Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 726701, 10 pages
http://dx.doi.org/10.1155/2013/726701

Research Article

Reducing Chaos and Bifurcations in Newton-Type Methods

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Received 15 October 2012; Revised 25 February 2013; Accepted 16 April 2013

Academic Editor: Shukai Duan

We study the dynamics of some Newton-type iterative methods when they are applied of polynomials degrees two and three. The methods are free of high-order derivatives which are the main limitation of the classical high-order iterative schemes. The iterative schemes consist of several steps of damped Newton's method with the same derivative. We introduce a damping factor in order to reduce the *bad* zones of convergence. The conclusion is that the damped schemes become real alternative to the classical Newton-type method since both chaos and bifurcations of the original schemes are reduced. Therefore, the new schemes can be utilized to obtain good starting points for the original schemes.

1. Introduction

One of the most important and usual problems in numerical analysis is finding the solutions of a nonlinear equation:

$$f(x) = 0, \tag{1}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^r function, with $r \geq 1$. To approximate the solutions of these equations, iterative methods can be used. There are many iterative methods with different properties but the most commonly used is Newton's iterative method:

$$N_f(x) = x - \frac{f(x)}{f'(x)}. \tag{2}$$

This method and its variants have been studied by many authors since it was proposed by Newton in 1669. It is well known that under certain assumptions related to the starting point x_0 and to the function f the method provides a sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges quadratically to a solution of (1).

The classical third-order schemes, such as Halley or Chebyshev methods, evaluate second-order Fréchet derivatives. These evaluations are very time consuming for systems of equations. Indeed, let us observe that, for a nonlinear system of m equations and m unknowns, the second Fréchet derivative has m^3 entries. In particular, these methods are hardly used in practice. For that reason, we are interested in the study of methods that increase the order but evaluating only first-order Fréchet derivatives. The methods consist of two (or more) steps of the Newton method having the same derivative, other main advantage of these methods relays on the fact that if we consider a system of equations only one *LU* decomposition is necessary in each iteration. Because of these two properties the schemes are considered a real alternative to the classical Newton method.

On the other hand, we can find a rigorous and useful procedure to find the best scheme in families of Newton-type methods with frozen derivatives in [1]. For most of the cases, the most efficient candidate of the family is the two-step iterative method. We will study all this family but with special attention to the two-step scheme.

In this paper, our main purpose is to study the dynamics of the discrete system defined by the methods but introducing a damping factor $\lambda \in (0, 1]$. We are interested to search chaos or sensitive dependence on initial conditions for this discrete dynamical system. In [2], Hurley and Martin showed that the classical Newton iterative method exhibits chaos for a large class of functions (see also [3]). The introduction of the damping factor not only reduces the order of convergence but also reduces the chaos and the bifurcations as we will see in Sections 3 and 4. In particular, we can utilize these schemes as starting points for the classical Newton schemes. We conclude this paper by comparing the damped method with the original one and the successive damped high-order methods that come from using more Newton's steps.

The study of the dynamics of the iterative methods attracts the attention of many groups [4–8].

The paper is organized as follows. In Section 2 we introduce our new scheme and present the associated scaling theorem. The scaling theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations of general maps, to the study of specific families of iterations of simpler maps. The dynamics for polynomials of degree-two and three, having chaotic dynamics in some cases, are presented in Sections 3 and 4, respectively. In these sections, we observe that the inclusion of the damping factor reduces the bad zone of convergence. In Section 5 we give the main conclusions drawn from the study and we summarize the benefits to include the damping factor. Finally, we propose the use of damped methods in order to find good starting points for the original methods.

2. A Damped Newton-Type Iterative Method

Our first goal is to analyze the dynamics of a modification of the following third-order iterative root-finding method [9]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}. \tag{3}$$

This iterative root-finding algorithm is defined by the following iterative function:

$$M_f(x) = N_f(x) - \frac{f(N_f(x))}{f'(x)}, \tag{4}$$

where $N_f(x) = x - u_f(x)$ and $u_f(x) = f(x)/f'(x)$. In other words, $x_{n+1} = M_f(x_n)$. The dynamics of this method were studied in [10]. Now, we introduce a damping factor $\lambda \in (0,1]$ and we analyze its influence on the dynamics in the iterative method (4). By introducing the damping factor the root-finding algorithm has the following form:

$$y_n = x_n - \lambda\frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = y_n - \lambda\frac{f(y_n)}{f'(x_n)}. \tag{5}$$

Thus the following iterative function describes the root-finding algorithm:

$$M_{\lambda,f}(x) = N_{\lambda,f}(x) - \lambda\frac{f(N_{\lambda,f}(x))}{f'(x)}, \quad 0 < \lambda \le 1, \tag{6}$$

where $N_{\lambda,f}(x) = x - \lambda f(x)/f'(x)$.

Notice that it is easy to proof that the roots of the function f are fixed points of $M_{\lambda,f}$. Moreover, we have that

$$\begin{aligned} M'_{\lambda,f}(x) = {} & 1 - \lambda\frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2} \\ & - \lambda\Bigg(\Bigg(f'\left(x - \lambda\frac{f(x)}{f'(x)}\right) \\ & \times\left(1 - \lambda\frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2}\right) f'(x) \\ & - f\left(x - \lambda\frac{f(x)}{f'(x)}\right) f''(x)\Bigg) \\ & \times\left(f'(x)^2\right)^{-1}\Bigg). \end{aligned} \tag{7}$$

In particular, if α is a simple root of f, that is, $f(\alpha) = 0$ and $f'(\alpha) \ne 0$, then $M'_{\lambda,f}(\alpha) = (1-\lambda)^2$. Thus every simple root of f is an attracting point of $M_{\lambda,f}$ and for $\lambda = 1$ a superattracting fixed point. If β is a multiple root of multiplicity m of f, then it is an attracting fixed point for every value of λ. There are fixed points of $M_{\lambda,f}(x)$ different from the roots. These points are called *extraneous fixed points*. In [10], Amat et al. showed that this method for $\lambda = 1$ (without damping factor) presents chaos and bifurcations when it is applied to a one-parameter family of cubic polynomials. Chaos behavior are related to the existence of points x^* such that $f'(x^*) = 0$ and $f(x^*) \ne 0$.

When we apply the iterative method $M_{\lambda,f}$ to a polynomial, we may have some problems, since we obtain a rational map, say $M_{\lambda,p}(x) = P(x)/Q(x)$, where P and Q, are polynomials, that we may suppose without common factors. The difficulty arises at those points where the evaluation of numerator is nonzero and the denominator is zero, one such points correspond to the poles of the iterative method.

In order to study the dynamics of $M_{\lambda,f}$ by means of graphical analysis we consider it as a map $\widetilde{M}_{\lambda,f}$ on $[0,1[$. For this, let $G : \mathbb{R} \to]0,1[$ be given by $G(x) = (1/\pi)\arctan(x) + 1/2$. This map is an homeomorphism from $\mathbb{R}$ into $]0,1[$. Define the maps $\widetilde{M}_{\lambda,f} :]0,1[\to]0,1[$ by $\widetilde{M}_{\lambda,f} = (G \circ M_{\lambda,f} \circ G^{-1})(x)$, where $M_{\lambda,f}$ is the iterative root-finding method introduced above for a map $f : \mathbb{R} \to \mathbb{R}$. We may extend $\widetilde{M}_{\lambda,f}$ to maps from $[0,1]$ into itself. We use the same notation for this extension. Note that the extended function has fixed points at $x = 0$ and at $x = 1$ and that these fixed points are repelling.

Now we have the following useful result.

Theorem 1 (the scaling theorem). *Let $f(x)$ be an analytic function, and let $T(x) = \alpha x + \beta$, with $\alpha \ne 0$, be an affine map. Let $g(x) = (f \circ T)(x)$. Then $T \circ M_{\lambda,g} \circ T^{-1}(x) = M_{\lambda,f}(x)$; that is, $M_{\lambda,g}$ and $M_{\lambda,f}$ are affine conjugated by T.*

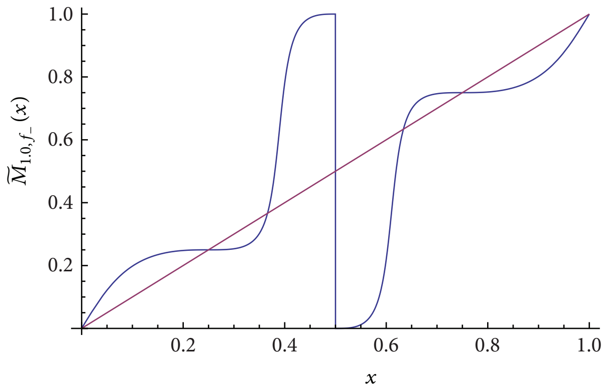

FIGURE 1: $\widetilde{M}_{1.0,f_-}$ as a circle map.

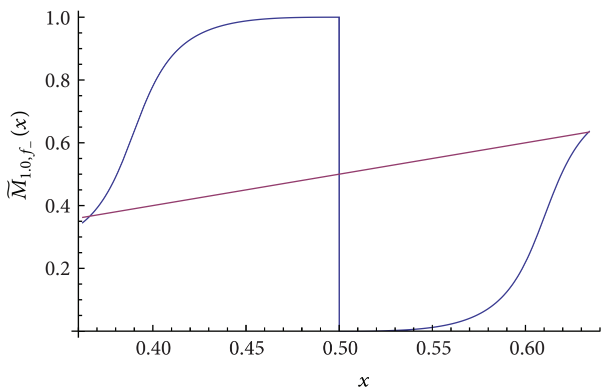

FIGURE 2: $\widetilde{M}_{1.0,f_-}$ restricted to the interval I.

Proof. We have

$$M_{\lambda,g}\left(T^{-1}(x)\right) = T^{-1}(x) - \lambda u_g\left(T^{-1}(x)\right) - \lambda \frac{g\left(T^{-1}(x) - \lambda u_g\left(T^{-1}(x)\right)\right)}{g'\left(T^{-1}(x)\right)}. \tag{8}$$

On the other hand, since $g\circ T^{-1}(x) = f(x)$ and $(g\circ T^{-1})'(x) = (1/\alpha)g'(T^{-1}(x))$, then $g'(T^{-1}(x)) = \alpha(g\circ T^{-1})'(x) = \alpha f'(x)$, and by an easy induction process it follows that $g^{(k)}(T^{-1}(x)) = \alpha^k f^{(k)}(x)$. Hence, $u_g(T^{-1}(x)) = (1/\alpha)u_f(x)$.

Substituting these quantities in $M_{\lambda,g}(T^{-1}(x))$ we obtain

$$\begin{aligned}
T\circ M_{\lambda,g}\circ T^{-1}(x) &= T\left(M_{\lambda,g}\left(T^{-1}(x)\right)\right) \\
&= \alpha M_{\lambda,g}\left(T^{-1}(x)\right) + \beta \\
&= \alpha T^{-1}(x) - \alpha\lambda u_g\left(T^{-1}(x)\right) - \alpha\lambda \frac{g\left(T^{-1}(x) - \lambda u_g\left(T^{-1}(x)\right)\right)}{g'\left(T^{-1}(x)\right)} + \beta \\
&= \alpha T^{-1}(x) - \lambda u_f(x) - \lambda \frac{g\left(T^{-1}(x) - (\lambda/\alpha)u_f(x)\right)}{f'(x)} + \beta \\
&= x - \lambda u_f(x) - \lambda \frac{g\left(T^{-1}(x) - (\lambda/\alpha)u_f(x)\right)}{f'(x)}.
\end{aligned} \tag{9}$$

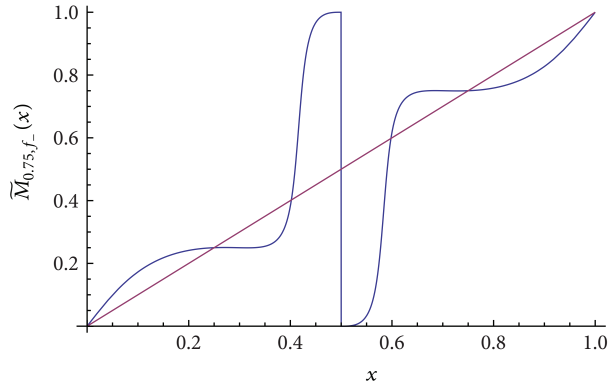

FIGURE 3: Graphics of $\widetilde{M}_{\lambda,f_-}$ with $\lambda = 0.75$, where the BZ interval is $I_{0.75} = [0.401996\cdots, 0.598004\cdots]$.

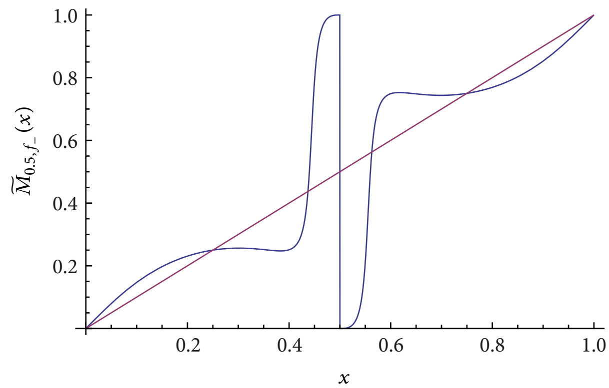

FIGURE 4: Graphics of $\widetilde{M}_{\lambda,f_-}$ with $\lambda = 0.5$, where the BZ interval is $I_{0.5} = [0.437167\cdots, 0.562833\cdots]$.

Finally, by a comparison of Taylor series expansions of f and g, we obtain

$$\begin{aligned}
g\left(T^{-1}(x) - \frac{\lambda}{\alpha}u_f(x)\right) &= g\left(T^{-1}(x)\right) - \frac{\lambda}{\alpha}g'\left(T^{-1}(x)u_f(x)\right)\cdots \\
&= f(x) - \lambda\alpha f'(x)\frac{1}{\alpha}u_f(x) + \cdots \\
&= f\left(x - \lambda uf(x)\right).
\end{aligned} \tag{10}$$

Therefore, $T\circ M_{\lambda,g}\circ T^{-1}(x) = M_{\lambda,f}(x)$.

This ends the proof. □

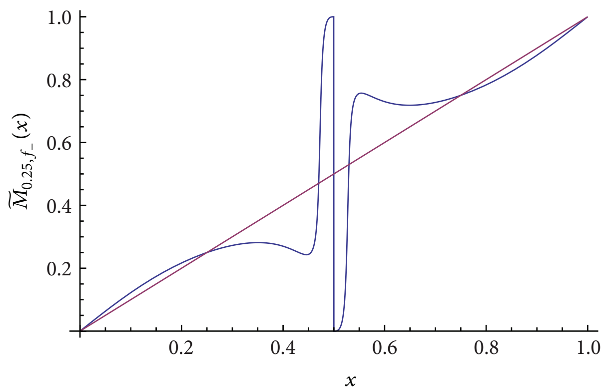

FIGURE 5: Graphics of $\widetilde{M}_{\lambda,f_-}$ with $\lambda = 0.25$, where the BZ interval is $I_{0.25} = [0.470144\cdots, 0.529856\cdots]$.

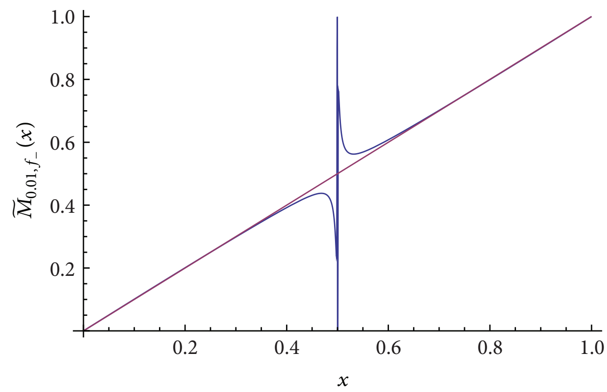

FIGURE 6: Graphics of $\widetilde{M}_{\lambda,f_-}$ with $\lambda = 0.01$, where the BZ interval is $I_{0.01} = [0.498872\cdots, 0.501128\cdots]$.

The theorem remains valid for any nonzero constant c such that $g(x) = c(f \circ T)(x)$.

The Scaling Theorem allows up to suitable change of coordinates, to reduce the study of the dynamics of iterations $M_{\lambda,f}$, to the study of specific families of iterations of simpler maps. For example, each quadratic polynomial $f(x) = ax^2 + bx + c$, with $a \neq 0$, that we suppose that is monic, by an affine change of variables reduces to one of the following polynomials:

(i) $f_-(x) = x^2 - 1$,

(ii) $f_+(x) = x^2 + 1$,

(iii) $f_0(x) = x^2$,

depending on the number of real roots of $f(x) = 0$ (two, one (double), or not real roots). Therefore the study of the dynamics of $M_{\lambda,f}$, when it is applied to a quadratic polynomial, reduces to the study of the dynamics of M_{λ,f_s} where $s = +, 0, -$, up to affine change of variables. On the other hand, the study of the dynamics of the iterative method $\widetilde{M}_{\lambda,f}$ reduces to the study of the dynamics of the interval map $\widetilde{M}_{\lambda,f_s}$. Similarly, any cubic polynomial reduces to one of the simplest cubic polynomials:

(I) $f_-(x) = x^3 - x$,

(II) $f_+(x) = x^3 + x$,

(III) $f_0(x) = x^3$,

(IV) $f_\gamma(x) = x^3 + \gamma x + 1$.

The last one is an appropriate rescaling that puts $M_{\lambda,f}$ inside the conjugacy class.

3. Quadratic Polynomials

Following the analysis of Section 2, we have to study three cases.

Case 1 ($f_0(x) = x^2$). In this case form (6) we have

$$M_{\lambda,f_0}(x) = -\frac{1}{8}x(-2+\lambda)(4+(-2+\lambda)\lambda). \tag{11}$$

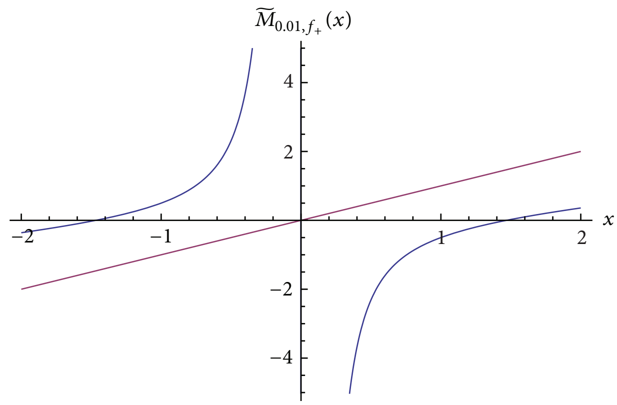

FIGURE 7: Graphics of $M_{1.0,f_+}$.

This function for $\lambda \in (0, 1]$ is a linear contraction. The unique fixed point are $x = 0$, which is a global attractor, but not super-attracting. Therefore, its dynamics is trivial.

Case 2 ($f_-(x) = x^2 - 1$). For $f_-(x) = x^2 - 1$ from (6) and (7) we have that

$$\begin{aligned} M_{\lambda,f_-}(x) &= -\Big(\big(\lambda^3 + x^4(-2+\lambda)(4+(-2+\lambda)\lambda) \\ &\quad -2x^2\lambda(4+(-2+\lambda)\lambda)\big) \times \big(8x^3\big)^{-1}\Big), \\ M'_{\lambda,f_-}(x) &= \Big(-\big(-3\lambda^3 + x^4(-2+\lambda)(4+(-2+\lambda)\lambda) \\ &\quad +2x^2\lambda(4+(-2+\lambda)\lambda)\big) \times \big(8x^4\big)^{-1}\Big). \end{aligned} \tag{12}$$

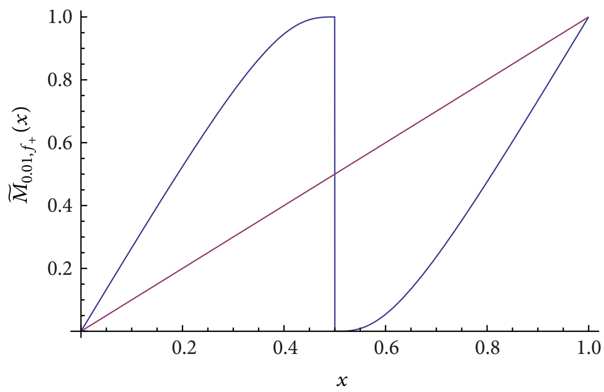

FIGURE 8: Graphics of $\widetilde{M}_{1.0,f_+}$ as a circle map.

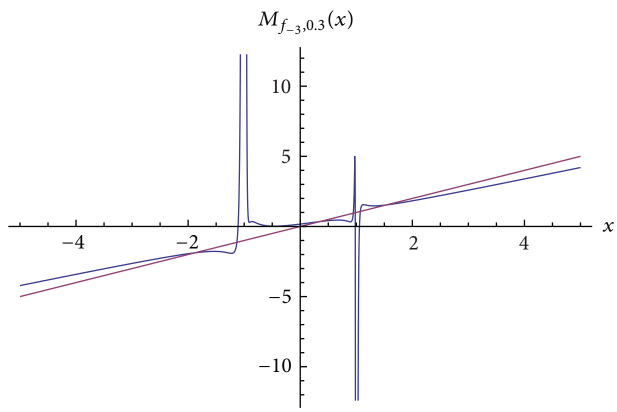

FIGURE 10: $M_{f_{-3},\lambda}$ for $\lambda = 0.3$.

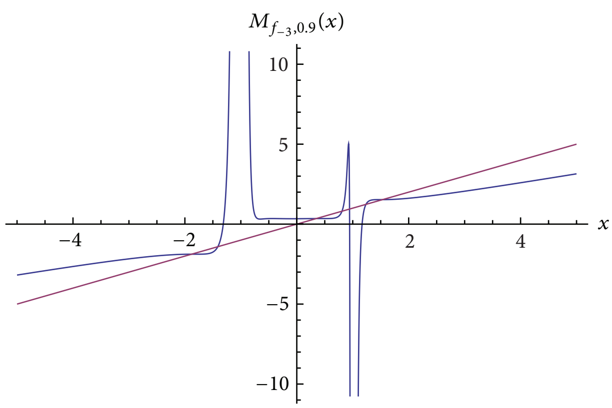

FIGURE 9: $M_{f_{-3},\lambda}$ for $\lambda = 0.9$.

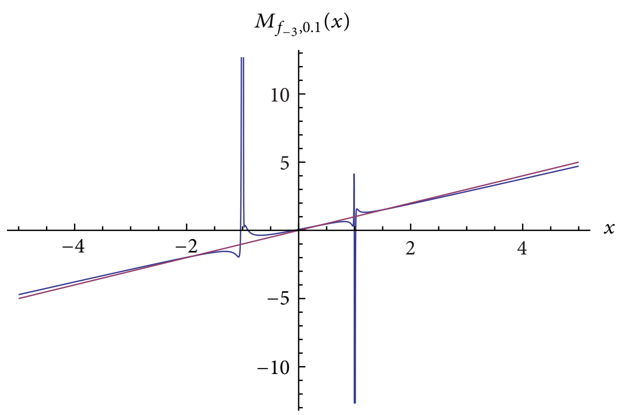

FIGURE 11: $M_{f_{-3},\lambda}$ for $\lambda = 0.1$.

Now the fixed points of $M_{\lambda,f_-}(x)$ are $x_{1,2} = \pm 1$ and $x_{3,4} = \pm\lambda/\sqrt{8-4\lambda+\lambda^2}$. The points $x_{1,2}$ are the roots of f_-, and we have that $M'_{\lambda,f_-}(x_{1,2}) = (\lambda-1)^2$; thus $x_{1,2}$ are attracting fixed points for $\lambda \in (0,1)$, and superattracting for $\lambda = 1$. On the other hand, we have that $M'_{\lambda,f_-}(x_{3,4}) = -15 + 16/\lambda + 6\lambda - \lambda^2$. Thus, the extraneous fixed points $x_{3,4}$ are repelling. Note that $M_{\lambda,f_-}(x)$ has a vertical asymptote at $x = 0$.

Let $\lambda = 1$ and $I = [p_1, p_2]$ be the interval determined by the repelling fixed points $p_1 = G(x_3) = 0.36614\ldots$ and $p_2 = G(x_4) = 0.63386\ldots$ of $\widetilde{M}_{1.0,f_-}$. Figure 1 shows $\widetilde{M}_{1.0,f_-}$ and Figure 2 shows a zoom of the restriction of the function to the interval I.

It is clear that $\widetilde{M}_{1.0,f_-}$ restricted to the interval I has an invariant Cantor set Δ_- of zero Lebesgue measure of nonescaping points; that is, Δ_- is the set of points whose orbits remain in the interval I under iteration by $\widetilde{M}_{1.0,f_-}$. In other words, these points are not attracted to one of the superattracting fixed points of $\widetilde{M}_{1.0,f_-}$ (note that when $\lambda = 1$ the fixed points are superattracting).

From the above analysis, we have that the orbit of any point $x \in [0,1] \setminus (\Delta_- \cup p_1, p_2)$ is attracted to one of the superattracting fixed points of $\widetilde{M}_{1.0,f_-}$. Consequently, the orbit of any point in $\mathbb{R} \setminus (G^{-1}(\Delta_-) \cup \{x_3, x_4\})$ is attracted to one of the fixed points of $M_{1.0,f_-}$.

Thus we can define the "bad zone" as $BZ = \Delta_- \cup \{p_1, p_2\}$ since if $x \in BZ$, then it is not attracted by any of the attracting points of $\widetilde{M}_{\lambda,f_-}$.

Lemma 2. *The "bad zone" of the damping two-step Newton-like method defined in* (6) *applied to quadratic polynomials with two different roots decreases with the damping parameter* λ.

Proof. We begin by calculating

$$p_1 = G(x_3) = \frac{1}{2} - \frac{\arctan\left[\lambda/\sqrt{8+(-4+\lambda)\lambda}\right]}{\pi},$$
$$p_2 = G(x_4) = \frac{1}{2} + \frac{\arctan\left[\lambda/\sqrt{8+(-4+\lambda)\lambda}\right]}{\pi}. \tag{13}$$

So we have that

$$I_\lambda = \left[\frac{1}{2} - \frac{\arctan\left[\lambda/\sqrt{8+(-4+\lambda)\lambda}\right]}{\pi}, \frac{1}{2} + \frac{\arctan\left[\lambda/\sqrt{8+(-4+\lambda)\lambda}\right]}{\pi}\right]. \tag{14}$$

Thus I is an interval centered in 0.5 and width $2(\arctan(\lambda/\sqrt{8+(-4+\lambda)\lambda})/\pi) = 2a$. It is easy to see that

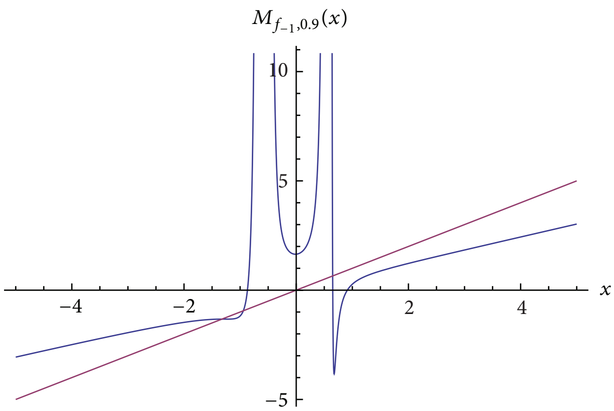

FIGURE 12: $M_{f_{-1},\lambda}$ for $\lambda = 0.9$.

Δ_- decreases when a decreases. Taking into account that the functions $(\lambda/\sqrt{8+(-4+\lambda)\lambda})(x)$ and $\arctan(x)$ are increasing functions, it follows that a decreases as λ so with smaller λ we have smaller BZ. □

Thus we have seen analytically and graphically, in Figures 3, 4, 5, and 6, that the size of $(\Delta_- \cup \{p_1, p_2\})$ decreases with the damping factor λ and as a consequence we can state that the chaotic zone decreases with λ.

Case 3 ($f_+(x) = x^2 + 1$). In this case, the equation f_+ has no real roots, thus $M_{\lambda,f_+}(x)$ has not real fixed points, and its dynamics is chaotic. Figures 7 and 8 illustrate this behavior.

4. Cubic Polynomials

In this section we consider cubic polynomials. As we have seen in Section 2, by an affine change of coordinates, every cubic polynomial f reduces to one of the simplest f_0, f_-, f_+ or to a member of the one-parameter family of cubic polynomials f_γ. Therefore, we analyze the dynamics of these simpler cubic polynomials.

Case 4 ($f_+(x) = x^3 + x$). In this case, f has only one real root at $x = 0$, and the method (6) has the following form:

$$M_{f_+,\lambda} = x - \frac{(x+x^3)\lambda}{1+3x^2} - \left(\lambda\left(x - \frac{(x+x^3)\lambda}{1+3x^2} + \left(x - \frac{(x+x^3)\lambda}{1+3x^2}\right)^3\right) \times \left(1+3x^2\right)^{-1}\right). \quad (15)$$

Since $f_+'(x) = 1+3x^2 > 0$ for all x, it follows that f_+ has no critical points; hence, $M_{f_+,\lambda}(x)$ is a global homeomorphism from $\mathbb{R}$ into itself and its dynamics is trivial.

Case 5 ($f_-(x) = x^3 - x$). In this case f has 3 real roots, $x = 0$, $x = 1$ and $x = -1$. $M_{f_-,\lambda}(x)$ is given by (6)

$$M_{f_-,\lambda}(x) = x - \frac{x(-1+x^2)\lambda}{-1+3x^2} - \left(\lambda\left(-x + \frac{x(-1+x^2)\lambda}{-1+3x^2} + \left(x - \frac{x(-1+x^2)\lambda}{-1+3x^2}\right)^3\right)\left(-1+3x^2\right)^{-1}\right). \quad (16)$$

Notice that $M_{f_-,\lambda}(x)$ has two asymptotes at the points $a_1 = \sqrt{3}/3$ and $a_2 = -\sqrt{3}/3$. We have that when $x \to a_1$, then $M_{f_-,\lambda}(x) \to -\infty$, and when $x \to a_2$, then $M_{f_-,\lambda}(x) \to \infty$. If we now calculate the fixed points of $M_{f_-,\lambda}(x)$, there exist 9 fixed points which are the three roots of $f(x_1 = -1, x_2 = 0, x_3 = 1)$ and the following extraneous fixed points:

$$\begin{aligned}
x_4 &= -\sqrt{\frac{3}{9-3\lambda+\lambda^2} - \frac{\lambda}{2(9-3\lambda+\lambda^2)} + \frac{\lambda^2}{2(9-3\lambda+\lambda^2)} - \frac{\lambda\sqrt{9-2\lambda+\lambda^2}}{2(9-3\lambda+\lambda^2)}},\\
x_5 &= \sqrt{\frac{3}{9-3\lambda+\lambda^2} - \frac{\lambda}{2(9-3\lambda+\lambda^2)} + \frac{\lambda^2}{2(9-3\lambda+\lambda^2)} - \frac{\lambda\sqrt{9-2\lambda+\lambda^2}}{2(9-3\lambda+\lambda^2)}},\\
x_6 &= -\sqrt{\frac{3}{9-3\lambda+\lambda^2} - \frac{\lambda}{2(9-3\lambda+\lambda^2)} + \frac{\lambda^2}{2(9-3\lambda+\lambda^2)} + \frac{\lambda\sqrt{9-2\lambda+\lambda^2}}{2(9-3\lambda+\lambda^2)}},\\
x_7 &= \sqrt{\frac{3}{9-3\lambda+\lambda^2} - \frac{\lambda}{2(9-3\lambda+\lambda^2)} + \frac{\lambda^2}{2(9-3\lambda+\lambda^2)} + \frac{\lambda\sqrt{9-2\lambda+\lambda^2}}{2(9-3\lambda+\lambda^2)}},
\end{aligned} \quad (17)$$

where $x_8 = -\sqrt{(-2+\lambda)/(-6+\lambda)}$ and $x_9 = \sqrt{(-2+\lambda)/(-6+\lambda)}$. Evaluating all these points in $M'_{f_-,\lambda}$ (7), we obtain that the first 3 ones (the roots of f) are attracting fixed points for every $\lambda \neq 1$ and superattracting for $\lambda = 1$. The

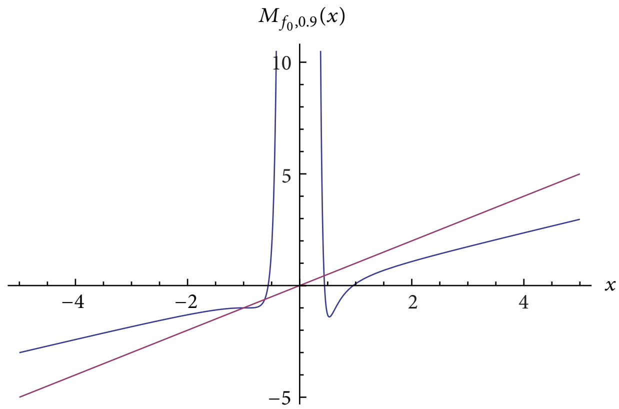

Figure 13: $M_{f_0,\lambda}$ for $\lambda = 0.9$.

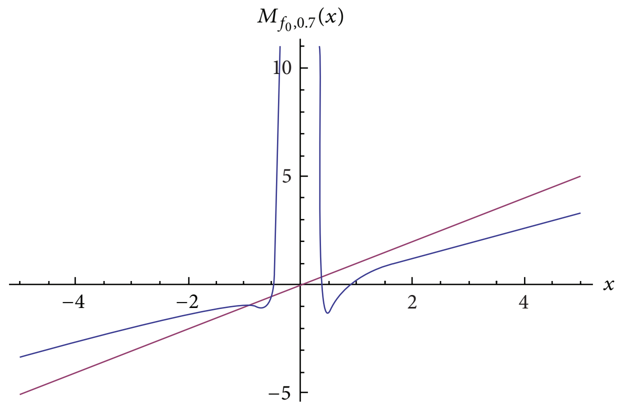

Figure 14: $M_{f_0,\lambda}$ for $\lambda = 0.7$.

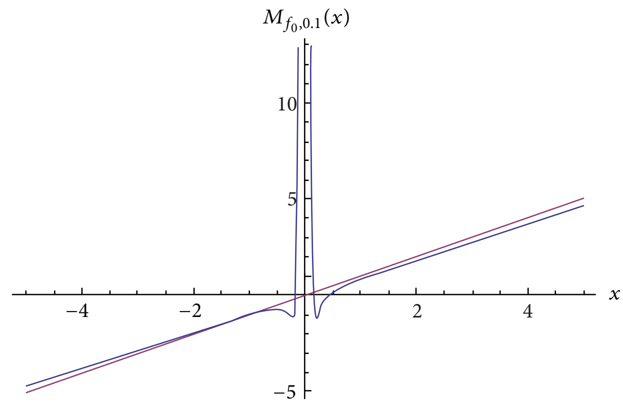

Figure 15: $M_{f_0,\lambda}$ for $\lambda = 0.1$.

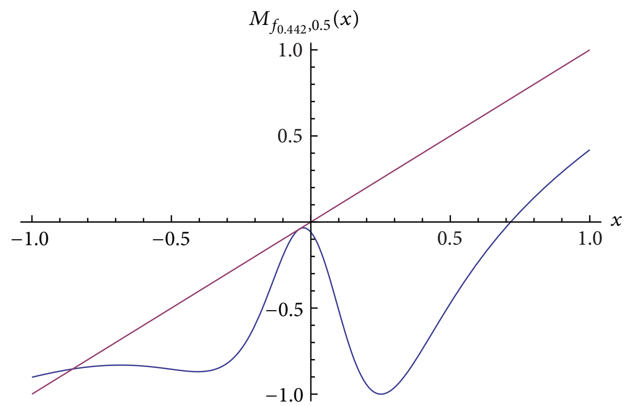

Figure 16: $M_{f_\gamma,\lambda}$ for $\lambda = 0.5$ and $\gamma = \gamma_{0.5} = 0.442$.

extraneous fixed points are repelling ones since $M'_{f_-,\lambda}(x_{1,2,3}) = (-1+\lambda)^2$:

$$\begin{aligned} &M'_{f_-,\lambda}(x_{4,5}) \\ &\quad = -\left(9\left(-3+\sqrt{9+(-2+\lambda)\lambda}\right)\right. \\ &\qquad +\lambda\left(47+\lambda\left(-13+5\lambda\right.\right. \\ &\qquad\qquad \left.\left.\left.+3\sqrt{9+(-2+\lambda)\lambda}\right)\right)\right)(4\lambda)^{-1}, \\ &M'_{f_-,\lambda}(x_{6,7}) \\ &\quad = \left(9\left(3+\sqrt{9+(-2+\lambda)\lambda}\right)\right. \\ &\qquad +\lambda\left(-47+\lambda\left(13-5\lambda\right.\right. \\ &\qquad\qquad \left.\left.\left.+3\sqrt{9+(-2+\lambda)\lambda}\right)\right)\right)(4\lambda)^{-1}, \\ &M'_{f_-,\lambda}(x_{8,9}) = 13+(-8+\lambda)\lambda. \end{aligned} \tag{18}$$

In particular, the dynamics is trivial.

Case 6 ($f_0(x) = x^3$). The function f_0 only has one triple root at $x = 0$. We have that (6) has the following form:

$$M_{f_0,\lambda} = \frac{1}{81}x(-3+\lambda)\left(-27+(-3+\lambda)^2\lambda\right), \tag{19}$$

which is a linear contraction for every value of $\lambda \in (0,1]$. The only fixed point is attracting, but not superattracting and the dynamics of $M_{f_0,\lambda}$ is trivial.

Case 7 ($f_\gamma(x) = x^3 + \gamma x + 1$). We have from (6) that

$$\begin{aligned} M_{f_\gamma,\lambda}(x) &= x - \frac{\left(1+x^3+x\gamma\right)\lambda}{3x^2+\gamma} \\ &\quad - \left(\lambda\left(1+\gamma\left(x-\frac{\left(1+x^3+x\gamma\right)\lambda}{3x^2+\gamma}\right)\right.\right. \\ &\qquad \left.+\left(x-\frac{\left(1+x^3+x\gamma\right)\lambda}{3x^2+\gamma}\right)^3\right) \\ &\qquad \left.\times\left(3x^2+\gamma\right)^{-1}\right). \end{aligned} \tag{20}$$

We are going to analyze in this section the dynamics of $M_{f_\gamma,\lambda}(x)$ depending on the two parameters γ and λ. Our main purpose is to compare the results by fixing the parameter γ for different values of the damping parameter λ and to study how it changes the dynamics.

Table 1: Approximations of γ_λ for some values of λ.

λ	γ_λ
0.01	0.00795
0.1	0.081
0.2	0.1657
0.3	0.254
0.4	0.346
0.5	0.442
0.6	0.543
0.7	0.649
0.8	0.762
0.9	0.881
1.0	1.00768

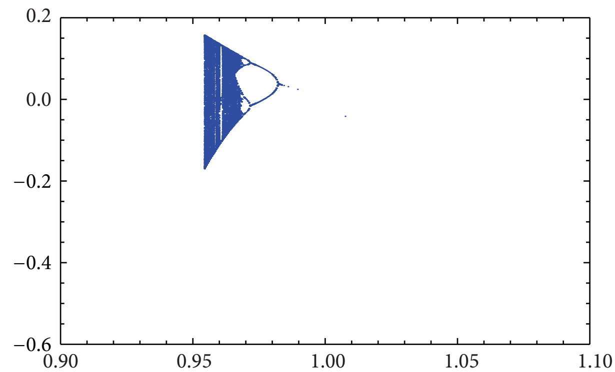

Figure 17: Feigembaun diagram for $\lambda = 1.0$.

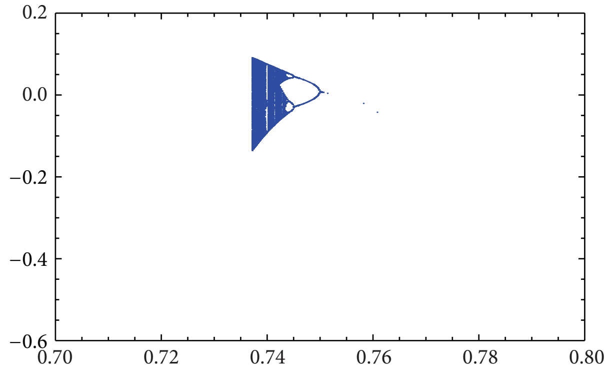

Figure 18: Feigembaun diagram for $\lambda = 0.8$.

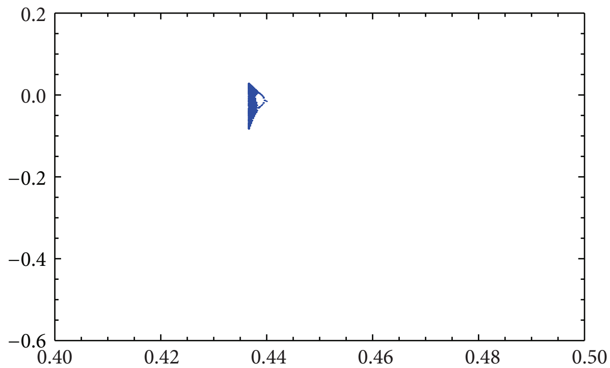

Figure 19: Feigembaun diagram for $\lambda = 0.5$.

We begin by studying the case $\gamma < 0$. In this case, our function f_γ has two critical points: $\gamma_{+,-} = \pm\sqrt{-\gamma/3}$. Let be $\gamma_* = -3/\sqrt[3]{4} \approx -1.88988$. Notice that f_{γ_*} has a double root at the positive critical point γ_+. Hence, this value of the parameter will be very significant in the study of the dynamics.

We have three different possibilities.

(i) If $\gamma < \gamma_*$, the polynomial f_γ has three real roots.

(ii) If $\gamma = \gamma_*$, the polynomial f_γ has one positive double real root and a simple one which is negative.

(iii) If $\gamma > \gamma_*$, the polynomial f_γ has only one real root.

For $\gamma < \gamma_*$ the iterative method $M_{f_\gamma,\lambda}(x)$ has two vertical asymptotes. $M_{f_\gamma,\lambda}(x)$ has 7 fixed points, the 3 real roots of the polynomial (attracting fixed points), and the extraneous fixed points (repelling). See Figure 9 as an example. As we can observe in Figures 10, and 11 if the damping factor decreases, then extraneous fixed points tend to be closer to the asymptotes.

For $\gamma_* \le \gamma < 0$ the iterative method $M_{f_\gamma,\lambda}(x)$ has two vertical asymptotes at γ_- and γ_+, as we can see in Figure 12. Notice that $\lim_{x\to\gamma_-^\pm} M_{f_\gamma,0}(x) = \infty$ and $\lim_{x\to\gamma_+^\pm} M_{f_\gamma,0}(x) = \infty$. $M_{f_\gamma,\lambda}(x)$ has 3 fixed points, the one which corresponds to the real solution of the polynomial f_γ is attracting and two repelling extraneous fixed points. Again we obtain that when the damping factor decreases the chaotic zone also decreases.

Now we are going to consider the simplest case which corresponds to $\gamma = 0$, and we have

$$\begin{aligned} M_{f_0,\lambda}(x) &= \left(\left(3x^3(-3+\lambda)\lambda^3 + \lambda^4 + 3x^6\lambda\left(-18+9\lambda-6\lambda^2+\lambda^3\right)\right.\right. \\ &\quad \left.\left. + x^9\left(81-54\lambda+27\lambda^2-9\lambda^3+\lambda^4\right)\right) \right. \\ &\quad \left. \times\left(81x^8\right)^{-1}\right). \end{aligned} \tag{21}$$

This map has one attracting point at $x_1 = -1$ which corresponds to the root of f_0. Moreover it has another two extraneous fixed points which are the real solutions of the equation:

$$\lambda^3 + \left(-9\lambda^2 + 2\lambda^3\right)x^3 + \left(-54 + 27\lambda - 9\lambda^2 + \lambda^3\right)x^6 = 0. \tag{22}$$

As we see in Figure 13 both extraneous fixed points are repelling.

Now if we focus our attention on the damping factor, we obtain that when λ decreases, the extraneous fixed points tend to be closer to the asymptote $x = 0$. This behavior can be

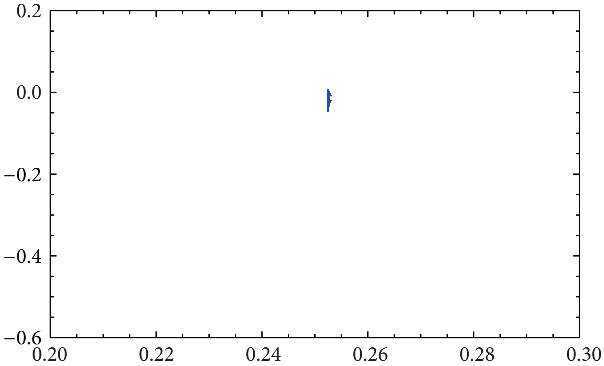

FIGURE 20: Feigembaun diagram for $\lambda = 0.3$.

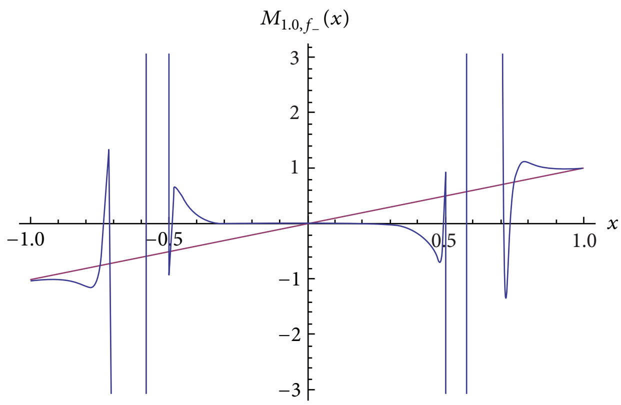

FIGURE 21: $M_{f_-,\lambda}$ for $\lambda = 1.0$ and $f_-(x) = x^3 - x$ with three Newton steps.

observed in Figures 14 and 15. Notice that $x = 0$ is the critical point of f_0.

The last study is related with $\gamma > 0$. Notice that for, every value of $\gamma \neq 0$, $M_{f_\gamma,\lambda}$ presents a maximum which increases when γ decreases. Moreover there exists a value γ_λ that depends on the value of the damping factor, such that, for each $\gamma > \gamma_\lambda$, $M_{f_\gamma,\lambda}$ has only one fixed point (the root of the function) and the dynamics is trivial. In Table 1, it shows the different approximations of γ_λ for some values of λ. Figure 16 illustrates this case.

When $\gamma = \gamma_\lambda$, $M_{f_\gamma,1}$ presents another fixed point which is a saddle node (as Amat et al. showed in [10]) and when the parameter γ decreases, it appears bifurcations. If we now see Figures 17, 18, 19, and 20, it is clear that the problems decrease with λ; even for small values ($\lambda < 0.2$) they seem they disappear.

Finally, we conclude with the following conjecture: *if* $\lambda = n/2^k$ *with* $n, k \in \mathbb{N}$, *then* $M_{f_\gamma,\lambda}$ *has only three fixed points.*

5. Concluding Remarks

In this work we have analyzed the dynamics of a new damped third-order Newton-like method. We have stated the following conclusions drawn from the study

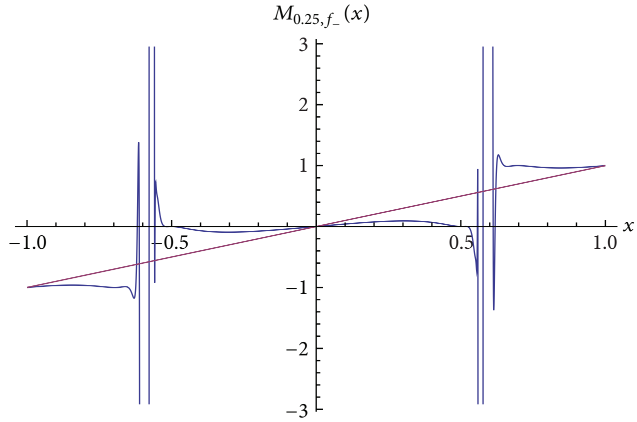

FIGURE 22: $M_{f_-,\lambda}$ for $\lambda = 0.25$ and $f_-(x) = x^3 - x$ with three Newton steps.

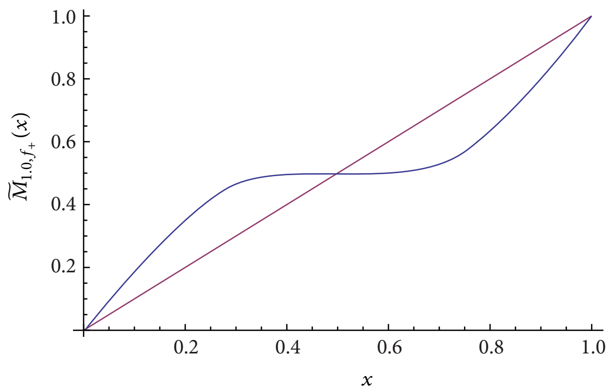

FIGURE 23: $\widetilde{M}_{f_+,\lambda}$ for $\lambda = 1.0$ and $f_+(x) = x^3 + x$ with three Newton steps.

(i) The connected component of the basin of attraction increases when the damping factor decreases.

(ii) The sizes of the chaotic and the bifurcation zones decrease when the damping factor decreases.

(iii) The nonconvergence zone decreases when the damping factor decreases.

In addition, we have performed a similar study including more Newton's steps. The conclusions remain valid. Figures 21, 22, 23, and 24 show examples of the behavior with more Newton's steps. Therefore, we can propose the use of damped methods in order to find good starting points for the original methods. We would like to analyze these hybrid methods in a forthcoming paper. We are also interested to analyze the situation of nonfixed damping factors.

Acknowledgments

The authors would like to thank the referees and the editor of AAA for their suggestions in the first draft of this paper. Research of the first two authors is supported in part by

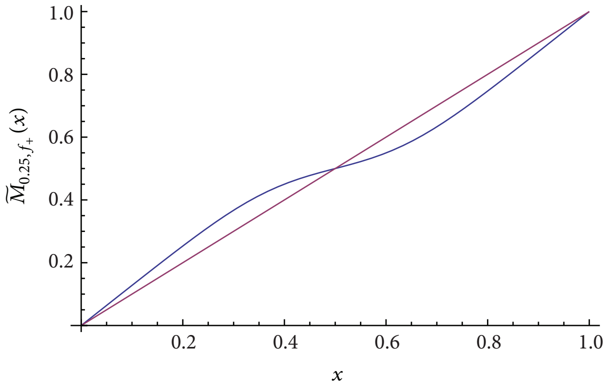

FIGURE 24: $\widetilde{M}_{f_+,\lambda}$ for $\lambda = 0.25$ and $f_+(x) = x^3 + x$ with three Newton steps.

FEDER MTM 2010-17508 and 08662/PI/08. Research of the third author is supported in part by MTM2011-28636-C02-01.

References

[1] S. Amat, S. Busquier, Á. Grau, and M. Grau-Sánchez, "Maximum efficiency for a family of Newton-like methods with frozen derivatives and some applications," *Applied Mathematics and Computation*, vol. 219, no. 15, pp. 7954–7963, 2013.

[2] M. Hurley and C. Martin, "Newton's algorithm and chaotic dynamical systems," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 2, pp. 238–252, 1984.

[3] M. Hurley, "Attracting orbits in Newton's method," *Transactions of the American Mathematical Society*, vol. 297, no. 1, pp. 143–158, 1986.

[4] A. Cordero, J. R. Torregrosa, and P. Vindel, "Dynamics of a family of Chebyshev-Halleytype methods," *Applied Mathematics and Computation*, vol. 219, no. 16, pp. 8568–8583, 2013.

[5] J. M. Gutiérrez, M. A. Hernández, and N. Romero, "Dynamics of a new family of iterative processes for quadratic polynomials," *Journal of Computational and Applied Mathematics*, vol. 233, no. 10, pp. 2688–2695, 2010.

[6] G. Honorato, S. Plaza, and N. Romero, "Dynamics of a higher-order family of iterative methods," *Journal of Complexity*, vol. 27, no. 2, pp. 221–229, 2011.

[7] S. Plaza and N. Romero, "Attracting cycles for the relaxed Newton's method," *Journal of Computational and Applied Mathematics*, vol. 235, no. 10, pp. 3238–3244, 2011.

[8] W. Yang, "Symmetries of the Julia sets of Newton's method for multiple root," *Applied Mathematics and Computation*, vol. 217, no. 6, pp. 2490–2494, 2010.

[9] F. A. Potra and V. Pták, *Nondiscrete Induction and Iterative Processes*, vol. 103 of *Research Notes in Mathematics*, Pitman, Boston, Mass, USA, 1984.

[10] S. Amat, S. Busquier, and S. Plaza, "Chaotic dynamics of a third-order Newton-type method," *Journal of Mathematical Analysis and Applications*, vol. 366, no. 1, pp. 24–32, 2010.