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Research Article

Nonuniform Continuity of the Osmosis K(2, 2) Equation

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The qualitative theory of differential equations is applied to the osmosis K(2, 2) equation. The parametric conditions of existence of the smooth periodic travelling wave solutions are given. We show that the solution map is not uniformly continuous by using the theory of Himonas and Misiolek. The proof relies on a construction of smooth periodic travelling waves with small amplitude.

1. Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics. Travelling wave solution is an important type of solution for the nonlinear partial differential equation and many nonlinear partial differential equations have been found to have a variety of travelling wave solutions.

The well-known Korteweg-de-Vries equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

was first derived by Boussinesq in 1877, and later by Korteweg and de Vries in 1895, as an approximate description of surface water waves propagating in a canal. This equation has since found application to a range of problems in solid and fluid mechanics as well as plasma physics and astrophysics. The KdV equation has smooth solitary wave solutions and smooth periodic wave solutions [1]. Bona and Smith [2] considered the Cauchy problem for (1). Bona [3] investigated the stability of solitary waves of (1). Angulo Pava et al. [4] studied stability of cnoidal waves of (1).

The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{2}$$

was proposed by Camassa and Holm [5] as a model equation for unidirectional nonlinear dispersive waves in shallow water. This equation has attracted a lot of attention over the past decade due to its interesting mathematical properties. The Camassa-Holm equation has been found to has peakons, cuspons, stumpons, and composite wave solutions [6–11]. Himonas and Misiołek [12] showed that for $s \geq 2$ the solution map $u_0 \to u$ for the Camassa-Holm equation is not uniformly continuous from any bounded set in $H^s(S)$ into $C([0,T], H^s(S))$. A key step in the proof of that result is a construction of a sequence of smooth travelling waves. Himonas et al. [13] extend the result to the range $3/2 < s < 2$. Their proof is based on the approximation of solutions by terms containing high and low frequencies and exploring the conservation of the H^1 norm.

The Degasperis-Procesi equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{3}$$

was originally derived by Degasperis and Procesi. Zhang and Qiao [14] gave smooth and cusped soliton solutions of the Degasperis-Procesi equation. Liu and Yin [15] proved that the first blowup in finite time to (3) must occur as wave breaking, and shock waves possibly appear afterwards. Christov and Hakkaev [16] considered the problem of the uniformly continuity of Degasperis-Procesi equation.

In 1993, Rosenau and Hyman [17] introduced a genuinely nonlinear dispersive equation, a special type of KdV equation, of the form

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad (4)$$

where both the convection term $(u^m)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. If $m = n = 2$, then there exits special form

$$u_t + (u^2)_x + \lambda(u^2)_{xxx} = 0. \quad (5)$$

When $\lambda = 1$, then (5) becomes the K(2, 2) equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \quad (6)$$

Rosenau and Hyman derived solutions called compactons for (6). For $\lambda = -1$, Xu and Tian [18] introduced the osmosis K(2, 2) equation

$$u_t + (u^2)_x - (u^2)_{xxx} = 0, \quad (7)$$

where the negative coefficient of dispersion term denotes the contracting dispersion. They obtained the peaked solitary wave solution and the periodic cusp wave solution for (7). Zhou et al. [19] obtained two new types of travelling wave solutions called kink-like and antikink-like wave solutions. Zhou and Tian [20] obtained the analytic expressions of soliton solution of (7) by using the bifurcation method of dynamical systems. Deng and Han [21] successfully found a peaked wave solution of (7) by using the first-integral method. Deng et al. [22] obtained some new exact travelling-wave solutions and stationary-wave solutions by using the auxiliary elliptic equation method. Recently, Chen and Li [23] obtained single peak solitary wave solutions of the osmosis K(2, 2) equation.

To the best of our knowledge, the problems of the well-posedness and the uniformly continuity of (7) have not yet been considered. Applying Kato's theory for abstract quasilinear evolution equation of hyperbolic type [16], ones may obtain the local wellposedness for (7). Here, we do not consider the wellposedness for (7). Following [12], we consider the problem of the uniformly continuity of (7) by constructing two sequences of solutions. We hope to extend the result to the range $s < 2$ by using approximate solutions and delicate commutator and multiplier estimates in the future. Our main result is the following theorem.

Theorem 1. *For any $s \geq 2$, the solution map $u_0 \to u$ for (7) is not uniformly continuous from any bounded set in $H^s(S)$ into $C([0,T], H^s(S))$, where $S = R/2\pi Z$. More precisely, for each $s \geq 2$ there exist constants c_1 and c_2 and two sequences of smooth solutions u_n and v_n of (7) such that for any $t \in [0,1]$,*

$$\begin{gathered} \sup_n \|u_n(t)\|_{H^s} + \sup_n \|v_n(t)\|_{H^s} \leq c_1, \\ \lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{H^s} = 0, \\ \liminf_n \|u_n(t) - v_n(t)\|_{H^s} \geq c_2 \sin\left(\frac{t}{2}\right). \end{gathered} \quad (8)$$

The paper is organized as follows. In Section 2, we discuss the dynamical behavior of solutions of the K(2, 2) equation (5) and give parameter condition of existence of the smooth periodic travelling wave solutions. In Section 3, we provide a precise estimate of periods of the periodic travelling wave solutions. In Section 3, we establish upper bounds for these solutions in H^s-norms. The last section contains the proof of the main result.

2. Dynamical Analysis of Travelling Waves

In this section we investigate the periodic travelling wave solutions of (5). Note that if $u(x,t)$ is a classical solution of (5), then so is the function

$$u_c(x,t) = cu(x,ct), \quad \text{for any constant } c. \quad (9)$$

If $u(x,t) = \varphi(\xi) = \varphi(x-t)$ is to be a solution to (5), the function φ must satisfy the ordinary differential equation

$$-\varphi' + (\varphi^2)' + \lambda(\varphi^2)''' = 0. \quad (10)$$

Integrating this equation gives

$$2\lambda\varphi\varphi'' = a + \varphi - \varphi^2 - 2\lambda(\varphi')^2, \quad (11)$$

where a is an integration constant. Equation (11) is equivalent to the planar system

$$\begin{aligned} \frac{d\varphi}{d\xi} &= y, \\ \frac{d\varphi}{d\xi} &= \frac{a + \varphi - \varphi^2 - 2\lambda y^2}{2\lambda\varphi}, \end{aligned} \quad (12)$$

with the first integral

$$H(\varphi, y) = \lambda\varphi^2 y^2 + \frac{1}{4}\varphi^4 - \frac{1}{3}\varphi^3 - \frac{a}{2}\varphi^2 = h, \quad (13)$$

where h is also an integral constant. As well known, system (12) has a periodic solution if and only if it has a center. Using qualitative theory of differential equations [24, 25], we can easily verify the following statement.

Proposition 2. *System (12) has a center if and only if λ, a satisfies one of the following parameter set (see Figures 1, 2, 3, 4, 5, and 6):*

$$\begin{aligned} A &= \left\{(\lambda, a) \mid \lambda > 0, -\frac{1}{4} < a < 0\right\}; \\ B &= \{(\lambda, a) \mid \lambda > 0, a = 0\}; \\ C &= \{(\lambda, a) \mid \lambda > 0, a > 0\}; \\ D &= \left\{(\lambda, a) \mid \lambda < 0, -\frac{2}{9} < a < 0\right\}; \\ E &= \left\{(\lambda, a) \mid \lambda < 0, a = -\frac{2}{9}\right\}; \\ F &= \left\{(\lambda, a) \mid \lambda < 0, -\frac{1}{4} < a < -\frac{2}{9}\right\}. \end{aligned} \quad (14)$$

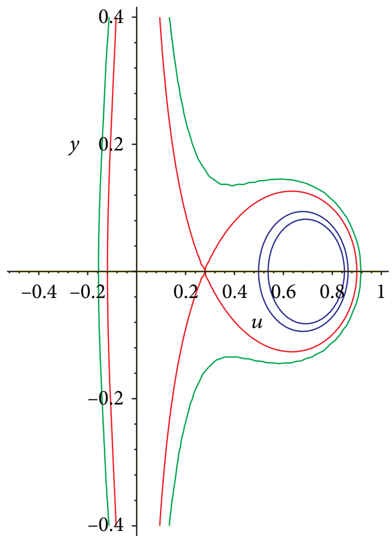

FIGURE 1: Phase portrait of system (12) for $(\lambda, a) \in A$.

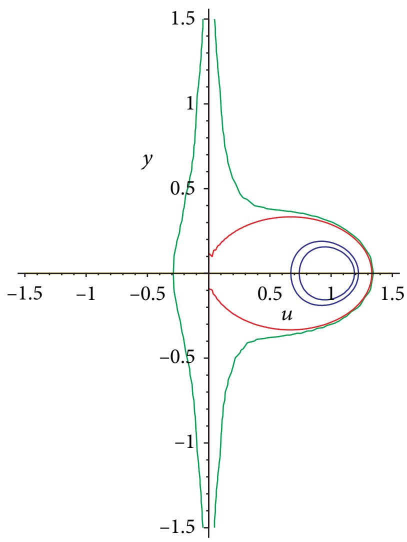

FIGURE 2: Phase portrait of system (12) for $(\lambda, a) \in B$.

3. Estimates of Periods of Solutions

In this section we construct a family of smooth travelling wave solutions of suitably high frequency and provide a precise estimate of their periods.

Let m and M be the minimum and maximum of the function φ, correspondingly; that is, $m \le \varphi \le M$ (see Figure 7). We assume that $0 \ll m = \delta < M = \delta + \epsilon < 1$ for sufficiently small $\epsilon, \delta > 0$. We assume that $\lambda = -1$, since there are similar results for $\lambda = 1$. Equation (13) gives

$$\varphi^2 y^2 = \frac{1}{4}\varphi^4 - \frac{1}{3}\varphi^3 - \frac{a}{2}\varphi^2 - h. \tag{15}$$

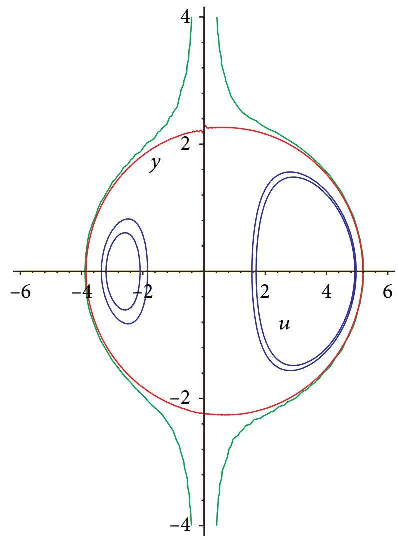

FIGURE 3: Phase portrait of system (12) for $(\lambda, a) \in C$.

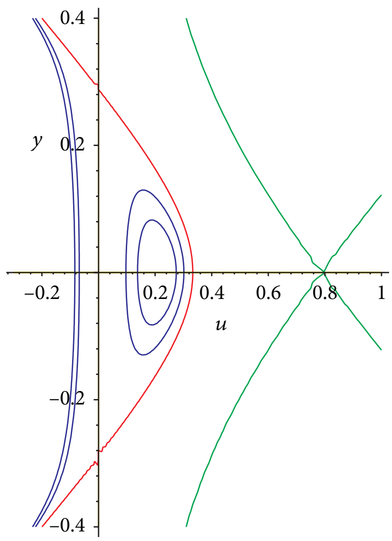

FIGURE 4: Phase portrait of system (12) for $(\lambda, a) \in D$.

Expressing a, h through M and m we find

$$a = \frac{3(M+m)\left(M^2+m^2\right) - 4\left(M^2+Mm+m^2\right)}{6(M+m)}, \tag{16}$$

$$h = \frac{M^2 m^2 (4 - 3(M+m))}{12(M+m)}. \tag{17}$$

Then (15) becomes

$$y^2 = \frac{V(\varphi)}{4\varphi^2}, \tag{18}$$

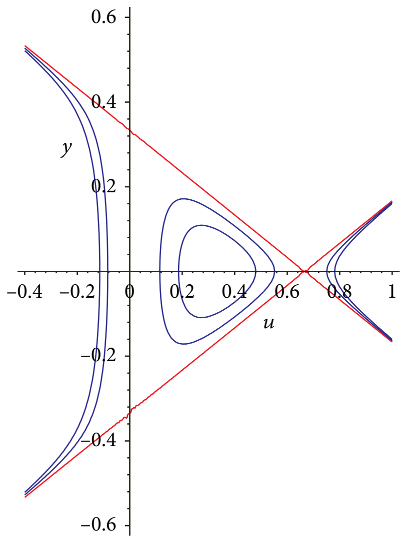

Figure 5: Phase portrait of system (12) for $(\lambda, a) \in E$.

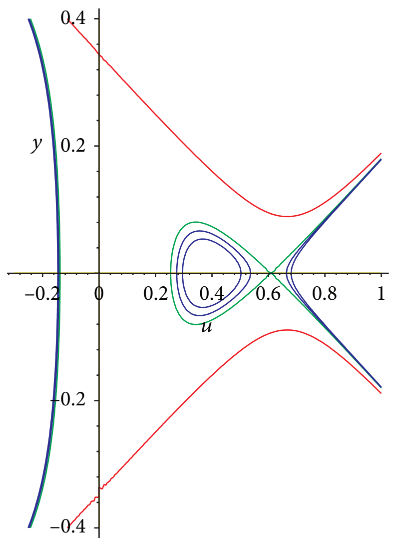

Figure 6: Phase portrait of system (12) for $(\lambda, a) \in F$.

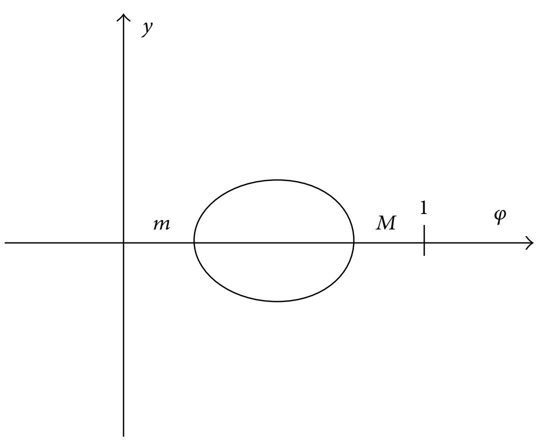

Figure 7: The periodic solution of (18).

where

$$V(\varphi) = (M-\varphi)(\varphi-m)\left(-\varphi^2 + p\varphi + q\right),$$
$$p = \frac{4}{3} - M - m, \tag{19}$$
$$q = \frac{Mm(4-3(M+m))}{3(M+m)}.$$

In comparison with Camassa-Holm equation, the function $V(\varphi)$ is a quartic function rather than cubic function. It can be seen that (18) admits to a nonconstant solution with period $2l$ for certain $l > 0$, which satisfies the following initial value problem:

$$\varphi'' = -\frac{1}{6} + \frac{1}{4}\varphi + \frac{h}{\varphi^3}, \qquad \varphi(0) = \delta, \qquad \varphi'(0) = 0. \tag{20}$$

The above discussion can be summarized as follows.

Proposition 3. *For any $\epsilon, \delta < 1/6$, there exists a positive number $l = l(\epsilon, \delta)$ and even $2l$ periodic smooth function $\varphi = \varphi(\xi)$ which solves* (18) *and* (20)*. The function $\varphi = \varphi(\xi)$ satisfies*

$$m = \delta \le \varphi \le \delta + \epsilon = M \tag{21}$$

and $u(x,t) = \varphi(x-t)$, where $\varphi(\xi) = \varphi(x-t)$ is a travelling wave solution of the osmosis K(2, 2) equation (7)*.*

The next proposition gives precise estimates for the period of the solution φ in terms of the parameters ϵ and δ.

Proposition 4. *The period $2l$ of function φ depends continuously on parameters ϵ and δ and satisfies*

$$\sqrt{\frac{3}{8}}\pi\sqrt{\delta+\epsilon} \le l \le 2\sqrt{2}\pi\sqrt{\delta+\epsilon}. \tag{22}$$

Proof. The half period can be expressed as

$$l = \int_m^M \frac{d\xi}{d\varphi}d\varphi = \int_m^M \frac{2\varphi d\varphi}{\sqrt{(M-\varphi)(\varphi-m)f(\varphi)}}, \tag{23}$$

where $f(\varphi) = -\varphi^2 + p\varphi + q$. The function f is increasing in the interval $(-\infty, \varphi_*]$, where $\varphi_* = 2/3 - (M+m)/2$. On the other hand,

$$\begin{aligned} \varphi_* - M &= \frac{2}{3} - \frac{M+m}{2} - M \\ &= \frac{1}{6}(4 - 9M - 3m) \\ &\ge \frac{1}{6}(4 - 12M) \\ &\ge \frac{1}{6}(4 - 12(\delta+\epsilon)) > 0. \end{aligned} \tag{24}$$

Thus, $f_{\min} = f(m)$ and $f_{\max} = f(M)$ in interval $[m, M]$. Then

$$l \leq \frac{2}{\sqrt{f(m)}} \int_m^M \frac{\varphi d\varphi}{\sqrt{(M-\varphi)(\varphi-m)}} = \frac{\pi}{\sqrt{f(m)}}(M+m), \tag{25}$$

$$l \geq \frac{2}{\sqrt{f(M)}} \int_m^M \frac{\varphi d\varphi}{\sqrt{(M-\varphi)(\varphi-m)}} = \frac{\pi}{\sqrt{f(M)}}(M+m). \tag{26}$$

The estimate for $f(M)$ is $f(M) \leq (8/3)M$. Then from (26) it follows the lower bound

$$\begin{aligned} l &\geq \frac{\pi}{\sqrt{f(M)}}(M+m) \geq \frac{\pi}{\sqrt{(8/3)M}}(M+m) \\ &\geq \sqrt{\frac{3}{8}}\pi\sqrt{M} = \sqrt{\frac{3}{8}}\pi\sqrt{\delta+\epsilon}. \end{aligned} \tag{27}$$

Estimating $f(m)$ we have $f(m) \geq 2m(2/3 - M - m)$. Choose M and m in such a way to achieve

$$f(m) \geq 2m\left(\frac{2}{3} - M - m\right) \geq \frac{M}{2}. \tag{28}$$

Then with the help of (25) it follows

$$\begin{aligned} l &\leq \frac{\pi}{\sqrt{f(m)}}(M+m) \leq \frac{\pi}{\sqrt{M/2}}(M+m) \\ &\leq 2\sqrt{2}\pi\sqrt{M} = 2\sqrt{2}\pi\sqrt{\delta+\epsilon}. \end{aligned} \tag{29}$$

Combining (27) and (29), we complete the proof of the proposition. □

4. Sobolev Estimates of Solutions

We write $l \simeq \sqrt{\delta+\epsilon}$ for the sake of (22). Since $l(\epsilon, \delta)$ is continuous for n sufficiently large, we can find ϵ and δ such that

$$l = \frac{\pi}{n}, \tag{30}$$

$$n \simeq \frac{1}{\sqrt{\delta+\epsilon}}, \tag{31}$$

where $\delta^s = \epsilon^2, s \geq 2$. Hence, we have constructed high-frequency solution $\varphi = \varphi_n(\xi)$ with the period $T = 2\pi/n$.

Next we need some estimates in order to obtain upper bounds for these solutions. We start with L^∞ estimates of the derivatives.

Proposition 5. *Suppose $\delta \geq \epsilon$. Then for any $k = 2, 3, \ldots$, there exists a constant $c_k > 0$ such that*

$$\left|\varphi^{(k)}(\xi)\right| \leq \frac{c_k}{\left(\sqrt{\delta}\right)^{k-2}}. \tag{32}$$

For $k = 1$ we have

$$\left|\varphi'(\xi)\right| \leq \frac{2\sqrt{3}}{3}\frac{\epsilon}{\sqrt{\delta}}. \tag{33}$$

Proof. With the help of (18), the first derivative is estimated as follows:

$$\begin{aligned} \left|\varphi'\right| &= \frac{\sqrt{(M-\varphi)(\varphi-m)f(\varphi)}}{2\varphi} \leq \frac{(M-m)\sqrt{f_{\max}}}{2m} \\ &\leq \frac{(M-m)\sqrt{(8/3)M}}{2m} \leq \frac{2\sqrt{3}}{3}\frac{\epsilon}{\sqrt{\delta}}. \end{aligned} \tag{34}$$

For $k = 2$ using (20) we have

$$\begin{aligned} \left|\varphi''\right| &\leq \frac{1}{6} + \frac{1}{4}|\varphi| + \frac{|h|}{|\varphi^3|} \leq \frac{1}{6} + \frac{1}{4}(\delta+\epsilon) + \frac{|h|}{|\delta^3|} \\ &= \frac{1}{6} + \frac{1}{4}(\delta+\epsilon) + \frac{\delta^2(\delta+\epsilon)^2(4-6\delta-3\epsilon)}{12\delta^3(2\delta+\epsilon)} \\ &\leq \frac{11}{12}. \end{aligned} \tag{35}$$

Next, we proceed by induction and assume that (32) is true for all positive integers up to $k+2$. To estimate $(k+3)$-order derivative we have from (20)

$$\varphi^3\varphi'' = \frac{1}{4}\varphi^4 - \frac{1}{6}\varphi^3 + h. \tag{36}$$

Differentiate both sides of (36) and divide by φ^2 to obtain

$$\varphi\varphi''' = -3\varphi'\varphi'' + \varphi\varphi' - \frac{1}{2}\varphi'. \tag{37}$$

Taking k derivatives and using Leibniz rule we have

$$\begin{aligned} &\sum_{j=0}^{k}\binom{k}{j}\varphi^{(k-j)}\varphi^{(j+3)} \\ &\quad = -3\sum_{j=0}^{k}\binom{k}{j}\varphi^{(k-j+1)}\varphi^{(j+2)} \\ &\qquad + \sum_{j=0}^{k}\binom{k}{j}\varphi^{(k-j)}\varphi^{(j+1)} - \frac{1}{2}\varphi^{(k+1)}. \end{aligned} \tag{38}$$

Now, $\varphi^{(k+3)}$ can be expressed as follows:

$$\begin{aligned} \varphi^{(k+3)} = \frac{1}{\varphi}\Bigg(&-\sum_{j=0}^{k-1}\binom{k}{j}\varphi^{(k-j)}\varphi^{(j+3)} \\ &- 3\sum_{j=0}^{k}\binom{k}{j}\varphi^{(k-j+1)}\varphi^{(j+2)} \\ &+ \sum_{j=0}^{k}\binom{k}{j}\varphi^{(k-j)}\varphi^{(j+1)} - \frac{1}{2}\varphi^{(k+1)}\Bigg). \end{aligned} \tag{39}$$

The last equation together with the induction hypothesis gives

$$\begin{aligned}
&\left|\varphi^{(k+3)}\right| \\
&\leq \frac{1}{\delta}\sum_{j=0}^{k-1}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+3}}{(\sqrt{\delta})^{j+3-2}} \\
&\quad+\frac{3}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j+1}}{(\sqrt{\delta})^{k-j+1-2}}\frac{c_{j+2}}{(\sqrt{\delta})^{j+2-2}} \\
&\quad+\frac{1}{\delta}\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\frac{c_{j+1}}{(\sqrt{\delta})^{j+1-2}}+\frac{1}{2\delta}\frac{c_{k+1}}{(\sqrt{\delta})^{k+1-2}},
\end{aligned}\tag{40}$$

and therefore we have

$$\begin{aligned}
\left|\varphi^{(k+3)}\right| \leq \frac{c'_{k+3}}{\delta}\Bigg(&\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-1}}+\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-1}} \\
&+\sum_{j=0}^{k}\binom{k}{j}\frac{1}{(\sqrt{\delta})^{k-3}}+\frac{1}{(\sqrt{\delta})^{k-1}}\Bigg).
\end{aligned}\tag{41}$$

Thus

$$\left|\varphi^{(k+3)}(\xi)\right| \leq \frac{c_{k+3}}{(\sqrt{\delta})^{(k+3)-2}},\tag{42}$$

which complete the proof. □

Further, we proceed with L^2 estimates.

Proposition 6. *For any $k = 2, 3, \ldots$, there exists a constant $c_k > 0$ such that*

$$\left\|\varphi^{(k)}\right\|^2_{L^2(-l,l)} \leq \frac{c_k}{\delta^{k-1}}\left\|\varphi'\right\|^2_{L^2(-l,l)}.\tag{43}$$

For $k = 1$ we have

$$\int_{-l}^{l}(\varphi')^2 d\xi \leq \frac{\sqrt{3}\pi}{6}\frac{\epsilon^2}{\sqrt{\delta}}.\tag{44}$$

Proof. For the first derivative we get

$$\begin{aligned}
&\int_{-l}^{l}(\varphi')^2 d\xi \\
&= \int_{\delta}^{\delta+\epsilon}\frac{\sqrt{(\delta+\epsilon-\varphi)(\varphi-\delta)(-\varphi^2+p\varphi+q)}}{\varphi}d\varphi \\
&\leq \frac{f_{\max}}{\delta}\int_{\delta}^{\delta+\epsilon}\sqrt{(\delta+\epsilon-\varphi)(\varphi-\delta)}d\varphi \\
&\leq \frac{\sqrt{(8/3)M}}{\delta}\cdot\frac{\epsilon^2}{8}\cdot\pi \leq \frac{\sqrt{3}\pi}{6}\frac{\epsilon^2}{\sqrt{\delta}}.
\end{aligned}\tag{45}$$

For the second derivative using symmetry, periodicity, and integrating by parts, we have

$$\begin{aligned}
\int_0^l (\varphi'')^2 d\xi &= -\int_0^l \varphi'''(\xi)\varphi'(\xi)\,d\xi \\
&= -\int_{\delta}^{\delta+\epsilon}\left(\frac{1}{4}-\frac{3h}{\varphi^4}\right)\varphi' d\varphi.
\end{aligned}\tag{46}$$

Since $\delta \leq \varphi \leq \delta+\epsilon$, using (17), the last equality gives

$$\begin{aligned}
\int_0^l (\varphi'')^2 d\xi &\leq \left(\frac{3|h|}{\delta^4}-\frac{1}{4}\right)\int_{\delta}^{\delta+\epsilon}\varphi' d\varphi \\
&= \left(\frac{3|h|}{\delta^4}-\frac{1}{4}\right)\left\|\varphi'\right\|^2_{L^2(0,l)} \\
&= \left(\frac{(\delta+\epsilon)^2(4-6\delta-3\epsilon)}{4\delta^2(2\delta+\epsilon)}-\frac{1}{4}\right)\left\|\varphi'\right\|^2_{L^2(0,l)} \\
&\leq \frac{2}{\delta}\left\|\varphi'\right\|^2_{L^2(0,l)}.
\end{aligned}\tag{47}$$

Proceeding by induction and using the expression for φ^{k+3} in Proposition 5, we obtain

$$\begin{aligned}
&\left\|\varphi^{(k+3)}\right\|_{L^2(-l,l)} \\
&\leq \frac{1}{\delta}\Bigg(\sum_{j=0}^{k-1}\binom{k}{j}\left\|\varphi^{(k-j)}\varphi^{(j+3)}\right\|_{L^2(-l,l)} \\
&\quad+3\sum_{j=0}^{k}\binom{k}{j}\left\|\varphi^{(k-j+1)}\varphi^{(j+2)}\right\|_{L^2(-l,l)} \\
&\quad+\sum_{j=0}^{k}\binom{k}{j}\left\|\varphi^{(k-j)}\varphi^{(j+1)}\right\|_{L^2(-l,l)}+\frac{1}{2}\left\|\varphi^{(k+1)}\right\|_{L^2(-l,l)}\Bigg) \\
&\leq \frac{1}{\delta}\Bigg(\sum_{j=0}^{k-1}\binom{k}{j}\left\|\varphi^{(k-j)}\right\|_{L^\infty}\left\|\varphi^{(j+3)}\right\|_{L^2(-l,l)} \\
&\quad+3\sum_{j=0}^{k}\binom{k}{j}\left\|\varphi^{(k-j+1)}\right\|_{L^\infty}\left\|\varphi^{(j+2)}\right\|_{L^2(-l,l)} \\
&\quad+\sum_{j=0}^{k}\binom{k}{j}\left\|\varphi^{(k-j)}\right\|_{L^\infty}\left\|\varphi^{(j+1)}\right\|_{L^2(-l,l)}+\frac{1}{2}\left\|\varphi^{(k+1)}\right\|_{L^2(-l,l)}\Bigg).
\end{aligned}\tag{48}$$

Applying the estimates of Proposition 5 and using the induction hypothesis, we estimate further as follows:

$$\begin{aligned}
&\left\|\varphi^{(k+3)}\right\|_{L^2(-l,l)} \\
&\le \frac{1}{\delta}\left(\sum_{j=0}^{k-1}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\cdot\frac{c_{j+3}}{(\sqrt{\delta})^{j+2}}\left\|\varphi'\right\|_{L^2(-l,l)}\right. \\
&\quad + 3\sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j+1}}{(\sqrt{\delta})^{k-j-1}}\cdot\frac{c_{j+2}}{(\sqrt{\delta})^{j+1}}\left\|\varphi'\right\|_{L^2(-l,l)} \\
&\quad + \sum_{j=0}^{k}\binom{k}{j}\frac{c_{k-j}}{(\sqrt{\delta})^{k-j-2}}\cdot\frac{c_{j+1}}{(\sqrt{\delta})^{j}}\left\|\varphi'\right\|_{L^2(-l,l)} \\
&\quad \left. + \frac{1}{2}\frac{c_{k+1}}{(\sqrt{\delta})^{k}}\left\|\varphi'\right\|_{L^2(-l,l)}\right).
\end{aligned} \tag{49}$$

Therefore, we have

$$\left\|\varphi^{(k+3)}\right\|_{L^2(-l,l)} \le \frac{c_{k+3}}{\sqrt{\delta}^{(k+3)-1}}\left\|\varphi'\right\|_{L^2(-l,l)}. \tag{50}$$

□

We close the section with an interpolation argument to obtain the estimates for noninteger values of the Sobolev index in the norm

$$\|f\|_{H^s}^2 = \sum_{\xi\in Z}\left(1+\xi^2\right)^s\left|\hat{f}(\xi)\right|^2, \tag{51}$$

where $\hat{f}(\xi)$ is the Fourier transform of f.

Proposition 7 (see [12]). *Let $\varphi = \varphi_n$ be the $2\pi/n$ periodic smooth solution constructed in Proposition 3 for any $s \ge 2$, there exists a constant $c_s > 0$ such that*

$$\|\varphi\|_{H^s(-\pi,\pi)}^2 \le c_s\left(\frac{1}{\delta^{s-1}}\left\|\varphi'\right\|_{L^2(-\pi,\pi)}^2 + (\delta+\epsilon)^2\right). \tag{52}$$

5. Proof of Main Theorem

Define two sequences of travelling wave solutions

$$u_n(x,t) = \varphi_n(x-t), \qquad v_n(x,t) = c_n\varphi_n(x-c_nt), \tag{53}$$

and pick

$$c_n = 1 + \frac{1}{n}. \tag{54}$$

We show that these sequences are bounded, their difference goes to zero at time zero and stays away from zero at any other time.

It is sufficient to estimate the H^s-norm of v_n since it is bigger than the H^s-norm of u_n. From (53), we have

$$\begin{aligned}
\|v_n(t)\|_{H^s(-\pi,\pi)}^2 &= \|c_n\varphi_n\|_{H^s(-\pi,\pi)}^2 \\
&\le c_n^2c_s\left(\frac{1}{\delta^{s-1}}\left\|\varphi'\right\|_{L^2(-\pi,\pi)}^2 + (\delta+\epsilon)^2\right).
\end{aligned} \tag{55}$$

Also, using (31) and the first of the estimates in (44) of Proposition 6, we find that

$$\left\|\varphi_n'\right\|_{L^2(-\pi,\pi)}^2 = n\left\|\varphi_n'\right\|_{L^2(-\pi/n,\pi/n)}^2 \le \frac{1}{\sqrt{\delta+\epsilon}}\cdot\frac{\epsilon^2}{\sqrt{\delta}}. \tag{56}$$

Combining this inequality, we get

$$\|v_n(t)\|_{H^s(-\pi,\pi)}^2 = \|c_n\varphi_n\|_{H^s(-\pi,\pi)}^2 \le c_n^2c_s\frac{\epsilon^2}{\delta^s}, \tag{57}$$

where $s \ge 2$. Note that it is already chosen $\delta^s = \epsilon^2$. Hence, both sequences of smooth solutions are bounded. Further,

$$\begin{aligned}
\|v_n(0) - u_n(0)\|_{H^s(S)}^2 &= \|c_n\varphi_n - \varphi_n\|_{H^s(S)}^2 \\
&= (c_n-1)^2\|\varphi_n\|_{H^s(S)}^2 \simeq \frac{1}{n^2} \longrightarrow 0.
\end{aligned} \tag{58}$$

Finally, the behavior at time $t > 0$ can be established as follows:

$$\begin{aligned}
&\|v_n(t) - u_n(t)\|_{H^s(S)}^2 \\
&= \sum_{\xi\in Z}\left(1+\xi^2\right)^s\left|c_n\hat{\varphi}_n(\cdot - c_nt)(\xi) - \hat{\varphi}_n(\cdot - t)(\xi)\right|^2,
\end{aligned} \tag{59}$$

where $\hat{\varphi}_n(\cdot - c_nt)(\xi)$ denotes the Fourier transform of the function $\varphi_n(x-c_nt)$ with respect to x; that is,

$$\begin{aligned}
\hat{\varphi}_n(\cdot - c_nt)(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{-ix\xi}\varphi_n(x-c_nt)\,dx \\
&= \frac{e^{-ic_nt\xi}}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{-ix\xi}\varphi_n(x)\,dx \\
&= e^{-ic_nt\xi}\hat{\varphi}_n(\xi)
\end{aligned} \tag{60}$$

after a change of variables. Therefore,

$$\begin{aligned}
&\|v_n(t) - u_n(t)\|_{H^s(S)}^2 \\
&= \sum_{\xi\in Z}\left(1+\xi^2\right)^s\left|\left(e^{-it\xi/n}-1\right) + \frac{1}{n}e^{-it\xi/n}\right|^2\left|\hat{\varphi}_n(\xi)\right|^2.
\end{aligned} \tag{61}$$

Keep only the term that corresponds to $\xi = n$ gives the inequality

$$\begin{aligned}
&\|v_n(t) - u_n(t)\|_{H^s(S)}^2 \\
&\ge \left(1+n^2\right)^s\left|\left(e^{-it}-1\right) + \frac{1}{n}e^{-it}\right|^2\left|\hat{\varphi}_n(n)\right|^2.
\end{aligned} \tag{62}$$

Since φ_n is $2\pi/n$-periodic, we have

$$\hat{\varphi}_n(n) = \frac{n}{\sqrt{2\pi}}\int_{-(\pi/n)}^{\pi/n}e^{-inx}\varphi_n(x)\,dx. \tag{63}$$

Also using the fact that $\varphi_n(x)$ is an even function and integrating by parts, we obtain

$$\begin{aligned}
\hat{\varphi}_n(n) &= \frac{2n}{\sqrt{2\pi}}\int_0^{\pi/n}\cos(nx)\varphi_n(x)\,dx \\
&= \frac{2}{\sqrt{2\pi}}\int_0^{\pi/n}\sin(nx)\varphi_n'(x)\,dx.
\end{aligned} \tag{64}$$

Therefore,

$$\|v_n(t) - u_n(t)\|_{H^s(S)}^2 \geq \frac{2}{\pi}\left(1+n^2\right)^s \left|(e^{-it}-1) + \frac{1}{n}e^{-it}\right|^2 |B_n|^2, \quad (65)$$

where

$$B_n = \int_0^{\pi/n} \sin(nx)\, \varphi_n'(x)\, dx. \quad (66)$$

Then, in the same line as Lemma 4.1 in [12] the integral for B_n can be estimated as

$$B_n \geq c_0 \epsilon. \quad (67)$$

Returning to (65) one gets

$$\|v_n(t) - u_n(t)\|_{H^s(S)}^2 \geq n^{2s}\epsilon^2 \left|(e^{-it}-1) + \frac{1}{n}e^{-it}\right|^2. \quad (68)$$

Thus, the desired estimate is obtained as in [12] using (31) and $\delta^s = \epsilon^2$.

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