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Research Article

Sharp Bounds for the Weighted Geometric Mean of the First Seiffert and Logarithmic Means in terms of Weighted Generalized Heronian Mean

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Optimal bounds for the weighted geometric mean of the first Seiffert and logarithmic means by weighted generalized Heronian mean are proved. We answer the question: for $\alpha \in (0,1)$, what the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that the double inequality, $H_{p(\alpha)}(a,b) < P^{\alpha}(a,b)L^{1-\alpha}(a,b) < H_{q(\alpha)}(a,b)$, holds for all $a,b > 0$ with $a \neq b$ are. Here, $P(a,b)$, $L(a,b)$, and $H_{\omega}(a,b)$ denote the first Seiffert, logarithmic, and weighted generalized Heronian means of two positive numbers a and b, respectively.

1. Introduction

Recently, means has been the subject of intensive research. In particular, many remarkable inequalities for the Seiffert, logarithmic, and Heronian mean can be found in the literature [1–11]. In the paper [1], authors proved the following optimal inequalities:

Let $a > 0, b > 0, a \neq b$ then

$$\begin{aligned} &H_{\delta}(a,b) < P(a,b) < H_{\beta}(a,b) \quad \text{for } \delta \geq \pi - 2,\ \beta \leq 1, \\ &\qquad \delta = \pi - 2,\ \beta = 1 \text{ are the best constants.} \\ &H_{\gamma}(a,b) < L(a,b) < H_{\tau}(a,b) \quad \text{for } \gamma = +\infty,\ \tau \leq 4, \\ &\qquad \gamma = +\infty,\ \tau = 4 \text{ are the best constants.} \end{aligned} \tag{1}$$

$P(a,b)$ is the first Seiffert mean, which was introduced by Seiffert in [9]

$$\begin{aligned} P(a,b) &= \frac{a-b}{4\arctan\left(\sqrt{a/b}\right) - \pi} \\ &= \frac{a-b}{2\arcsin\left((a-b)/(a+b)\right)} \quad \text{for } a,b > 0,\ a \neq b. \end{aligned} \tag{2}$$

In [9], Seiffert proved that $L(a,b) < P(a,b) < I(a,b)$, where $I(a,b)$ is the identric mean

$$I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{1/(b-a)} \quad \text{if } a \neq b,\ I(a,a) = a. \tag{3}$$

$L(a,b)$ is the logarithmic mean

$$L(a,b) = \frac{a-b}{\log a - \log b} \quad \text{for } a,b > 0,\ a \neq b. \tag{4}$$

$G_{\alpha}(a,b)$ is the weighted geometric mean

$$G_{\alpha}(a,b) = a^{\alpha}b^{1-\alpha} \quad \text{for } a,b > 0,\ 0 \leq \alpha \leq 1. \tag{5}$$

$H_{\omega}(a,b)$ is the weighted generalized Heronian mean introduced by Janous [7]

$$\begin{aligned} H_{\omega}(a,b) &= \frac{a + \omega\sqrt{ab} + b}{\omega + 2} \quad \text{for } 0 \leq \omega < +\infty \\ &= \sqrt{ab} \quad \text{for } \omega = +\infty. \end{aligned} \tag{6}$$

It is well known, that $H_{\omega}(a,b)$ is a strictly decreasing continuous function of the argument ω. From this and from results of [1], it is natural to assume that there exist optimal functions $p(\alpha)$, $q(\alpha)$, $0 \le \alpha \le 1$ such that

$$H_{p(\alpha)}(a,b) < P^{\alpha}(a,b)\,L^{1-\alpha}(a,b) < H_{q(\alpha)}(a,b). \quad (7)$$

The purpose of this paper is to find the optimal functions. For some other details about means, see [1–11] and the related references cited there in.

2. Main Results

The main result of this paper is the following theorem.

Theorem 1. *Let $a,b>0$, $a \ne b$, $\alpha \in (0,1)$. Then*

$$H_{p}(a,b) < P^{\alpha}(a,b)\,L^{1-\alpha}(a,b) < H_{q}(a,b), \quad \text{for } p=+\infty, \quad q \le q(\alpha), \quad (8)$$

where $p = p(\alpha) = +\infty$, $q(\alpha) = 2(2-\alpha)/(1+\alpha)$ are the best possible functions.

Proof. First, we prove the left inequality of (8). The inequalities (1) imply that

$$H_{+\infty}(a,b) < P^{\alpha}(a,b)\,L^{1-\alpha}(a,b) \quad \text{for } a,b>0, \quad a \ne b, \quad 0<\alpha<1. \quad (9)$$

From $\lim_{t\to 0^{+}} G(t,\alpha) = +\infty$ for $\alpha \in (0,1)$ (see (14)) we obtain that $p(\alpha) = +\infty$ is the optimal function.

Without loss of generality, we assume that $0<a<b$. Let $t = \sqrt{a/b}$; then $0<t<1$. The right inequality of (8) can be rewritten as

$$\frac{1}{b^{\alpha}}P^{\alpha}(a,b)\,\frac{1}{b^{1-\alpha}}L^{1-\alpha}(a,b) < \frac{1}{b}H_{q}(a,b) \quad \text{for } a,b>0, \quad a\ne b, \quad 0<\alpha<1. \quad (10)$$

Simple computations lead to

$$\frac{1-t^{2}}{(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha}} - \frac{t^{2}+qt+1}{q+2} < 0 \quad \text{for } 0<t<1, \quad 0<\alpha<1. \quad (11)$$

Then the inequality (11) is equivalent to

$$q\left(1-t^{2}-t(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha}\right) < \left(1+t^{2}\right)(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha} - 2\left(1-t^{2}\right). \quad (12)$$

Denote

$$s(t,\alpha) = 1-t^{2}-t(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha},$$
$$r(t) = \pi - 4\arctan t + 2\ln t, \qquad v(t) = t^{2}-2t\ln t - 1. \quad (13)$$

From $r(1)=0$ and $r'(t) = (2-4t+2t^{2})/(t+t^{3}) > 0$ we have $r(t)<0$ for $t\in(0,1)$. It implies $((\pi-4\arctan t)/(-2\ln t))^{\alpha} < 1$.From $v(1)=0$, $v'(1)=0$, $v''(t) = 2-2/t < 0$ we obtain $v'(t)>0$ and so $v(t)<0$. It implies that $s(t,\alpha)>0$ for t, $\alpha \in (0,1)$. This leads to

$$q < G(t,\alpha) = \frac{\left(1+t^{2}\right)(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha} - 2\left(1-t^{2}\right)}{1-t^{2}-t(\pi-4\arctan t)^{\alpha}(-2\ln t)^{1-\alpha}}. \quad (14)$$

If we show $G'_{t}(t,\alpha) < 0$ for $t,\alpha \in (0,1)$, then $q(\alpha) = \lim_{t\to 1^{-}} G(t,\alpha)$ will be the best function in (8). Simple computations lead to $G'_{t}(t,\alpha)<0$ which is equivalent to

$$H(t,\alpha) = 2(1-t)\ln t + \frac{4\alpha(1+t)(1-t)^{2}\ln t}{\pi-4\arctan t} - \frac{(1-\alpha)(1+t)(1-t)^{2}}{t} + 2(1+t)\ln^{2}t\left(\frac{\pi-4\arctan t}{-2\ln t}\right)^{\alpha} < 0. \quad (15)$$

Using the inequality $t^{\alpha} < 1-\alpha(1-t)$ for $t,\alpha\in(0,1)$ it suffices to show that

$$R(t,\alpha) = 2(1-t)\ln t + \frac{4\alpha(1+t)(1-t)^{2}\ln t}{\pi-4\arctan t} - \frac{(1-\alpha)(1+t)(1-t)^{2}}{t} + (2\ln t + \alpha(-2\ln t - \pi + 4\arctan t))(1+t)\ln t < 0. \quad (16)$$

It will be done, if we show $R(t,0)<0$ and $R(t,1)<0$. It follows from $R(t,\alpha)$ being a linear continuous function in the argument α

$$R(t,0) = 2(1-t)\ln t - \frac{(1+t)(1-t)^{2}}{t} + 2(1+t)\ln^{2}t < 0 \quad (17)$$

is equivalent to

$$s(t) = \frac{2(1-t)\ln t}{1+t} - \frac{(1-t)^{2}}{t} + 2\ln^{2}t < 0. \quad (18)$$

From $s(1)=0$ it suffices to show that $s'(t)>0$ which is equivalent to

$$v(t) = \ln t + \frac{\left(1+3t-3t^{2}-t^{3}\right)(1+t)}{4t\left(1+t+t^{2}\right)} > 0. \quad (19)$$

It follows from $v(1)=0$ and

$$v'(t) = \frac{w(t)}{4t^{2}\left(1+t+t^{2}\right)^{2}} < 0, \quad (20)$$

where $w(t) = -(1-t)^{2}(1-t^{2})^{2}$.

Next, we show that

$$R(t,1) = 4(1-t)\ln t + \frac{8(1+t)(1-t)^2\ln t}{\pi - 4\arctan t} - 2(\pi - 4\arctan t)(1+t)\ln t < 0. \tag{21}$$

The inequality (21) is equivalent to

$$\left(\pi - 4\arctan t - \frac{1-t}{1+t}\right)^2 < \left(\frac{1-t}{1+t}\right)^2 - 4(1-t)^2. \tag{22}$$

So, it suffices to show that

$$g(t) = -\pi + 4\arctan t + \frac{1-t}{1+t}\left(1 + \sqrt{1+4(1+t)^2}\right) > 0. \tag{23}$$

It is easy to see that

$$1 + \sqrt{1+4(1+t)^2} > 1 + 2(1+t) + \frac{1}{5(1+t)} = \frac{16+25t+10t^2}{5(1+t)} \quad \text{for } 0 \le t \le 1. \tag{24}$$

Because of

$$g(t) > g_1(t) = -\pi + 4\arctan t + \frac{1-t}{1+t}\left(\frac{16+25t+10t^2}{5(1+t)}\right) > 0, \tag{25}$$

it suffices to prove $g_1(t) > 0$ for $0 < t < 1$. From

$$g_1(t) = \frac{1}{5}g_2(t) = \frac{1}{5}\left(-5\pi + 20\arctan t + \frac{(1-t)\left(16+25t+10t^2\right)}{(1+t)^2}\right), \tag{26}$$

$\arctan(t) > t - t^3/3$, for $t \in (0,1)$, $g_2(1) = 0$ we have done it, if we show

$$g_3(t) = -5\pi + 20\left(t - \frac{t^3}{3}\right) + \frac{(1-t)\left(16+25t+10t^2\right)}{(1+t)^2} > 0, \tag{27}$$

on $(0, 0.67\rangle$ and $g_2'(t) < 0$ on $\langle 0.67, 1)$.

Simple computation gives

$$g_2'(t) = \frac{20}{1+t^2} - \left(\frac{23+39t+30t^2+10t^3}{(1+t)^3}\right). \tag{28}$$

The inequality $g_2'(t) < 0$ is equivalent to

$$ch(t) = -3 + 21t + 7t^2 - 29t^3 - 30t^4 - 10t^5 < 0. \tag{29}$$

From $ch'''(t) < 0$ we get $ch''(t)$ is a decreasing function. $ch''(0.67) = -324.3366$ implies $ch''(t) < 0$ on $\langle 0.67, 1)$. So, we obtain $ch'(t)$ is a decreasing function. From $ch'(0.67) = -54.8414$ we have $ch'(t) < 0$ on $\langle 0.67, 1)$. It implies that $ch(t)$ is a decreasing function. From $ch(0.67) = -1.9053$ we get $ch(t) < 0$ on $\langle 0.67, 1)$. So $g_2'(t) < 0$ on $\langle 0.67, 1)$.

Next, we show $g_3(t) > 0$ on $(0, 0.67\rangle$.

Simple computation gives

$$g_3(t) = \left(16 - 5\pi + t(29-10\pi) + t^2(25-5\pi) + t^3\frac{10}{3} - t^4\frac{40}{3} - t^5\frac{20}{3}\right)\left((1+t)^2\right)^{-1}. \tag{30}$$

The inequality $g_3(t) > 0$ is equivalent to

$$h(t) = 16 - 5\pi + t(29-10\pi) + t^2(25-5\pi) + t^3\frac{10}{3} - t^4\frac{40}{3} - t^5\frac{20}{3} > 0, \tag{31}$$

on $(0, 0.67\rangle$. From $h(0) = 16 - 5\pi > 0$, $h(0.1) = 0.1453$, $h(0.15) = 0.1427$, $h(0.67) = 0.2602$, $h'(0.15) = 0.3998$ it suffices to show that $h'(t) < 0$ on $(0, 0.1\rangle$; $h(t) > 0$ on $\langle 0.1, 0.15\rangle$ and $h'(t)$ has only one root in $(0.15, 0.67)$.

First, we show $h(t) > 0$ on $\langle 0.1, 0.15\rangle$. From $t^3 > 0.1t^2$, $t^4 < 0.15^2t^2$, $t^5 < 0.15^3t^2$ we have

$$h(t) > 16 - 5\pi + t(29-10\pi) + t^2\left(25 - 5\pi + 0.1\frac{10}{3} - 0.15^2\frac{40}{3} - 0.15^3\frac{20}{3}\right) > l(t), \tag{32}$$

where

$$l(t) = 16 - 5\pi + t(29-10\pi) + t^2 9.3. \tag{33}$$

It is easy to see that $l'(t) = 0$ for $t = (10\pi - 29)/18.6 = 0.1299$. From $l''(t) > 0$ on $\langle 0.1, 0.15\rangle$ and $l(0.1299) = 0.1351$ we have $l(t) > 0$. It implies $h(t) > 0$ on $\langle 0.1, 0.15\rangle$.

Next, we show $h'(t) < 0$ on $(0, 0.1\rangle$. Simple computation gives

$$h'(t) = (29-10\pi) + (50-10\pi)t + 10t^2 - \frac{160}{3}t^3 - \frac{100}{3}t^4 < j(t), \tag{34}$$

where

$$j(t) = (29-10\pi) + (50-10\pi)t + 10t^2, \tag{35}$$

$j(0) = 29 - 10\pi < 0$, $j''(t) > 0$, $j(0.1) = -0.45750$ imply $j(t) < 0$ so $h'(t) < 0$ on $(0, 0.1\rangle$.

Finally, we show that $h'(t)$ has only one root on $(0.15, 0.67)$. From $h''''(t) < 0$ we obtain $h'''(t)$ is a decreasing function. Because of $h'''(0.15) = -37$ we have $h'''(t) < 0$ on $(0.15, 0.67)$ so $h'(t)$ is a concave function. From $h'(0.15) = 0.3998$ and $h'(0.67) = -8.2333$ we have that $h'(t)$ has only one root on $(0.15, 0.67)$. It implies $h(t) > 0$ on $\langle 0.15, 0.67\rangle$. So, the proof of decreasing of $G(t,\alpha)$ is complete.

In what follows, we find the representation of the function $q(\alpha)$.

It is easy to see that

$$q(\alpha) = \lim_{t \to 1^-} \left(\left(\left(1+t^2\right)(\pi - 4\arctan t)^{\alpha} \times (-2\ln t)^{1-\alpha} - 2\left(1-t^2\right) \right) \times \left(1 - t^2 - t(\pi - 4\arctan t)^{\alpha} \times (-2\ln t)^{1-\alpha}\right)^{-1} \right). \quad (36)$$

Equation (36) can be rewritten as

$$q(\alpha) = \lim_{t \to 1^-} \frac{\left(1+t^2\right) Y(t,\alpha) U(t,\alpha) - 2(1+t)}{1 + t - tY(t,\alpha) U(t,\alpha)}, \quad (37)$$

where

$$Y(t,\alpha) = \left(\frac{\pi - 4\arctan t}{1-t}\right)^{\alpha}, \qquad U(t,\alpha) = \left(\frac{-2\ln t}{1-t}\right)^{1-\alpha}. \quad (38)$$

Simple computations give

$$Y(t,\alpha) = 2^{\alpha}\left(1 + \frac{1-t}{2} + \frac{(1-t)^2}{6} + y(t)(1-t)^3\right), \quad (39)$$

where $y(t)$ is a suitable function. Similarly we have

$$U(t,\alpha) = 2^{1-\alpha}\left(1 + \frac{1-t}{2} + \frac{(1-t)^2}{3} + u(t)(1-t)^3\right), \quad (40)$$

where $u(t)$ is a suitable function. Denote $S(t,\alpha) = Y(t,\alpha)U(t,\alpha)$. Then

$$S(t,\alpha) = 2\left(1 + \frac{1-t}{2} + \frac{(2-\alpha)(1-t)^2}{6} + s(t)(1-t)^3\right), \quad (41)$$

where $s(t)$ is a suitable function. Using the L'Hospital's rule we obtain

$$\begin{aligned} q(\alpha) &= \lim_{t \to 1^-} \frac{\left(1+t^2\right) S(t,\alpha) - 2(1+t)}{1 + t - tS(t,\alpha)} \\ &= \lim_{t \to 1^-} \frac{2S(t,\alpha) + 4tS'_t(t,\alpha) + \left(1+t^2\right) S''_{tt}(t,\alpha)}{-2S'_t(t,\alpha) - tS''_{tt}(t,\alpha)} \\ &= \frac{2(2-\alpha)}{1+\alpha}. \end{aligned} \quad (42)$$

The proof is complete. □

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