Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 721932, 4 pages
http://dx.doi.org/10.1155/2013/721932

Research Article

Kudriasov Type Univalence Criteria for Some Integral Operators

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Received 30 September 2013; Accepted 4 December 2013

Academic Editor: Mohamed Kamal Aouf

We consider some integral operators defined by analytic functions in the open unit disk and derive new univalence criteria for these operators, using Kudriasov condition for a function to be univalent.

1. Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

normalized by $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in U.

We consider the integral operators

$$H_n(z) = \left\{\beta \int_0^z u^{\beta-1}\left(\frac{f_1(u)}{u}\right)^{\gamma_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\gamma_n} du\right\}^{1/\beta},$$
$$T_n(z) = \left\{\beta \int_0^z u^{\beta-1}\left(f_1'(u)\right)^{\gamma_1} \cdots \left(f_n'(u)\right)^{\gamma_n} du\right\}^{1/\beta}, \quad (2)$$

for β, γ_j being complex numbers, $\beta \neq 0$, $j = \overline{1,n}$, and the functions $f_j \in \mathcal{A}$, $j = \overline{1,n}$.

Some univalence criteria for these integral operators were studied in [1]. Applying univalence conditions given by Kudriasov [2] and Pascu [3], we obtain new Kudriasov type univalence criteria for these two integral operators.

2. Preliminaries

Various generalizations of Becker's univalence criteria for analytic functions given in [4] were obtained by many authors. For example, the result obtained by Pascu in [3] is also known as an improvement of Becker's univalence criteria. This result or other similar generalizations of Becker's univalence criteria have been used further to derive new univalence criteria for integral operators (see, e.g., some relatively recent works as [1, 5, 6]). In this paper we use Pascu improvement of Becker's univalence criteria and also another univalence condition for a function to be univalent, given by Kudriasov in [2]. There are also some papers devoted to univalence criteria that use some Kudriasov type conditions (see, e.g., the work [1] containing a chapter dedicated to Kudriasov type univalence conditions and other papers as, e.g., [7–10]).

The following univalence criteria are given by Kudriasov for a regular function.

Lemma 1 (see [2]). *Let f be a regular function in U, $f(z) = z + a_2 z^2 + \cdots$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq K, \quad z \in U, \quad (3)$$

for all $z \in U$, where $K \cong 3.05$, the function f is univalent in U.

Remark 2. The constant K is a solution of the equation $8[x(x-2)^3]^{1/2} - 3(4-x)^2 = 12$. An approximation of this solution in MATLAB environment is 3.03902118847875. However, we call this constant in our further results like Kudriasov gave it, approximately equal to 3.05.

The improvement of Becker's univalence condition is given by Pascu for integral operators as follows.

Lemma 3 (see [3]). *Let α be a complex number, $\operatorname{Re}\alpha > 0$, and the function $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{4}$$

for all $z \in U$, then for every complex number β, $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1} f'(u)du\right]^{1/\beta} \tag{5}$$

is regular and univalent in U.

3. Main Results

Theorem 4. *Let α, γ_j be complex numbers, $\operatorname{Re}\alpha > 0$, $j = \overline{1,n}$, the functions $f_j \in \mathcal{A}$, $f_j = z + a_{2j}z^2 + \cdots$, $j = \overline{1,n}$, $n \in \mathbb{N}-\{0\}$, and K the positive real number $K \cong 3.05$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le K, \quad z \in U,\ j = \overline{1,n}, \tag{6}$$

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| \le \min\left\{\frac{\operatorname{Re}\alpha}{4}, \frac{1}{4}\right\}, \tag{7}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$, and for every complex number β, $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, the integral operator H_n is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$h_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\gamma_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\gamma_n} du. \tag{8}$$

The function h_n is regular in U and $h_n(0) = h_n'(0) - 1 = 0$. We have

$$\frac{zh_n''(z)}{h_n'(z)} = \gamma_1\left(\frac{zf_1'(z)}{f_1(z)} - 1\right) + \cdots + \gamma_n\left(\frac{zf_n'(z)}{f_n(z)} - 1\right), \tag{9}$$

for all $z \in U$.

From (9), we obtain further

$$\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le |\gamma_1|\left(\left|\frac{zf_1'(z)}{f_1(z)}\right| + 1\right) + \cdots + |\gamma_n|\left(\left|\frac{zf_n'(z)}{f_n(z)}\right| + 1\right). \tag{10}$$

By (6) and Lemma 1, we have $f_j \in \mathcal{S}$, $j = \overline{1,n}$, and hence we obtain

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \le \frac{1+|z|}{1-|z|}, \quad z \in U,\ j = \overline{1,n}. \tag{11}$$

From (10) and (11), we get

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\frac{2}{1-|z|}\left(|\gamma_1| + \cdots + |\gamma_n|\right) \tag{12}$$

for all $z \in U$.

Now we consider the following cases.

(1) $0 < \operatorname{Re}\alpha < 1$. The function $s : (0,1) \to \mathbb{R}$, $s(x) = 1 - a^{2x}$, $x = \operatorname{Re}\alpha$, $a = |z|$, $(0 \le a < 1)$ is increasing and we obtain

$$1 - |z|^{2\operatorname{Re}\alpha} \le 1 - |z|^2, \quad z \in U. \tag{13}$$

From (12) and (13), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le \frac{4}{\operatorname{Re}\alpha}\left(|\gamma_1| + \cdots + |\gamma_n|\right) \tag{14}$$

for all $z \in U$.

Using the hypothesis condition (7), from (14), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 1, \tag{15}$$

for all $z \in U$.

(2) $\operatorname{Re}\alpha \ge 1$. We notice that the function

$$q : [1,\infty) \longrightarrow \mathbb{R}, \qquad q(x) = \frac{1-a^{2x}}{x}, \qquad x = \operatorname{Re}\alpha, \qquad a = |z|, \quad (0 \le a < 1) \tag{16}$$

is decreasing function, and we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le 1 - |z|^2, \quad z \in U. \tag{17}$$

Using the last inequality in (12), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 4\left(|\gamma_1| + \cdots + |\gamma_n|\right) \tag{18}$$

for all $z \in U$.

Now using the hypothesis condition (7), from (18), we get

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le 1 \tag{19}$$

for all $z \in U$.

Hence, based on the conditions obtained in (15) and in (19), applying Lemma 3, we have that $H_n \in \mathcal{S}$. □

Theorem 5. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $\operatorname{Re}\alpha > 0$, the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \cdots$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, and K the positive real number, $K \cong 3.05$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le K, \quad z \in U, \; j = \overline{1,n}, \tag{20}$$

$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| \le \frac{(2\operatorname{Re}\alpha + 1)^{(2\operatorname{Re}\alpha+1)/(2\operatorname{Re}\alpha)}}{2K}, \tag{21}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$, and for every complex number β, $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, the integral operator T_n is in the class $\mathcal{S}$.

Proof. By (20) and Lemma 1, we obtain that $f_j \in \mathcal{S}$, $j = \overline{1,n}$.

We consider the function

$$p_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} \cdots \left(f_n'(u)\right)^{\gamma_n} du. \tag{22}$$

The function p_n is regular in U and $p_n(0) = p_n'(0) - 1 = 0$.

We have

$$\frac{zp_n''(z)}{p_n'(z)} = \gamma_1 \frac{zf_1''(z)}{f_1'(z)} + \cdots + \gamma_n \frac{zf_n''(z)}{f_n'(z)}, \tag{23}$$

for all $z \in U$.

Further we obtain

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \le \frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|z|\left[|\gamma_1|\left|\frac{f_1''(z)}{f_1'(z)}\right| + \cdots + |\gamma_n|\left|\frac{f_n''(z)}{f_n'(z)}\right|\right]. \tag{24}$$

From (24) and from the Kudriasov condition within the hypothesis, (20), we have

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| \le \left[\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|z|\right] \times \left(K|\gamma_1| + \cdots + K|\gamma_n|\right), \tag{25}$$

for all $z \in U$.

Let us consider the function $G : [0,1] \to \mathbb{R}$, $G(x) = ((1 - x^{2a})/a)x$, $x = |z|$, $a = \operatorname{Re}\alpha$. We have

$$\max_{x\in[0,1]} G(x) = \frac{2}{(2a+1)^{(2a+1)/(2a)}}. \tag{26}$$

By (25), (26), and (21) we obtain

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp_n''(z)}{p_n'(z)}\right| < 1, \tag{27}$$

for all $z \in U$.

Now from (27) and Lemma 3, it results that $T_n \in \mathcal{S}$. □

4. Corollaries

Corollary 1. *Let γ be complex number, $\operatorname{Re}[n(\gamma-1)+1] > 0$, the functions $f_j \in \mathcal{A}$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, and K the positive real number, $K \cong 3.05$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le K, \quad z \in U, \; j = \overline{1,n},$$
$$n|\gamma - 1| \le \min\left\{\frac{\operatorname{Re}[n(\gamma-1)+1]}{4}, \frac{1}{4}\right\}, \tag{28}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$, and the integral operator $I_{\alpha,n}$ defined by

$$I_{\alpha,n}(z) = \left\{[n(\gamma-1)+1]\int_0^z f_1^{\gamma-1}(u)\cdots f_n^{\gamma-1}(u)du\right\}^{1/(n(\gamma-1)+1)} \tag{29}$$

is in the class $\mathcal{S}$.

Proof. From (29), we have

$$I_{\alpha,n}(z) = \left\{[n(\gamma-1)+1]\int_0^z u^{n(\gamma-1)}\left(\frac{f_1(u)}{u}\right)^{\gamma-1} \cdots \left(\frac{f_n(u)}{u}\right)^{\gamma-1} du\right\}^{1/(n(\gamma-1)+1)}, \tag{30}$$

and for $\alpha = \beta = n(\gamma-1)+1$, $\gamma_1 = \gamma_2 = \cdots = \gamma_n = \gamma - 1$, from Theorem 4, we obtain Corollary 1. □

Corollary 2. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < \operatorname{Re}\alpha \le 1$, the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \cdots$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, and K the positive real number $K \cong 3.05$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le K, \quad z \in U, \; j = \overline{1,n},$$
$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| \le \frac{\operatorname{Re}\alpha}{4}, \tag{31}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$ and the integral operator L_n defined by

$$L_n(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\gamma_1} \cdots \left(\frac{f_n(u)}{u}\right)^{\gamma_n} du \tag{32}$$

belongs to the class $\mathcal{S}$.

Corollary 3. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $0 < \operatorname{Re}\alpha \le 1$, the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + \cdots$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, and K the positive real number, $K \cong 3.05$.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le K, \quad z \in U, \; j = \overline{1,n},$$
$$|\gamma_1| + |\gamma_2| + \cdots + |\gamma_n| \le \frac{(2\operatorname{Re}\alpha + 1)^{(2\operatorname{Re}\alpha+1)/(2\operatorname{Re}\alpha)}}{2K}, \tag{33}$$

then $f_j \in \mathcal{S}$, $j = \overline{1, n}$, *and the integral operator* G_n *defined by*

$$G_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} \cdots \left(f_n'(u)\right)^{\gamma_n} du \tag{34}$$

is in the class $\mathcal{S}$.

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