result.

Theorem 12. *Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping on the complete gauge space $\mathbb{E}$. Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$ and $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ be a given function. Suppose that the following conditions hold:*

(i) *T is an α-ψ_λ-contraction;*

(ii) T *is* α*-admissible;*

(iii) *there exist* $N \in \mathbb{N}$ *and* $x_0 \in \mathbb{E}$ *such that* $Tx_0 \in x_0[N, \alpha]$;

(iv) T *is* N*-*α*-Picard continuous from* x_0.

Then T *has a fixed point.*

Proof. Let $\lambda \in \Lambda$ and $\varepsilon > 0$. From condition (iii) and Lemma 9, we have $T^2x_0 \in Tx_0[N, \alpha]$ and

$$d_\lambda\left(Tx_0, T^2x_0\right) \leq p_\lambda^N\left(Tx_0, T^2x_0\right) \leq \psi_\lambda\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right). \tag{30}$$

Again, from Lemma 9, we have $T^3x_0 \in T^2x_0[N, \alpha]$ and

$$\begin{aligned} d_\lambda\left(T^2x_0, T^3x_0\right) &\leq p_\lambda^N\left(T^2x_0, T^3x_0\right) \\ &\leq \psi_\lambda^2\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right). \end{aligned} \tag{31}$$

Continuing this process, by induction, for $n \geq 2$, we have $T^{n+1}x_0 \in T^nx_0[N, \alpha]$ and

$$\begin{aligned} d_\lambda\left(T^nx_0, T^{n+1}x_0\right) &\leq p_\lambda^N\left(T^nx_0, T^{n+1}x_0\right) \\ &\leq \psi_\lambda^n\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right). \end{aligned} \tag{32}$$

Thus, $\{T^nx_0\} \in \mathscr{T}_N(T, \alpha, x_0)$ and, for $m \geq 1$,

$$\begin{aligned} d_\lambda\left(T^nx_0, T^{n+m}x_0\right) &\leq \sum_{i=0}^{m-1} d_\lambda\left(T^{n+i}x_0, T^{n+i+1}x_0\right) \\ &\leq \sum_{i=0}^{m-1} \psi_\lambda^{n+i}\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right) \\ &\leq \sum_{j\geq n} \psi_\lambda^{j}\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right). \end{aligned} \tag{33}$$

From condition (C2), we have

$$\sum_{j\geq n} \psi_\lambda^{j}\left(p_\lambda^N\left(x_0, Tx_0\right) + \varepsilon\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{34}$$

which implies that $\{T^nx_0\}$ is a Cauchy sequence in the complete gauge space $\mathbb{E}$. Since T is N-α-Picard continuous from x_0, the limit of $\{T^nx_0\}$ is a fixed point of T. □

Corollary 13. *Let* $T : \mathbb{E} \to \mathbb{E}$ *be a self-mapping on the complete gauge space* $\mathbb{E}$*. Let* $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$ *and* $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ *be a given function. Suppose that the following conditions hold:*

(i) T *is an* α*-*ψ_λ*-contraction;*

(ii) T *is* α*-admissible;*

(iii) *there exist* $N \in \mathbb{N}$ *and* $x_0 \in \mathbb{E}$ *such that* $Tx_0 \in x_0[N, \alpha]$;

(iv) T *is continuous.*

Then T *has a fixed point.*

Proof. Let $\{x_n\} \in \mathscr{T}_N(T, \alpha, x_0)$ be such that $x_n \to x \in \mathbb{E}$. Since T is continuous, we have $x_{n+1} = Tx_n \to Tx$. Since $\mathbb{E}$ is endowed with a separating gauge structure, we have $x = Tx$. The conclusion follows from Theorem 12. □

Corollary 14. *Let* $T : \mathbb{E} \to \mathbb{E}$ *be a self-mapping on the complete gauge space* $\mathbb{E}$*. Let* $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$ *and* $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ *be a given function. Suppose that the following conditions hold:*

(i) T *is an* α*-*ψ_λ*-contraction;*

(ii) T *is* α*-admissible;*

(iii) *there exist* $N \in \mathbb{N}$ *and* $x_0 \in \mathbb{E}$ *such that* $Tx_0 \in x_0[N, \alpha]$;

(iv) *for every* $\{x_n\} \in \mathscr{T}_N(T, \alpha, x_0)$ *such that* $x_n \to x \in \mathbb{E}$*, there exist a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *and* $k_0 \in \mathbb{N}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for* $k \geq k_0$.

Then T *has a fixed point.*

Proof. Let $\{x_n\} \in \mathscr{T}_N(T, \alpha, x_0)$ be such that $x_n \to x \in \mathbb{E}$. From condition (iv), there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and $k_0 \in \mathbb{N}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for $k \geq k_0$. Since T is an α-ψ_λ-contraction, for all $\lambda \in \Lambda$ and $k \geq k_0$, we have

$$\begin{aligned} d_\lambda\left(Tx_{n(k)}, Tx\right) &\leq \alpha\left(x_{n(k)}, x\right) d_\lambda\left(Tx_{n(k)}, Tx\right) \\ &\leq \psi_\lambda\left(d_\lambda\left(x_{n(k)}, x\right)\right). \end{aligned} \tag{35}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$x_{n(k)+1} = Tx_{n(k)} \longrightarrow Tx, \quad \text{as } k \longrightarrow \infty. \tag{36}$$

Since $\mathbb{E}$ is endowed with a separating gauge structure, we have $x = Tx$. The conclusion follows from Theorem 12. □

For $T : \mathbb{E} \to \mathbb{E}$, we denote by $\mathrm{Fix}(T)$ the set of fixed points of T; that is,

$$\mathrm{Fix}\,(T) := \{x \in \mathbb{E} : x = Tx\}. \tag{37}$$

The next result gives us a sufficient condition that ensures the uniqueness of the fixed point.

Theorem 15. *Suppose that all the conditions of Theorem 12 are satisfied. Moreover, suppose that*

(v) *for every* $(x, y) \in \mathrm{Fix}(T) \times \mathrm{Fix}(T)$ *with* $x \neq y$*, there exists* $N(x, y) \in \mathbb{N}$ *such that* $y \in x[N(x, y), \alpha]$.

Then T *has a unique fixed point.*

Proof. From Theorem 12, The mapping T has at least one fixed point. Suppose that $u, v \in \mathbb{E}$ are two fixed points of T with $u \neq v$. From the condition (v), there exists $N(u, v) \in \mathbb{N}$ such that $v \in u[N(u, v), \alpha]$. Let $\lambda \in \Lambda$ and $\varepsilon > 0$. From Lemma 9, we have

$$\begin{aligned} d_\lambda\left(u, v\right) &\leq p_\lambda^{N(u,v)}\left(u, v\right) \\ &\leq \psi_\lambda^n\left(p_\lambda^{N(u,v)}\left(u, v\right) + \varepsilon\right), \end{aligned} \tag{38}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we obtain that $d_\lambda(u,v) = 0$ for all $\lambda \in \Lambda$, which is a contradiction with $u \neq v$ (since we have a separating gauge structure). We deduce that $u = v$. □

The following result follows immediately from Theorems 12 and 15 with $N = 1$ and $\alpha(x,y) = 1$ for every $x, y \in \mathbb{E}$.

Corollary 16. *Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping on the complete gauge space $\mathbb{E}$. Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$. Suppose that for all $\lambda \in \Lambda$, for all $x, y \in \mathbb{E}$, one has*

$$d_\lambda(Tx,Ty) \le \psi_\lambda(d_\lambda(x,y)). \tag{39}$$

Then T has a unique fixed point.

Corollary 17. *Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping on the complete gauge space $\mathbb{E}$. Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$ and $\alpha : \mathbb{E}\times\mathbb{E} \to [0,\infty)$ be a given function. Suppose that the following conditions hold:*

(i) *T is an α-ψ_λ-contraction;*

(ii) *T is α-admissible;*

(iii) *there exists $x_0 \in \mathbb{E}$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iv) *T is continuous*

(or) for any sequence $\{x_n\} \subset \mathbb{E}$ such that $x_n = Tx_{n-1}$, $x_n \to x \in \mathbb{E}$ and $\alpha(x_{n-1}, Tx_{n-1}) \ge 1$ for $n \in \mathbb{N}$, there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and $k_0 \in \mathbb{N}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for $k \ge k_0$.

Then T has a fixed point. Moreover, if

(v) *for every $(x,y) \in \mathrm{Fix}(T)\times\mathrm{Fix}(T)$ with $x \neq y$, there exists $z \in \mathbb{E}$ such that $\alpha(x,z) \ge 1$ and $\alpha(z,y) \ge 1$,*

one has uniqueness of the fixed point.

Proof. The existence follows from Theorem 12 with $N = 1$. The uniqueness follows from Theorem 15 with $N(x,y) = 2$. □

Corollary 18. *Let $\preceq$ be a partial order on the complete gauge space $\mathbb{E}$. Let $T : \mathbb{E} \to \mathbb{E}$ be a self-mapping and $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset \Psi$. Suppose that the following conditions hold:*

(i) *for all $\lambda \in \Lambda$, for all $x, y \in \mathbb{E}$ such that x and y are comparable, one has*

$$d_\lambda(Tx,Ty) \le \psi_\lambda(d_\lambda(x,y)); \tag{40}$$

(ii) *$x, y \in \mathbb{E}$, x and y are comparable $\Rightarrow$ Tx and Ty are comparable;*

(iii) *there exists $x_0 \in \mathbb{E}$ such that x_0 and Tx_0 are comparable;*

(iv) *T is continuous,*

(or) for any sequence $\{x_n\} \subset \mathbb{E}$ such that $x_n = Tx_{n-1}$, $x_n \to x \in \mathbb{E}$, x_{n-1} and $x_n = Tx_{n-1}$ are comparable for $n \in \mathbb{N}$, there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ and $k_0 \in \mathbb{N}$ such that $x_{n(k)}$ and x are comparable for $k \ge k_0$.

Then T has a fixed point. Moreover, if

(v) *for every $(x,y) \in \mathrm{Fix}(T)\times\mathrm{Fix}(T)$ with $x \neq y$, there exists $z \in \mathbb{E}$ such that x and z are comparable, z and y are comparable,*

one has uniqueness of the fixed point.

Proof. It follows from Corollary 17 with

$$\alpha(x,y) := \begin{cases} 1, & \text{if } (x \preceq y) \vee (y \preceq x), \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

□

3. Applications

In this section, we are interested in the study of the existence of solutions to the nonlinear integral equation on the real axis

$$x(t) = h(t) + \int_0^t F(t,s,x(s))\,ds, \quad t \ge 0, \tag{42}$$

where $h \in C([0,\infty),\mathscr{E})$ and $F \in C([0,\infty)\times[0,\infty)\times\mathscr{E},\mathscr{E})$. Here, $\mathscr{E}$ is a Banach space with respect to a given norm $\|\cdot\|_{\mathscr{E}}$.

Let $\mathbb{E} := C([0,\infty),\mathscr{E})$ and the family of pseudonorms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ defined by

$$\|x\|_n := \max_{t\in[0,n]} \|x(t)\|_{\mathscr{E}} e^{-\tau t}, \quad \tau > 0. \tag{43}$$

For every $n \in \mathbb{N}$, define now

$$d_n(x,y) := \|x-y\|_n, \quad \text{for } x, y \in \mathbb{E}. \tag{44}$$

Then $\mathscr{D} := \{d_n\}_{n\in\mathbb{N}}$ is a separating gauge structure on $\mathbb{E}$.

We have the following existence result.

Theorem 19. *Suppose that the following conditions hold:*

(i) *there exist a nonempty set $\Gamma \subseteq \mathscr{E}\times\mathscr{E}$ and a constant $k < \tau$ such that*

$$\|F(t,s,u) - F(t,s,v)\|_{\mathscr{E}} \le k\|u-v\|_{\mathscr{E}}, \tag{45}$$

for all $t, s \ge 0$, $(u,v) \in \Gamma$;

(ii) *there exists a nonempty set $\Gamma_{\mathbb{E}} \subseteq \mathbb{E}\times\mathbb{E}$ such that*

$$(x,y) \in \Gamma_{\mathbb{E}} \Longleftrightarrow (x(t), y(t)) \in \Gamma, \quad \forall t \ge 0; \tag{46}$$

(iii) *for all $(x,y) \in \Gamma_{\mathbb{E}}$, one has*

$$\left(h(t) + \int_0^t F(t,s,x(s))\,ds, h(t) + \int_0^t F(t,s,y(s))\,ds\right) \in \Gamma, \tag{47}$$

for all $t \ge 0$;

(iv) *there exists $x_0 \in \mathbb{E}$ such that*

$$\left(x_0(t), h(t) + \int_0^t F(t,s,x_0(s))\,ds\right) \in \Gamma, \tag{48}$$

for all $t \ge 0$;

(v) *if* $\{x_p\} \subset \mathbb{E}$ *is a sequence such that* $(x_p, x_{p+1}) \in \Gamma_{\mathbb{E}}$ *for* $p \in \mathbb{N}$ *and* $x_p \to x \in \mathbb{E}$ *(with respect to* $\mathscr{D}$*), then there exist a subsequence* $\{x_{p(k)}\}$ *of* $\{x_p\}$ *and* $k_0 \in \mathbb{N}$ *such that*

$$\left(h(t) + \int_0^t F\left(t, s, x_{p(k)}(s)\right) ds, x(t)\right) \in \Gamma, \qquad (49)$$

for all $k \geq k_0$, $t \geq 0$.

Then (42) *has at least one solution in* $\mathbb{E}$.

Proof. Consider the mapping $T : \mathbb{E} \to \mathbb{E}$ defined by

$$Tx(t) := h(t) + \int_0^t F(t, s, x(s)) ds, \quad t \geq 0, \qquad (50)$$

for all $x \in \mathbb{E}$. We have to prove that T has at least a fixed point.

Define the function $\alpha : \mathbb{E} \times \mathbb{E} \to [0, \infty)$ by

$$\alpha(x, y) := \begin{cases} 1, & \text{if } (x, y) \in \Gamma_{\mathbb{E}}, \\ 0, & \text{otherwise.} \end{cases} \qquad (51)$$

We claim that for all $n \in \mathbb{N}$, for all $x, y \in \mathbb{E}$,

$$\alpha(x, y) d_n(Tx, Ty) \leq \psi_n(d_n(x, y)), \qquad (52)$$

where $\psi_n(t) = (k/\tau)t$ for all $n \in \mathbb{N}$, for all $t \geq 0$. Clearly, since $k < \tau$, $\{\psi_n\}_{n\in\mathbb{N}} \subset \Psi$. If $\alpha(x, y) = 0$, (52) holds immediately. So, suppose that $\alpha(x, y) \neq 0$; that is, $(x, y) \in \Gamma_{\mathbb{E}}$. Let $n \in \mathbb{N}$. From conditions (i) and (ii), for all $t \in [0, n]$, we have

$$\begin{aligned} \|Tx(t) - Ty(t)\|_{\mathscr{E}} &\leq \int_0^t \|F(t, s, x(s)) - F(t, s, y(s))\|_{\mathscr{E}} ds \\ &\leq \int_0^t ke^{\tau s}\left(\|x(s) - y(s)\|_{\mathscr{E}} e^{-\tau s}\right) ds \\ &\leq \frac{k}{\tau} d_n(x, y) e^{\tau t}, \end{aligned} \qquad (53)$$

which implies that

$$\alpha(x, y) d_n(Tx, Ty) = d_n(Tx, Ty) \leq \frac{k}{\tau} d_n(x, y). \qquad (54)$$

Thus, we proved (52).

We will prove that T is α-admissible. Let $(x, y) \in \mathbb{E} \times \mathbb{E}$ such that $\alpha(x, y) \geq 1$; that is, $(x, y) \in \Gamma_{\mathbb{E}}$. From condition (iii), we have $(Tx(t), Ty(t)) \in \Gamma$ for all $t \geq 0$, which implies from condition (ii) that $(Tx, Ty) \in \Gamma_E$; that is, $\alpha(Tx, Ty) \geq 1$. So, T is α-admissible.

From conditions (iv) and (ii), we have $(x_0, Tx_0) \in \Gamma_{\mathbb{E}}$, which is equivalent to say that $\alpha(x_0, Tx_0) \geq 1$.

Finally, condition (v) implies that for every $\{x_p\} \in \mathscr{T}_1(T, \alpha, x_0)$ such that $x_p \to x \in \mathbb{E}$, there exist a subsequence $\{x_{p(k)}\}$ of $\{x_p\}$ and $k_0 \in \mathbb{N}$ such that $\alpha(x_{p(k)}, x) \geq 1$ for $k \geq k_0$.

Now, All the hypotheses of Corollary 14 are satisfied; we deduce that T has at least a fixed point, which is a solution to (52). ☐

Theorem 20. *In addition to the assumptions of Theorem 19, suppose that*

(vi) *for all* $(x, y) \in \mathbb{E} \times \mathbb{E}$, *there exists* $z \in \mathbb{E}$ *such that* $(x, z) \in \Gamma_{\mathbb{E}}$ *and* $(z, y) \in \Gamma_{\mathbb{E}}$.

Then (42) *has one and only one solution in* $\mathbb{E}$.

Proof. It follows immediately from Theorem 15. ☐

Conflict of Interests

The authors declare that there is no competing/conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgment

This work is supported by the Research Center, College of Science, King Saud University.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] M. Edelstein, "An extension of Banach's contraction principle," *Proceedings of the American Mathematical Society*, vol. 12, pp. 7–10, 1961.

[3] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, "Fixed points for mappings satisfying cyclical contractive conditions," *Fixed Point Theory*, vol. 4, no. 1, pp. 79–89, 2003.

[4] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[5] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[6] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[7] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

[8] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[9] A. Petruşel and I. A. Rus, "Fixed point theorems in ordered L-spaces," *Proceedings of the American Mathematical Society*, vol. 134, no. 2, pp. 411–418, 2006.

[10] P. Salimi, C. Vetro, and P. Vetro, "Fixed point theorems for twisted (alpha, beta)-psi-contractive type mappings and applications," *Filomat*, vol. 27, no. 4, pp. 605–615, 2013.

[11] P. Salimi, C. Vetro, and P. Vetro, "Some new fixed point results in non-archimedean fuzzy metric spaces," *Nonlinear Analysis. Modelling and Control*, vol. 18, no. 3, pp. 344–358, 2013.

[12] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[13] P. Amiri, S. Rezapour, and N. Shahzad, "Fixed points of generalized α-ψ-contractions," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales*, vol. 105, no. 1, 2011.

[14] J. H. Asl, S. Rezapour, and N. Shahzad, "On fixed points of α-ψ-contractive multifunctions," *Fixed Point Theory and Applications*, vol. 2012, article 212, 2012.

[15] E. Karapınar and B. Samet, "Generalized α-ψ-contractive type mappings and related fixed point theorems with applications," *Abstract and Applied Analysis*, vol. 2012, Article ID 793486, 17 pages, 2012.

[16] P. Kumam, C. Vetro, and F. Vetro, "Fixed points for weak α-ψ-contractions in partial metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 986028, 9 pages, 2013.

[17] J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass, USA, 1966.