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Research Article

Piecewise Convex Technique for the Stability Analysis of Delayed Neural Network

Zixin Liu,[1,2] Jian Yu,[1] Daoyun Xu,[1] and Dingtao Peng[3]

[1] *College of Computer Science and Information, Guizhou University, Guiyang 550025, China*
[2] *School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550004, China*
[3] *College of Science, Guizhou University, Guiyang 550025, China*

Correspondence should be addressed to Zixin Liu; xinxin905@163.com

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On the basis of the fact that the neuron activation function is sector bounded, this paper transforms the researched original delayed neural network into a linear uncertain system. Combined with delay partitioning technique, by using the convex combination between decomposed time delay and positive matrix, this paper constructs a novel Lyapunov function to derive new less conservative stability criteria. The benefit of the method used in this paper is that it can utilize more information on slope of the activations and time delays. To illustrate the effectiveness of the new established stable criteria, one numerical example and an application example are proposed to compare with some recent results.

1. Introduction

As a special class of nonlinear dynamical systems, neural networks (NNs) have attracted considerable attention due to their extensive applications in pattern recognition, signal processing, associative memories, combinatorial optimization, and many other fields. However, time delay is frequently encountered in NNs due to the finite switching speed of amplifier and the inherent communication time of neurons, especially in the artificial neural network. And it is often an important source of instability and oscillations. It has been shown that the existence of time delay can change the topology of neural networks, and then change the dynamic behavior of neural networks, such as oscillation and chaos. Thus, it is significant to introduce time delay into the neural network model. Additionally, stochastic disturbances and parameter uncertainties can also destroy the convergence of a neural network system. This makes the design or performance for the corresponding closed-loop systems become difficult. Therefore, the equilibrium and stability properties of NNs with time delay have been widely considered by many researchers. Up to now, various stability conditions have been obtained, and many excellent papers and monographs have been available (see [1–8]). So far, these obtained stability results are classified into two types: delay independent and delay dependent. Since sufficiently considered the information of time delays, delay-dependent criteria may be less conservative than delay-independent ones when the size of time delay is small, and much attention has been paid to the delay-dependent category [9–12]. In order to utilize more information of time delay, delay interval is always divided into two or many subintervals with the same size [13, 14]. It has been shown that delay partitioning technique is effective, and the more delay subintervals are divided, the less conservatism of stable criterion may be. However, too many delay subintervals must increase the computational burden; how to balance these two contradictions is a very important issue. To solve this problem, weighting delay and convex analysis methods are widely employed [8, 13].

Additionally, as pointed out by Li et al. [15], the choice of an appropriate Lyapunov-Krasovskii functional (LKF) and the utilization of neuron activation function's information are very important for deriving less conservative stability criteria. Thus, recently, many authors were devoted to propose a new technique to establish less conservative stable results, such as discredited LKF, augmented LKF, free-weighting matrix LKF, weighting delay LKF, and delay-slope dependent LKF.

In view of the previous discussion, one can see that, to reduce criterion's conservatism, the crucial problem is how to effectively utilize the information of time delays and neuron activation function. Motivated by the preceding discussion, this paper mainly considers the effective utilization of time delay and neuron activation function's sector bound. By using the convex representation of the neuron activation function's sector bounds, we first transform the original nonlinear delayed system into a linear uncertain system. Then, a new LKF function is constructed to derive less conservative stable criteria. Different from previous LKF, this new LKF sufficiently employs the convex combination between decomposed time delay and positive matrix. Finally, one numerical example and an application example are presented to illustrate the validity of the main results.

Notation. The notations are used in our paper unless otherwise specified. $\|\cdot\|$ denotes a vector or a matrix norm; $\mathscr{R}$ and $\mathscr{R}^n$ are real and n-dimension real number sets, respectively; $\operatorname{diag}(\cdots)$ denotes the block diagonal matrix. Real matrix $P > 0\,(<0)$ denotes that P is positive definite (negative definite). $P > Q$ denotes that $P - Q$ is positive-definite. Consider $e_i \triangleq [\overbrace{0,\ldots,0,\underbrace{I}_{i},0,\ldots,0}^{11}]^T$ $(i = 1, 2, \ldots, 11)$, where I denotes identity matrix.

2. Preliminaries

Consider the following delayed neural networks:

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x(t-\tau(t))) + u, \quad (1)$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ denotes the neural state vector; $g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T$ denotes the neuron activation function; $u = [u_1, u_2, \ldots, u_n]^T$ is external input vector; $C = \operatorname{diag}(C_1, C_2, \ldots, C_n)$ with $C_i > 0$ describes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ represent the weighting and delayed weighting matrices, respectively; $g(x(t-\tau(t))) = [g_1(x_1(t-\tau(t))), g_2(x_2(t-\tau(t))), \ldots, g_n(x_n(t-\tau(t)))]^T$, $\tau(t)$ is time-varying continuous function which satisfies $0 \le \tau_l \le \tau(t) \le \tau_u$, $b_1 \le \dot{\tau}(t) \le b_2$, where τ_l, τ_u, b_1, and b_2 are given constants. Additionally, we always assume that each neuron activation function $g_i(\cdot)$ satisfies condition: $l_i^- \le (g_i(x)-g_i(y))/(x-y) \le l_i^+$, for all $x, y \in \mathscr{R}$, $x \ne y$, $i = 1, 2, \ldots, n$, where l_i^- and l_i^+ are known constant scalars. As pointed out in [16], under this assumption, system (1) has equilibrium point. Assume that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is an equilibrium point of system (1), and set $y_i(t) = x_i(t) - x_i^*$, $f_i(y_i(t)) = g_i(y_i(t) + x_i^*) - g_i(x_i^*)$. Then, system (1) can be transformed into the following form:

$$\frac{dy(t)}{dt} = -Cy(t) + Af(y(t)) + Bf(y(t-\tau(t))), \quad (2)$$

where $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$, $f(y(t)) = [f_1(y_1(t)), f_2(y_2(t)), \ldots, f_n(y_n(t))]^T$. From the previous assumption, for any $y_i(t) \in \mathscr{R}$, $y_i(t) \ne 0$, function $f_i(\cdot)$ satisfies $l_i^- \le (f_i(y_i(t)))/(y_i(t)) \le l_i^+$, $f_i(0) = 0$, $i = 1, 2, \ldots, n$.

Notice that the nonlinear function $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ can be rewritten as a convex combination form with the sector bounds as follows:

$$\begin{aligned} f_i(y_i(t)) &= (\lambda_i(y_i(t)) l_i^- + (1-\lambda_i(y_i(t))) l_i^+)\, y_i(t), \\ f_i(y_i(t-\tau(t))) &= (\overline{\lambda}_i(y_i(t-\tau(t))) l_i^- \\ &\quad + (1-\overline{\lambda}_i(y_i(t-\tau(t)))) l_i^+)\, y_i(t-\tau(t)), \end{aligned} \quad (3)$$

where $\lambda_i(y_i) = (f_i(y_i(t)) - l_i^- y_i(t))/(l_i^+ - l_i^-)y_i(t)$, $\overline{\lambda}_i(y_i(t-\tau(t))) = (f_i(y_i(t-\tau(t))) - l_i^- y_i(t-\tau(t)))/(l_i^+ - l_i^-)y_i(t-\tau(t))$ satisfying $0 \le \lambda_i(y_i(t)), \overline{\lambda}_i(y_i(t-\tau(t))) \le 1$. Namely, $f_i(y(t)) = \Lambda_i(y_i(t))y_i(t)$, $f_i(y(t-\tau(t))) = \overline{\Lambda}_i(y_i(t-\tau(t)))y_i(t-\tau(t))$, where $\Lambda_i(y_i(t)), \overline{\Lambda}_i(y_i(t-\tau(t)))$, are elements of a convex hull $\operatorname{Co}\{l_i^-, l_i^+\}$.

Set $L = \operatorname{diag}(l_1, l_2, \ldots, l_n)$, where $l_i = \max\{|l_i^-|, |l_i^+|\}$. Obviously, $-1 \le (\lambda_i(y_i(t))l_i^- + (1-\lambda_i(y_i(t)))l_i^+)/l_i, (\overline{\lambda}_i(y_i(t-\tau(t)))l_i^- + (1-\overline{\lambda}_i(y_i(t-\tau(t))))l_i^+)/l_i \le 1$.

Define $\Delta_i = (\lambda_i(y_i(t))l_i^- + (1-\lambda_i(y_i(t)))l_i^+)/l_i$, $\overline{\Delta}_i = (\overline{\lambda}_i(y_i(t-\tau(t)))l_i^- + (1-\overline{\lambda}_i(y_i(t-\tau(t))))l_i^+)/l_i$, $\Delta = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$, $\overline{\Delta} = \operatorname{diag}(\overline{\Delta}_1, \overline{\Delta}_2, \ldots, \overline{\Delta}_n)$; then nonlinearities $f(y(t))$ and $f(y(t-\tau(t)))$ can be expressed as $f(y(t)) = L\Delta y(t)$, $f(y(t-\tau(t))) = L\overline{\Delta}y(t-\tau(t))$, where Δ and $\overline{\Delta}$ satisfy $\Delta^T\Delta \le I$, $\overline{\Delta}^T\overline{\Delta} \le I$. And the system (2) can be rewritten as the following delayed uncertain system:

$$\dot{y}(t) = (-C + AL\Delta)\, y(t) + BL\overline{\Delta} y(t-\tau(t)). \quad (4)$$

Remark 1. Different from previous work, in this paper, by using the convex expression $f_i(y(t)) = \Lambda_i(y_i(t))y_i(t)$, $f_i(y(t-\tau(t))) = \overline{\Lambda}_i(y_i(t-\tau(t)))y_i(t-\tau(t))$, we transform the original nonlinear system (2) into a linear system with parameter uncertain system (4). As a result, the stability problem of delayed neural network system (1) can be transformed into the robust stability problem of uncertain linear system (4).

Let $L^- = \operatorname{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \operatorname{diag}(l_1^+, l_2^+, \ldots, l_n^+)$. For further discussion, the following lemmas are needed.

Lemma 2 (see [16]). *Let $f_1, f_2, \ldots, f_N : \mathscr{R}^m \mapsto \mathscr{R}$ have positive values in an open subset D of $\mathscr{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies*

$$\lim_{\{\alpha_i \mid \alpha_i > 0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i \ne j} g_{i,j}(t) \quad (5)$$

subject to

$$\left\{ g_{i,j} : \mathscr{R}^m \longmapsto \mathscr{R}, g_{j,i}(t) \triangleq g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}. \quad (6)$$

Lemma 3 (see [17]). *Given the symmetric matrix P_1 and any real matrices P_2, P_3 of appropriate dimensions, then*

$$P_1 + P_2\Delta P_3 + P_3^T\Delta^T P_2^T < 0 \tag{7}$$

for all $\Delta \in \Theta$ satisfying $\Delta^T\Delta \le I$ if and only if there exists $S \in S_\Delta$ such that

$$\begin{bmatrix} P_1 + P_3^T S P_3 & P_2 \\ P_2^T & -S \end{bmatrix} < 0, \tag{8}$$

where $S_\Delta =: \{\mathrm{diag}(s_1 I, \ldots, s_k I, S_1, \ldots, S_l) : S_i > 0, k, l \in N\}$.

Lemma 4 ([18], Jensen inequality). *Consider two scalars $a < b$ and a positive definite matrix $R \in \mathscr{R}^{n\times n}$. For any conditions function $\omega : [a,b] \to \mathscr{R}^n$ and any strictly positive condition $f : [a,b] \to \mathscr{R}$, the following inequality holds:*

$$\int_a^b w^T(s) f(s) R w(s)\, ds \ge \left(\int_a^b w(s)\, ds\right)^T \left(\int_a^b (f(s))^{-1} ds\right)^{-1} R \left(\int_a^b w(s)\, ds\right). \tag{9}$$

Lemma 5 (see [19]). *For any constant matrix $Z \in \mathscr{R}^{n\times n}$, $Z = Z^T > 0$, scalars $h_2 > h_1 > 0$, such that the following integrations are well defined; then*

$$\begin{aligned} &-\frac{1}{2}\left(h_2^2 - h_1^2\right)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} x^T(s) Z x(s)\, ds\, d\theta \\ &\le -\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} x^T(s)\, ds\, d\theta Z \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} x(s)\, ds\, d\theta. \end{aligned} \tag{10}$$

3. Stability Analysis

Set $\overline{\tau} = (\tau_u + \tau_l)/2$, $\underline{\tau} = (\tau_u - \tau_l)/2$, and $Q_1, Q_2, Q_3, P_1 > P_2$, $P_3 > P_4$, $P_5 > P_6$ are positive matrices. Let

$$\begin{aligned} Q(\tau(t)) &= \begin{cases} \dfrac{\tau(t) - \tau_l}{\underline{\tau}} Q_1 + \dfrac{\overline{\tau} - \tau(t)}{\underline{\tau}} Q_2 : \tau_l \le \tau(t) \le \overline{\tau}, \\ \dfrac{\tau(t) - \overline{\tau}}{\underline{\tau}} Q_3 + \dfrac{\tau_u - \tau(t)}{\underline{\tau}} Q_1 : \overline{\tau} < \tau(t) \le \tau_u, \end{cases} \\ P(s) &= \begin{cases} \dfrac{s - t + \tau_u}{\underline{\tau}} P_1 + \dfrac{t - \overline{\tau} - s}{\underline{\tau}} P_2 : t - \tau_u \le s < t - \overline{\tau}, \\ \dfrac{s - t + \overline{\tau}}{\underline{\tau}} P_3 + \dfrac{t - \tau_l - s}{\underline{\tau}} P_4 : t - \overline{\tau} \le s < t - \tau_l, \\ \dfrac{s - t + \tau_l}{\tau_l} P_5 + \dfrac{t - s}{\tau_l} P_6 : t - \tau_l \le s < t. \end{cases} \end{aligned} \tag{11}$$

Consider a new class of Lyapunov functional candidate as follows:

$$V(y(t)) = V_1(y(t)) + V_2(y(t)), \tag{12}$$

where

$$\begin{aligned} V_1(y(t)) &= y^T(t) Q(\tau(t)) y(t), V_2(y(t)) \\ &= \int_{\tau_l}^{\tau_u}\int_{t-\theta}^{t} \dot{y}^T(s) P(s) \dot{y}(s)\, ds\, d\theta. \end{aligned} \tag{13}$$

Define $V(t, x_t, \dot{x}_t) = \limsup_{s\to 0^+}(1/s)[V(t+s, x_{t+s}, \dot{x}_{t+s}) - V(t, x_t, \dot{x}_t)]$; we can obtain $\dot{V}_1(y(t)) = y^T(t)\dot{Q}(\tau(t)y(t)) + 2\dot{y}^T(t)[\chi(\tau(t))[((\tau(t) - \tau_l)/\underline{\tau})Q_1 + ((\overline{\tau} - \tau(t))/\underline{\tau})Q_2] + (1 - \chi(\tau(t)))[((\tau(t) - \overline{\tau})/\underline{\tau})Q_3 + ((\tau_u - \tau(t))/\underline{\tau})Q_1]]y(t)$, where $\dot{Q}(\tau(t))_{|\tau \ne \underline{\tau}} = \dot{\tau}(t)[\chi(\tau(t))((Q_1 - Q_2)/\underline{\tau}) + (1 - \chi(\tau(t)))((Q_3 - Q_1)/\underline{\tau})]$,

$$\chi(\tau(t)) = \begin{cases} 1 : \text{ if } \tau(t) \in [\tau_l, \underline{\tau}] \\ 0 : \text{ otherwise.} \end{cases} \tag{14}$$

Notice that

$$\begin{aligned} V_2(y(t)) &= \int_{t-\tau_l}^{t}\int_{\tau_l}^{\tau_u} \dot{y}^T(s) P(s) \dot{y}(s)\, d\theta\, ds \\ &\quad + \int_{t-\tau_u}^{t-\tau_l}\int_{t-s}^{\tau_u} \dot{y}^T(s) P(s) \dot{y}(s)\, d\theta\, ds \\ &= 2\underline{\tau}\int_{t-\tau_l}^{t} \dot{y}^T(s)\left[\frac{s - t + \tau_l}{\tau_l} P_5 + \frac{t - s}{\tau_l} P_6\right]\dot{y}(s)\, ds \\ &\quad + \int_{t-\tau_u}^{t-\overline{\tau}} (\tau_u + s - t)\, \dot{y}^T(s) \\ &\qquad \times \left[\frac{s - t + \tau_u}{\underline{\tau}} P_1 + \frac{t - \overline{\tau} - s}{\underline{\tau}} P_2\right]\dot{y}(s)\, ds \\ &\quad + \int_{t-\overline{\tau}}^{t-\tau_l} (\tau_u + s - t)\, \dot{y}^T(s) \\ &\qquad \times \left[\frac{s - t + \overline{\tau}}{\underline{\tau}} P_3 + \frac{t - \tau_l - s}{\underline{\tau}} P_4\right]\dot{y}(s)\, ds. \end{aligned} \tag{15}$$

Set $\xi^T(t) = [y^T(t), y^T(t-\tau(t)), y^T(t-\tau_l), y^T(t-\tau_u), y^T(t-\overline{\tau}), \dot{y}^T(t), \dot{y}^T(t-\tau_l), \dot{y}^T(t-\overline{\tau}), \int_{t-\overline{\tau}}^{t-\tau(t)} y^T(s)ds, \int_{t-\tau(t)}^{t-\tau_l} y^T(s)ds, \int_{t-\tau_u}^{t-\overline{\tau}} y^T(s)ds, \int_{t-\overline{\tau}}^{t-\tau_l} y^T(s)ds]$. It yields that

$$\begin{aligned} \dot{V}_2(y(t)) &= \xi^T(t)\left[2\underline{\tau} e_6^T P_5 e_6 - e_7^T\left(2\underline{\tau} P_6 - 2\underline{\tau} P_3\right) e_7 \right. \\ &\qquad \left. - e_8^T\left(\underline{\tau} P_4 - \underline{\tau} P_1\right) e_8\right]\xi(t) \\ &\quad - \int_{t-\tau_u}^{t-\overline{\tau}} \dot{y}^T(s)\left[\frac{s - t + \tau_u}{\underline{\tau}} P_1 + \frac{t - \overline{\tau} - s}{\underline{\tau}} P_2\right]\dot{y}(s)\, ds \\ &\quad - \frac{1}{\underline{\tau}}\int_{t-\tau_u}^{t-\overline{\tau}} (\tau_u + s - t)\, \dot{y}^T(s)\left[P_1 - P_2\right]\dot{y}(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&-\int_{t-\overline{\tau}}^{t-\tau_l} \dot{y}^T(s)\left[\frac{s-t+\overline{\tau}}{\underline{\tau}}P_3+\frac{t-\tau_l-s}{\underline{\tau}}P_4\right]\dot{y}(s)\,ds \\
&-\frac{1}{\underline{\tau}}\int_{t-\overline{\tau}}^{t-\tau_l}(\tau_u+s-t)\,\dot{y}^T(s)\left[P_3-P_4\right]\dot{y}(s)\,ds \\
&-\frac{2\underline{\tau}}{\tau_l}\int_{t-\tau_l}^{t}\dot{y}^T(s)\left[P_5-P_6\right]\dot{y}(s)\,ds \\
&\triangleq \xi^T(t)\left[2\underline{\tau}e_6^TP_5e_6-e_7^T\left(2\underline{\tau}P_6-2\underline{\tau}P_3\right)e_6\right. \\
&\qquad\left.-e_8^T\left(\underline{\tau}P_4-\underline{\tau}P_1\right)e_8\right]\xi(t) \\
&+I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{16}
$$

We start with the case $\tau(t)\in[\tau_l,\overline{\tau}]$, where $\chi(\tau(t))=1$. Therefore

$$
\begin{aligned}
\dot{V}_1(y(t))=\xi^T(t)&\left[\frac{\dot{\tau}(t)}{\underline{\tau}}e_1^T(Q_1-Q_2)e_1^T\right. \\
&\left.+2e_6^T\left(\frac{\tau(t)-\tau_l}{\underline{\tau}}Q_1+\frac{\overline{\tau}-\tau(t)}{\underline{\tau}}Q_2\right)e_1\right] \\
&\times\xi(t).
\end{aligned}
\tag{17}
$$

Notice that $I_1=-(1/\underline{\tau})[\int_{t-\tau_u}^{t-\overline{\tau}}\int_{t-s}^{\tau_u}\dot{y}^T(s)P_1\dot{y}(s)d\theta\,ds+\int_{t-\tau_u}^{t-\overline{\tau}}\int_{s-t}^{-\overline{\tau}}\dot{y}^T(s)P_2\dot{y}(s)d\theta\,ds]=-(1/\underline{\tau})[\int_{\overline{\tau}}^{\tau_u}\int_{t-\theta}^{t-\overline{\tau}}\dot{y}^T(s)P_1\dot{y}(s)ds\,d\theta+\int_{-\tau_u}^{-\overline{\tau}}\int_{t-\tau_u}^{t+\theta}\dot{y}^T(s)P_2\dot{y}(s)d\theta\,ds]$. $I_2=-(1/\underline{\tau})\int_{t-\tau_u}^{t-\overline{\tau}}\int_{t-s}^{\tau_u}\dot{y}^T(s)[P_1-P_2]\dot{y}(s)d\theta\,ds=-(1/\underline{\tau})\int_{\overline{\tau}}^{\tau_u}\int_{t-\theta}^{t-\overline{\tau}}\dot{y}^T(s)[P_1-P_2]\dot{y}(s)ds\,d\theta$, $\int_{t-\tau_u}^{t-\overline{\tau}}(s-t+\tau_u)\dot{y}(s)ds=(\tau_u-\overline{\tau})y(t-\overline{\tau})-\int_{t-\tau_u}^{t-\overline{\tau}}y(s)ds$, $\int_{t-\tau_u}^{t-\overline{\tau}}(t-\overline{\tau}-s)\dot{y}(s)ds=-(\tau_u-\overline{\tau})y(t-\tau_u)+\int_{t-\tau_u}^{t-\overline{\tau}}y(s)ds$; from Lemma 5, it yields that

$$
\begin{aligned}
I_1+I_2\le\;&-\frac{4}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\xi^T(t)\left[\left(\tau_u-\overline{\tau}\right)e_5^T-e_{11}^T\right] \\
&\times P_1\left[\left(\tau_u-\overline{\tau}\right)e_5-e_{11}\right]\xi(t) \\
&-\frac{2}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\xi^T(t)\left[e_{11}^T-\left(\tau_u-\overline{\tau}\right)e_4^T\right] \\
&\times P_2\left[e_{11}-\left(\tau_u-\overline{\tau}\right)e_4\right]\xi(t) \\
&+\frac{2}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\xi^T(t)\left[\left(\tau_u-\overline{\tau}\right)e_5^T-e_{11}^T\right] \\
&\times P_2\left[\left(\tau_u-\overline{\tau}\right)e_5-e_{11}\right]\xi(t).
\end{aligned}
\tag{18}
$$

Notice that

$$
\begin{aligned}
I_3&=-\frac{1}{\underline{\tau}}\left[\int_{t-\overline{\tau}}^{t-\tau(t)}\int_{t-s}^{\overline{\tau}}\dot{y}^T(s)P_3\dot{y}(s)\,d\theta\,ds\right. \\
&\qquad\left.+\int_{t-\overline{\tau}}^{t-\tau(t)}\int_{s-t}^{-\tau_l}\dot{y}^T(s)P_4\dot{y}(s)\,d\theta\,ds\right] \\
&\quad-\frac{1}{\underline{\tau}}\left[\int_{t-\tau(t)}^{t-\tau_l}\int_{t-s}^{\overline{\tau}}\dot{y}^T(s)P_3\dot{y}(s)\,d\theta\,ds\right. \\
&\qquad\left.+\int_{t-\tau(t)}^{t-\tau_l}\int_{s-t}^{-\tau_l}\dot{y}^T(s)P_4\dot{y}(s)\,d\theta\,ds\right] \\
&=-\frac{1}{\underline{\tau}}\left[\int_{\tau(t)}^{\overline{\tau}}\int_{t-\theta}^{t-\tau(t)}\dot{y}^T(s)P_3\dot{y}(s)\,ds\,d\theta\right. \\
&\qquad\left.+\int_{-\overline{\tau}}^{-\tau(t)}\int_{t-\overline{\tau}}^{t+\theta}\dot{y}^T(s)P_4\dot{y}(s)\,ds\,d\theta\right] \\
&\quad-\frac{1}{\underline{\tau}}\left[\int_{\tau_l}^{\tau(t)}\int_{t-\theta}^{t-\tau_l}\dot{y}^T(s)P_3\dot{y}(s)\,ds\,d\theta\right. \\
&\qquad\left.+\int_{-\tau(t)}^{-\tau_l}\int_{t-\tau(t)}^{t+\theta}\dot{y}^T(s)P_4\dot{y}(s)\,ds\,d\theta\right].
\end{aligned}
\tag{19}
$$

Thus, from Lemma 5, it yields that

$$
\begin{aligned}
I_3\le\;&-\frac{2}{\underline{\tau}\left(\overline{\tau}^2-\tau(t)^2\right)}\xi^T(t)\left[\left(\overline{\tau}-\tau(t)\right)e_2^T-e_9^T\right] \\
&\times P_3\left[\left(\overline{\tau}-\tau(t)\right)e_2-e_9\right]\xi(t) \\
&-\frac{2}{\underline{\tau}\left(-\tau(t)^2\right)}\xi^T(t)\left[-\left(\overline{\tau}-\tau_l\right)e_5^T+e_9^T\right] \\
&\times P_4\left[-\left(\overline{\tau}-\tau_l\right)e_5+e_9\right]\xi(t) \\
&-\frac{2}{\underline{\tau}\left(\tau(t)^2-\tau_l^2\right)}\xi^T(t)\left[\left(\overline{\tau}-\tau_l\right)e_3^T-e_{10}^T\right] \\
&\times P_3\left[\left(\overline{\tau}-\tau_l\right)e_3-e_{10}\right]\xi(t) \\
&-\frac{2}{\underline{\tau}\left(\tau(t)^2-\tau_l^2\right)}\xi^T(t)\left[-\left(\tau(t)-\tau_l\right)e_2^T+e_{10}^T\right] \\
&\times P_4\left[-\left(\tau(t)-\tau_l\right)e_2+e_{10}\right]\xi(t).
\end{aligned}
\tag{20}
$$

Set $E_1=\begin{bmatrix}(\overline{\tau}-\tau(t))e_2^T-e_9^T\\ -(\overline{\tau}-\tau_l)e_5^T+e_9^T\\ (\overline{\tau}-\tau_l)e_3^T-e_{10}^T\\ -(\tau(t)-\tau_l)e_2^T+e_{10}^T\end{bmatrix}$, $\widetilde{P}=1/(\underline{\tau}^2(\overline{\tau}+\tau_l))\begin{bmatrix}P_3 & S_{1,2} & S_{1,3} & S_{1,4}\\ * & P_4 & S_{2,2} & S_{2,3}\\ * & * & P_3 & S_{3,3}\\ * & * & * & P_4\end{bmatrix}$, where $S_{i,j}$ $(1\le i\le j\le 3)$ are arbitrary matrices. If $\widetilde{P}>0$, since $(\overline{\tau}^2-\tau(t)^2)/(\underline{\tau}(\overline{\tau}+\tau_l))+(\tau(t)^2-\tau_l^2)/(\underline{\tau}(\overline{\tau}+\tau_l))=1$, from Lemma 2, one can obtain

$$
I_3\le-\xi^T(t)E_1^T\widetilde{P}E_1\xi(t).
\tag{21}
$$

Note that inequality (21) holds for all $\tau(t) \in [\tau_l, \overline{\tau}]$. When $\tau(t) = \tau_l$ and $\overline{\tau}$, inequality (21) still holds; it yields that

$$I_3 \leq -\xi^T(t)\left(E_1^{(1)}\right)^T \widetilde{P} E_1^{(1)} \xi(t), \tag{22a}$$

$$I_3 \leq -\xi^T(t)\left(E_1^{(2)}\right)^T \widetilde{P} E_1^{(2)} \xi(t), \tag{22b}$$

where

$$E_1^{(1)} = \begin{bmatrix} (\overline{\tau}-\tau_l)e_3^T - e_9^T \\ -(\overline{\tau}-\tau_l)e_5^T + e_9^T \\ (\overline{\tau}-\tau_l)e_3^T - e_{10}^T \\ e_{10}^T \end{bmatrix}, \qquad E_1^{(2)} = \begin{bmatrix} -e_9^T \\ -(\overline{\tau}-\tau_l)e_5^T + e_9^T \\ (\overline{\tau}-\tau_l)e_3^T - e_{10}^T \\ -(\overline{\tau}-\tau_l)e_5^T + e_{10}^T \end{bmatrix}. \tag{23}$$

Using the fact that $I_4 = -(1/\underline{\tau})[\int_{t-\overline{\tau}}^{t-\tau(t)}(\tau_u+s-t)\dot{y}^T(s)[P_3-P_4]\dot{y}(s)ds + \int_{t-\tau(t)}^{t-\tau_l}(\tau_u+s-t)\dot{y}^T(s)[P_3-P_4]\dot{y}(s)ds]$ and τ_u+s-t is a strictly positive continuous function on $[t-\overline{\tau}, t-\tau_l]$, by Lemma 4, one can obtain

$$\begin{aligned} I_4 \leq & -\frac{1}{\underline{\tau}\ln 2}\xi^T(t) \\ & \times \left[\left[e_2^T - e_5^T\right][P_3-P_4][e_2-e_5] \right. \\ & \left. + \left[e_3^T - e_2^T\right][P_3-P_4][e_3-e_2]\right]\xi(t). \end{aligned} \tag{24}$$

$$I_5 \leq -\frac{2\underline{\tau}}{\tau_l^2}\xi^T(t)\left[e_1^T - e_3^T\right][P_5-P_6][e_1-e_3]\xi(t).$$

Furthermore, from (4), for arbitrary matrices M_1 and M_2 of appropriate dimensions, the following equalities hold:

$$\begin{aligned} & 2\xi^T(t)e_1^T M_1[-Ce_1-e_6]\xi(t) \\ & \quad + \xi^T(t)e_1^T M_1\left[AL\Delta e_1 + BL\overline{\Delta}e_2\right]\xi(t) = 0, \end{aligned} \tag{25a}$$

$$\begin{aligned} & 2\xi^T(t)e_6^T M_2[-Ce_1-e_6]\xi(t) \\ & \quad + \xi^T(t)e_6^T M_2\left[AL\Delta e_1 + BL\overline{\Delta}e_2\right]\xi(t) = 0. \end{aligned} \tag{25b}$$

Define

$$\begin{aligned} & E_2^{(1)} = 2e_6^T Q_2 e_1, \qquad E_2^{(2)} = 2e_6^T Q_1 e_1, \\ & E_2^{(3)} = 2e_6^T\left(\frac{\tau(t)-\tau_l}{\underline{\tau}}Q_1 + \frac{\overline{\tau}-\tau(t)}{\underline{\tau}}Q_2\right)e_1, \end{aligned} \tag{26a}$$

$$\begin{aligned} & E_3^{(1)} = \frac{b_1}{\underline{\tau}}e_1^T(Q_1-Q_2)e_1, \qquad E_3^{(2)} = \frac{b_2}{\underline{\tau}}e_1^T(Q_1-Q_2)e_1, \\ & E_3^{(3)} = \frac{\dot{\tau}(t)}{\underline{\tau}}e_1^T(Q_1-Q_2)e_1. \end{aligned} \tag{26b}$$

Therefore

$$\begin{aligned} E^1 = {} & 2\underline{\tau}e_6^T P_5 e_6 - e_7^T\left(2\underline{\tau}P_6 - 2\underline{\tau}P_3\right)e_6 \\ & - e_8^T\left(\underline{\tau}P_4 - \underline{\tau}P_1\right)e_8 \\ & - \frac{4}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\left[(\tau_u-\overline{\tau})e_5^T - e_{11}^T\right] \\ & \times P_1\left[(\tau_u-\overline{\tau})e_5 - e_{11}\right] \\ & - \frac{2}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\left[e_{11}^T - (\tau_u-\overline{\tau})e_4^T\right] \\ & \times P_2\left[e_{11} - (\tau_u-\overline{\tau})e_4\right] \\ & + \frac{2}{\underline{\tau}\left(\tau_u^2-\overline{\tau}^2\right)}\left[(\tau_u-\overline{\tau})e_5^T - e_{11}^T\right] \\ & \times P_2\left[(\tau_u-\overline{\tau})e_5 - e_{11}\right] \\ & - \frac{1}{\underline{\tau}\ln 2}\left[\left[e_2^T-e_5^T\right][P_3-P_4][e_2-e_5]\right. \\ & \left. + \left[e_3^T-e_2^T\right][P_3-P_4][e_3-e_2]\right] \\ & - \frac{2\underline{\tau}}{\tau_l^2}\left[e_1^T-e_3^T\right][P_6-P_5][e_1-e_3] \\ & + e_1^T M_1[-Ce_1-e_6] + e_6^T M_2[-Ce_1-e_6] \\ & + [-Ce_1-e_6]^T M_1^T e_1 + [-Ce_1-e_6]^T M_2^T e_6. \end{aligned} \tag{27}$$

Moreover

$$\Phi = \left[e_1^T, e_2^T, e_1^T, e_2^T\right]^T, \qquad \widetilde{\Delta} = \operatorname{diag}\left\{\Delta, \Delta, \overline{\Delta}, \overline{\Delta}\right\}, \tag{28a}$$

$$\Psi = \left[e_1^T M_1 AL, e_1^T M_1 BL, e_6^T M_2 AL, e_6^T M_2 BL\right]. \tag{28b}$$

From (16)–(28b), we get that, along (4),

$$\dot{V}(y(t)) \leq \alpha_1\|x(t)\|^2, \tag{29}$$

for some scalar $\alpha_1 > 0$ if

$$\Pi_1 = E^1 + E_2^{(3)} + E_3^{(3)} - (E_1)^T\widetilde{P}E_1 + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0. \tag{30}$$

Inequality (30) leads, for $\tau(t) = \tau_l, \overline{\tau}$, to the following two inequalities:

$$\Pi_1^{(1)} = E^1 + E_2^{(1)} + E_3^{(3)} - \left(E_1^{(1)}\right)^T\widetilde{P}E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{31a}$$

$$\Pi_2^{(1)} = E^1 + E_2^{(2)} + E_3^{(3)} - \left(E_1^{(1)}\right)^T\widetilde{P}E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0. \tag{31b}$$

Inequalities (31a) and (31b) imply (30), because $\Pi_1 = ((\tau(t)-\tau_l)/\underline{\tau})\Pi_2^{(1)} + ((\overline{\tau}-\tau(t))/\underline{\tau})\Pi_1^{(1)}$ is convex in $\tau(t) \in [\tau_l, \overline{\tau}]$.

At the same time, inequalities (31a) and (31b) lead, for $\dot{\tau}(t) = b_1, b_2$, to the following inequalities:

$$\Pi_{(1)} \triangleq E^1 + E_2^{(1)} + E_3^{(1)} - \left(E_1^{(1)}\right)^T \widetilde{P} E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{32a}$$

$$\Pi_{(2)} \triangleq E^1 + E_2^{(1)} + E_3^{(2)} - \left(E_1^{(1)}\right)^T \widetilde{P} E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{32b}$$

$$\Pi_{(3)} \triangleq E^1 + E_2^{(2)} + E_3^{(1)} - \left(E_1^{(1)}\right)^T \widetilde{P} E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{32c}$$

$$\Pi_{(4)} \triangleq E^1 + E_2^{(2)} + E_3^{(2)} - \left(E_1^{(1)}\right)^T \widetilde{P} E_1^{(1)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0. \tag{32d}$$

Obviously, inequalities (32a)–(32d) imply (31a) and (31b) since $\Pi_1^{(1)} = ((b_2 - \dot{\tau})/(b_2 - b_1))\Pi_{(2)} + ((\dot{\tau} - b_1)/(b_2 - b_1))\Pi_{(1)}$, and $\Pi_2^{(1)} = ((b_2 - \dot{\tau})/(b_2 - b_1))\Pi_{(4)} + ((\dot{\tau} - b_1)/(b_2 - b_1))\Pi_{(3)}$ are convex in $\dot{\tau}(t) \in [b_1, b_2]$.

Similarly, there exists scalar $\alpha_2 > 0$ such that

$$\dot{V}\left(y(t)\right) \le \alpha_2 \|x(t)\|^2, \tag{33}$$

if

$$\overline{\Pi}_{(1)} \triangleq E^1 + E_2^{(1)} + E_3^{(1)} - \left(E_1^{(2)}\right)^T \widetilde{P} E_1^{(2)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{34a}$$

$$\overline{\Pi}_{(2)} \triangleq E^1 + E_2^{(1)} + E_3^{(2)} - \left(E_1^{(2)}\right)^T \widetilde{P} E_1^{(2)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{34b}$$

$$\overline{\Pi}_{(3)} \triangleq E^1 + E_2^{(2)} + E_3^{(1)} - \left(E_1^{(2)}\right)^T \widetilde{P} E_1^{(2)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0, \tag{34c}$$

$$\overline{\Pi}_{(4)} \triangleq E^1 + E_2^{(2)} + E_3^{(2)} - \left(E_1^{(2)}\right)^T \widetilde{P} E_1^{(2)} + \Psi\widetilde{\Delta}\Phi + \Phi^T\widetilde{\Delta}\Psi^T < 0. \tag{34d}$$

Similarly, set $\widetilde{\xi}^T(t) = [y^T(t), y^T(t-\tau(t)), y^T(t-\tau_l), y^T(t-\tau_u), y^T(t-\overline{\tau}), \dot{y}^T(t), \dot{y}^T(t-\tau_l), \dot{y}^T(t-\overline{\tau}), \int_{t-\tau_u}^{t-\tau(t)} y^T(s)ds, \int_{t-\tau(t)}^{t-\overline{\tau}} y^T(s)ds, \int_{t-\tau_u}^{t-\overline{\tau}} y^T(s)ds, \int_{t-\overline{\tau}}^{t-\tau_l} y^T(s)ds]$.

If $\tau(t) \in (\overline{\tau}, \tau_u]$, then, $\chi(\tau(t)) = 0$.

Therefore

$$\dot{V}_1\left(y(t)\right) = \widetilde{\xi}^T(t)\left[\frac{\dot{\tau}(t)}{\underline{\tau}} e_1^T \left(Q_3 - Q_1\right) e_1 + 2e_6^T \left(\frac{\tau(t) - \overline{\tau}}{\underline{\tau}} Q_3 + \frac{\tau_u - \tau(t)}{\underline{\tau}} Q_1\right) e_1\right] \times \widetilde{\xi}(t). \tag{35}$$

Moreover

$$\begin{aligned} I_1 = &-\frac{1}{\underline{\tau}} \int_{t-\tau_u}^{t-\tau(t)} \int_{t-s}^{\tau_u} \dot{y}^T(s) P_1 \dot{y}(s)\, d\theta\, ds \\ &-\frac{1}{\underline{\tau}} \int_{t-\tau(t)}^{t-\overline{\tau}} \int_{t-s}^{\tau_u} \dot{y}^T(s) P_1 \dot{y}(s)\, d\theta\, ds \\ &-\frac{1}{\underline{\tau}} \int_{t-\tau_u}^{t-\tau(t)} \int_{s-t}^{-\overline{\tau}} \dot{y}^T(s) P_2 \dot{y}(s)\, d\theta\, ds \\ &-\frac{1}{\underline{\tau}} \int_{t-\tau(t)}^{t-\overline{\tau}} \int_{s-t}^{-\overline{\tau}} \dot{y}^T(s) P_2 \dot{y}(s)\, d\theta\, ds. \\ I_2 = &-\frac{1}{\underline{\tau}} \int_{t-\tau_u}^{t-\tau(t)} \int_{t-s}^{\tau_u} \dot{y}^T(s) \left(P_1 - P_2\right) \dot{y}(s)\, d\theta\, ds \\ &-\frac{1}{\underline{\tau}} \int_{t-\tau(t)}^{t-\overline{\tau}} \int_{t-s}^{\tau_u} \dot{y}^T(s) \left(P_1 - P_2\right) \dot{y}(s)\, d\theta\, ds. \end{aligned} \tag{36}$$

Set

$$\widetilde{E}_1^T = \begin{bmatrix} \left(\tau_u - \tau(t)\right) e_2^T - e_9^T \\ \left(\tau(t) - \overline{\tau}\right) e_5^T - e_{10}^T \\ -\left(\tau_u - \tau(t)\right) e_4^T + e_9^T \\ -\left(\tau(t) - \overline{\tau}\right) e_2^T + e_{10}^T \\ \left(\tau_u - \tau(t)\right) e_2^T - e_9^T \\ \left(\tau(t) - \overline{\tau}\right) e_5^T - e_{10}^T \end{bmatrix},$$

$$\widetilde{\widetilde{P}} = \frac{2}{3\overline{\tau}^2\left(\tau_u + \overline{\tau}\right)} \begin{bmatrix} P_1 & \widetilde{S}_{1,2} & \widetilde{S}_{1,3} & \widetilde{S}_{1,4} & \widetilde{S}_{1,5} & \widetilde{S}_{1,6} \\ * & P_1 & \widetilde{S}_{2,2} & \widetilde{S}_{2,3} & \widetilde{S}_{2,4} & \widetilde{S}_{2,5} \\ * & * & P_2 & \widetilde{S}_{3,3} & \widetilde{S}_{3,4} & \widetilde{S}_{3,5} \\ * & * & * & P_2 & \widetilde{S}_{4,4} & \widetilde{S}_{4,5} \\ * & * & * & * & P_1 - P_2 & \widetilde{S}_{5,5} \\ * & * & * & * & * & P_1 - P_2 \end{bmatrix}, \tag{37}$$

where $\widetilde{S}_{i,j}$ $(1 \le i \le j \le 5)$ are arbitrary matrices. If $\widetilde{\widetilde{P}} > 0$, from Lemmas 5 and 2, similar to the proof of (19)–(21), one can obtain

$$I_1 + I_2 \le -\widetilde{\xi}^T(t)\, \widetilde{E}_1^T \widetilde{\widetilde{P}} \widetilde{E}_1 \widetilde{\xi}(t). \tag{38}$$

When $\tau(t) = \overline{\tau}$ and τ_u, inequality (38) still holds; it yields that

$$I_1 + I_2 \le -\widetilde{\xi}^T(t)\left(\widetilde{E}_1^{(1)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(1)} \widetilde{\xi}(t), \tag{39a}$$

$$I_1 + I_2 \le -\widetilde{\xi}^T(t)\left(\widetilde{E}_1^{(2)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(2)} \widetilde{\xi}(t), \tag{39b}$$

where

$$\widetilde{E}_1^{(1)} = \begin{bmatrix} (\tau_u - \overline{\tau}) e_5^T - e_9^T \\ -e_{10}^T \\ -(\tau_u - \overline{\tau}) e_4^T + e_9^T \\ e_{10}^T \\ (\tau_u - \overline{\tau}) e_5^T - e_9^T \\ -e_{10}^T \end{bmatrix},$$

$$\widetilde{E}_1^{(2)} = \begin{bmatrix} -e_9^T \\ (\tau_u - \overline{\tau}) e_5^T - e_{10}^T \\ e_9^T \\ -(\tau_u - \overline{\tau}) e_4^T + e_{10}^T \\ -e_9^T \\ (\tau_u - \overline{\tau}) e_5^T - e_{10}^T \end{bmatrix}. \quad (40)$$

One has

$$\begin{aligned} I_3 \le & -\frac{2}{\underline{\tau}(\overline{\tau}^2 - \tau_l^2)} \widetilde{\xi}^T(t) \left[(\overline{\tau} - \tau_l) e_3^T - e_{11}^T \right] \\ & \times P_3 \left[(\overline{\tau} - \tau_l) e_3 - e_{11} \right] \widetilde{\xi}(t) \\ & - \frac{2}{\underline{\tau}(\overline{\tau}^2 - \tau_l^2)} \widetilde{\xi}^T(t) \left[-(\overline{\tau} - \tau_l) e_4^T - e_{11}^T \right] \\ & \times P_4 \left[-(\overline{\tau} - \tau_l) e_4 - e_{11} \right] \widetilde{\xi}(t). \end{aligned} \quad (41)$$

$$I_4 \le -\frac{1}{\underline{\tau} \ln 2} \widetilde{\xi}^T(t) \left[e_3^T - e_5^T \right] \left[P_3 - P_4 \right] \left[e_3 - e_5 \right] \widetilde{\xi}(t).$$

$$I_5 \le -\frac{2\underline{\tau}}{\tau_l^2} \widetilde{\xi}^T(t) \left[e_1^T - e_3^T \right] \left[P_5 - P_6 \right] \left[e_1 - e_3 \right] \widetilde{\xi}(t).$$

Furthermore, from (4), for arbitrary matrices N_1 and N_2 of appropriate dimensions, the following equalities hold:

$$\begin{aligned} & 2\widetilde{\xi}^T(t) e_1^T N_1 \left[-Ce_1 - e_6 \right] \widetilde{\xi}(t) \\ & \quad + \widetilde{\xi}^T(t) e_1^T N_1 \left[AL\Delta e_1 + BL\overline{\Delta} e_2 \right] \widetilde{\xi}(t) = 0, \end{aligned} \quad (42a)$$

$$\begin{aligned} & 2\widetilde{\xi}^T(t) e_6^T N_2 \left[-Ce_1 - e_6 \right] \widetilde{\xi}(t) \\ & \quad + \widetilde{\xi}^T(t) e_6^T N_2 \left[AL\Delta e_1 + BL\overline{\Delta} e_2 \right] \widetilde{\xi}(t) = 0. \end{aligned} \quad (42b)$$

Define

$$\begin{aligned} & \widetilde{E}_2^{(1)} = 2e_6^T Q_1 e_1, \qquad \widetilde{E}_2^{(2)} = 2e_6^T Q_3 e_1, \\ & \widetilde{E}_2^{(3)} = 2e_6^T \left(\frac{\tau(t) - \overline{\tau}}{\underline{\tau}} Q_3 + \frac{\tau_u - \tau(t)}{\underline{\tau}} Q_1 \right), \end{aligned} \quad (43a)$$

$$\begin{aligned} & \widetilde{E}_3^{(1)} = \frac{b_1}{\underline{\tau}} e_1^T (Q_3 - Q_1) e_1, \qquad \widetilde{E}_3^{(2)} = \frac{b_2}{\underline{\tau}} e_1^T (Q_3 - Q_1) e_1, \\ & \widetilde{E}_3^{(3)} = \frac{\dot{\tau}(t)}{\underline{\tau}} e_1^T (Q_3 - Q_1) e_1. \end{aligned} \quad (43b)$$

Therefore

$$\begin{aligned} \widetilde{E}^1 = & 2\underline{\tau} e_6^T P_5 e_6 - e_7^T (2\underline{\tau} P_6 - 2\underline{\tau} P_3) e_6 - e_8^T (\underline{\tau} P_4 - \underline{\tau} P_1) e_8 \\ & - \frac{2}{\underline{\tau}(\overline{\tau}^2 - \tau_l^2)} \left[(\overline{\tau} - \tau_l) e_3^T - e_{11}^T \right] P_3 \left[(\overline{\tau} - \tau_l) e_3 - e_{11} \right] \\ & - \frac{2}{\underline{\tau}(\overline{\tau}^2 - \tau_l^2)} \left[-(\overline{\tau} - \tau_l) e_4^T - e_{11}^T \right] P_4 \left[-(\overline{\tau} - \tau_l) e_4 - e_{11} \right] \\ & - \frac{1}{\underline{\tau} \ln 2} \left[e_3^T - e_5^T \right] \left[P_3 - P_4 \right] \left[e_3 - e_5 \right] \\ & - \frac{2\underline{\tau}}{\tau_l^2} \left[e_1^T - e_3^T \right] \left[P_5 - P_6 \right] \left[e_1 - e_3 \right] \\ & + e_1^T N_1 \left[-Ce_1 - e_6 \right] + e_6^T N_2 \left[-Ce_1 - e_6 \right] \\ & + \left[-Ce_1 - e_6 \right]^T N_1^T e_1 + \left[-Ce_1 - e_6 \right]^T N_2^T e_6. \end{aligned} \quad (44)$$

Moreover

$$\widetilde{\Phi} = \left[e_1^T, e_2^T, e_1^T, e_2^T \right]^T, \qquad \widetilde{\Delta} = \operatorname{diag}\left\{ \Delta, \Delta, \overline{\Delta}, \overline{\Delta} \right\}, \quad (45a)$$

$$\widetilde{\Psi} = \left[e_1^T N_1 AL, e_1^T N_1 BL, e_6^T N_2 AL, e_6^T N_2 BL \right]. \quad (45b)$$

From (35)–(45b), similar to (30)–(32d), there exist α_3 and α_4 such that

$$\dot{V}(y(t)) \le \alpha_3 \|x(t)\|^2, \quad \text{or} \quad \dot{V}(y(t)) \le \alpha_4 \|x(t)\|^2, \quad (46)$$

if

$$\widetilde{\Pi}_{(1)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(1)} + \widetilde{E}_3^{(1)} - \left(\widetilde{E}_1^{(1)} \right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(1)} + \widetilde{\Psi} \widetilde{\Delta} \widetilde{\Phi} + \widetilde{\Phi}^T \widetilde{\Delta} \widetilde{\Psi}^T < 0, \quad (47a)$$

$$\widetilde{\Pi}_{(2)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(1)} + \widetilde{E}_3^{(2)} - \left(\widetilde{E}_1^{(1)} \right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(1)} + \widetilde{\Psi} \widetilde{\Delta} \widetilde{\Phi} + \widetilde{\Phi}^T \widetilde{\Delta} \widetilde{\Psi}^T < 0, \quad (47b)$$

$$\widetilde{\Pi}_{(3)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(2)} + \widetilde{E}_3^{(1)} - \left(\widetilde{E}_1^{(1)} \right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(1)} + \widetilde{\Psi} \widetilde{\Delta} \widetilde{\Phi} + \widetilde{\Phi}^T \widetilde{\Delta} \widetilde{\Psi}^T < 0, \quad (47c)$$

$$\widetilde{\Pi}_{(4)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(2)} + \widetilde{E}_3^{(2)} - \left(\widetilde{E}_1^{(1)} \right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(1)} + \widetilde{\Psi} \widetilde{\Delta} \widetilde{\Phi} + \widetilde{\Phi}^T \widetilde{\Delta} \widetilde{\Psi}^T < 0, \quad (47d)$$

or

$$\widetilde{\widetilde{\Pi}}_{(1)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(1)} + \widetilde{E}_3^{(1)} - \left(\widetilde{E}_1^{(2)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(2)} + \widetilde{\Psi}\widetilde{\Delta}\widetilde{\Phi} + \widetilde{\Phi}^T\widetilde{\Delta}\widetilde{\Psi}^T < 0, \quad (48a)$$

$$\widetilde{\widetilde{\Pi}}_{(2)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(1)} + \widetilde{E}_3^{(2)} - \left(\widetilde{E}_1^{(2)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(2)} + \widetilde{\Psi}\widetilde{\Delta}\widetilde{\Phi} + \widetilde{\Phi}^T\widetilde{\Delta}\widetilde{\Psi}^T < 0, \quad (48b)$$

$$\widetilde{\widetilde{\Pi}}_{(3)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(2)} + \widetilde{E}_3^{(1)} - \left(\widetilde{E}_1^{(2)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(2)} + \widetilde{\Psi}\widetilde{\Delta}\widetilde{\Phi} + \widetilde{\Phi}^T\widetilde{\Delta}\widetilde{\Psi}^T < 0, \quad (48c)$$

$$\widetilde{\widetilde{\Pi}}_{(4)} \triangleq \widetilde{E}^1 + \widetilde{E}_2^{(2)} + \widetilde{E}_3^{(2)} - \left(\widetilde{E}_1^{(2)}\right)^T \widetilde{\widetilde{P}} \widetilde{E}_1^{(2)} + \widetilde{\Psi}\widetilde{\Delta}\widetilde{\Phi} + \widetilde{\Phi}^T\widetilde{\Delta}\widetilde{\Psi}^T < 0. \quad (48d)$$

Theorem 6. *For given scalars $\tau_l \geq 0$, $\tau_u > 0$, b_1, b_2, $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, system* (1) *is globally asymptotically stable if there exist S^i, $S^j \in S_\Delta$ $(i, j = 1, 2, 3, 4)$ of appropriate dimensions such that the following conditions hold:*

$$\begin{bmatrix} \Pi_{(i)} + \Psi^T S^i \Psi & \Phi \\ \Phi^T & -S^i \end{bmatrix} < 0, \qquad \begin{bmatrix} \widetilde{\Pi}_{(j)} + \widetilde{\Psi}^T S^j \widetilde{\Psi} & \widetilde{\Phi} \\ \widetilde{\Phi}^T & -S^j \end{bmatrix} < 0, \quad (49)$$

where $S_\Delta =: \{\mathrm{diag}(s_1 I, \ldots, s_k I, S_1, \ldots, S_l) : S_i > 0, k, l \in N\}$.

Proof. By Lemma 3, the conditions in Theorem 6 are equivalent to (32a)–(32d) or (47a)–(47d). In view of the previous analysis from (30) to (32d) and (46) to (47d), one can obtain that there exist $\alpha_1 > 0$ and $\alpha_3 > 0$ such that $\dot{V}(y(t)) \leq -\min(\alpha_1, \alpha_3)\|x(t)\|^2$. By Lyapunov stable theory, system (1) is globally asymptotically stable, which completes the proof. □

Remark 7. Different from previous work, the LKF function in this paper is constructed by using the convex combination between decomposed time delay and positive matrix, which may reduce the conservatism of criterion.

Remark 8. From the proof of Theorem 6, one can see that, by using the different combinations among $\Pi_{(i)}$, $\overline{\Pi}_{(i)}$, and $\widetilde{\Pi}_{(i)}$, $\widetilde{\widetilde{\Pi}}_{(i)}$, we can establish different stable criteria as follows.

Corollary 9. *For given scalars $\tau_l \geq 0$, $\tau_u > 0$, b_1, b_2, $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, system* (1) *is globally asymptotically stable if there exist S^i, $S^j \in S_\Delta$ $(i, j = 1, 2, 3, 4)$ of appropriate dimensions such that the following conditions hold:*

$$\begin{bmatrix} \overline{\Pi}_{(i)} + \Psi^T S^i \Psi & \Phi \\ \Phi^T & -S^i \end{bmatrix} < 0, \qquad \begin{bmatrix} \widetilde{\Pi}_{(j)} + \widetilde{\Psi}^T S^j \widetilde{\Psi} & \widetilde{\Phi} \\ \widetilde{\Phi}^T & -S^j \end{bmatrix} < 0. \quad (50)$$

Corollary 10. *For given scalars $\tau_l \geq 0$, $\tau_u > 0$, b_1, b_2, $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, system* (1) *is globally asymptotically stable if there exist S^i, $S^j \in S_\Delta$ $(i, j = 1, 2, 3, 4)$ of appropriate dimensions such that the following conditions hold:*

$$\begin{bmatrix} \Pi_{(i)} + \Psi^T S^i \Psi & \Phi \\ \Phi^T & -S^i \end{bmatrix} < 0, \qquad \begin{bmatrix} \widetilde{\widetilde{\Pi}}_{(j)} + \widetilde{\Psi}^T S^j \widetilde{\Psi} & \widetilde{\Phi} \\ \widetilde{\Phi}^T & -S^j \end{bmatrix} < 0. \quad (51)$$

Corollary 11. *For given scalars $\tau_l \geq 0$, $\tau_u > 0$, b_1, b_2, $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, system* (1) *is globally asymptotically stable if there exist S^i, $S^j \in S_\Delta$ $(i, j = 1, 2, 3, 4)$ of appropriate dimensions such that the following conditions hold:*

$$\begin{bmatrix} \overline{\Pi}_{(i)} + \Psi^T S^i \Psi & \Phi \\ \Phi^T & -S^i \end{bmatrix} < 0, \qquad \begin{bmatrix} \widetilde{\widetilde{\Pi}}_{(j)} + \widetilde{\Psi}^T S^j \widetilde{\Psi} & \widetilde{\Phi} \\ \widetilde{\Phi}^T & -S^j \end{bmatrix} < 0. \quad (52)$$

Remark 12. Since the existence of items $\Psi^T S^i \Psi$ and $\Psi^T S^i \Psi$, the results established in Theorem 6 and Corollaries 9–11 are not LMI criteria. In order to use LMI toolbox in computing software, by using the lemma derived in [20], we further establish the following more practicable stable rules.

Theorem 13. *For given scalars $\tau_l \geq 0$, $\tau_u > 0$, b_1, b_2, $L^- = \mathrm{diag}(l_1^-, l_2^-, \ldots, l_n^-)$, $L^+ = \mathrm{diag}(l_1^+, l_2^+, \ldots, l_n^+)$, system* (1) *is globally asymptotically stable if there exist positive constants δ_i, $\delta_j > 0$ $(i, j = 1, 2, 3, 4)$ such that the following conditions hold:*

$$\begin{bmatrix} \delta_i \Pi_{(i)} & \Psi & \delta_i \Phi^T \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} \delta_j \widetilde{\Pi}_{(j)} & \widetilde{\Psi} & \delta_j \widetilde{\Phi}^T \\ * & -I & 0 \\ * & * & -I \end{bmatrix} < 0. \quad (53)$$

Remark 14. Similar to the analysis of Remark 12, the related practicable stable can also be established by using Corollaries 9–11, since the expressions are similar to Theorem 13, which are omitted here.

4. Illustrative Examples

In this section, two numerical examples are given to illustrate the effectiveness of the proposed method.

4.1. Numerical Examples

Example 1. Consider the delayed neural networks (1) with parameters given as

$$C = \mathrm{diag}(1.2769, 0.6231, 0.9230, 0.4480),$$
$$A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix}, \quad (54)$$
$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$g_1(s) = \tanh(-0.1137s)$, $g_2(s) = \tanh(-0.1279s)$, $g_3(s) = \tanh(-0.7994s)$, $g_4(s) = \tanh(-0.2368s)$. Obviously, $l_1^- = l_2^- = l_3^- = l_4^- = 0, l_1^+ = 0.1137, l_2^+ = 0.1279, l_3^+ = 0.7994, l_4^+ = 0.2368$, $L = \text{diag}(0, 0, 0, 0)$.

Similar to [15], our purpose is to estimate the allowable upper bounds delay τ_u under $\tau_l = 0$ such that the system (1) is globally asymptotically stable. For this example, when $\dot{\tau}(t) = 0$, the maximum allowable delay bound τ_u is 1.4224 in [9], 1.9321 in [10], 3.5841 in [11], 3.6156 in [13], and 3.7327 in [12]. Recently, by using delay-scope-dependent method, Li et al. improved the previous results further in [15] and gave out the maximum allowable delay bound τ_u as 3.8363. Applying Theorem 13 in this paper, the maximum allowable delay bound is 3.9221 with $\delta_i = \delta_j = 0.1$, which means that, for this example, the result obtained in this paper is less conservative that those established in [9–13, 15]. Additionally, for this example, the computed variables in [12, 15, 21] are 130, 198, and 86, respectively. In Theorem 13, the computed variables are 150, which is less computationally demanding than in [21], but heavier than in [12, 15]. If Q_1, Q_2, Q_3, P_1, P_2, P_3, P_4, P_5, and P_6 are all diagonal matrices, Theorem 13 established in this paper still holds. In this case, the computed variables in Theorem 13 are 96, which is less computationally demanding than in [12, 21], but heavier than in [15]. For the given initial value $[6, 7, -5 - 8]$, when $\tau_u = 3.9221$, the simulation result can be seen in Figure 1. Simulation result shows that, for the given parameters in Example 1, system (1) is asymptotically stable.

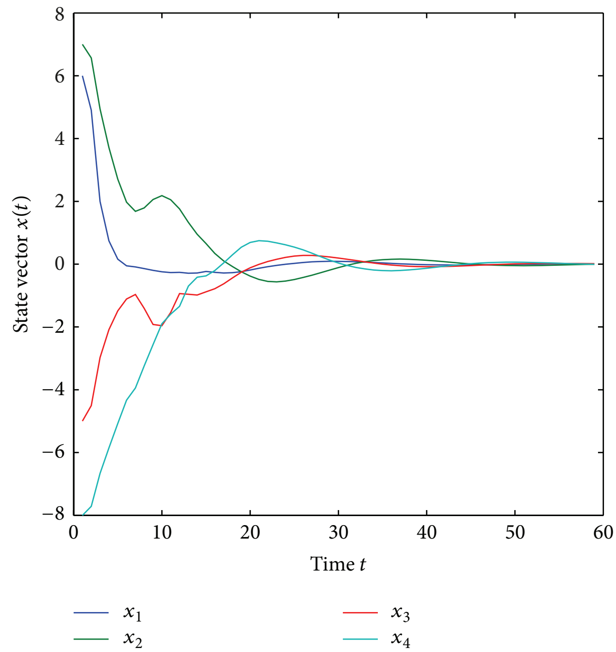

Figure 1: The state variables of system (1) in Example 1.

4.2. An Application Example

Example 2. Consider the continuous pH neutralization of an acid stream by a highly concentrated basic stream, which can be expressed in the following form [13]:

$$v\dot{y}(t) = -ay(t) - u(t), \qquad \text{pH} = w_2 \tanh(w_1 y(t)), \quad (55)$$

where v is the volume of the mixing tank, $y(t)$ is the strong acid, a is the acid flow rate, $u(t)$ is the manipulated variable representing the base flow rate, pH is the measured output signal, and w_2 and w_1 are some constants.

The purpose of this application is to find the maximum allowable upper bound of delay τ for a feedback gain K with output feedback controller $u = -K \times \text{pH}$ such that the closed-loop system is asymptotically stable. In order to do this, we can rewrite system (55) in the following form:

$$\dot{\tilde{y}}(t) = -C\tilde{y}(t) + Af(\tilde{y}(t)) + Bf(\tilde{y}(t-\tau)), \quad (56)$$

where $\tilde{y}(t) = w_1 y(t)$, $f(\tilde{y}(t)) = \tanh(\tilde{y}(t))$, $C = a/v$, $A = Kw_1 w_2/v$, $B = Kw_1 w_2 w_3/v$. For this application problem, [16, 17] gave out the maximum allowable upper bound of delay τ as 17.4956 when the parameters are given as $a = 5.8154$, $v = 1500.3732$, $w_1 = 28.9860$, $w_2 = -3.8500$, and $w_3 = 2.56$, and the feedback gain K is selected as $K = 0.5022$. Recently, by employing weighting-delay method, the maximum allowable upper bound of delay τ is improved to 18.2871 in [13]. Meanwhile, by using Theorem 13, the maximum allowable upper bound of delay τ is 18.7436. Namely, a little better result can be obtained by using our criteria. For a given initial value $\tilde{y}_0 = 0.5$, when $\tau_u = 18.7436$, the simulation result can be seen in Figure 2. Simulation result shows that, for the given parameters in Example 2, system (55) is asymptotically stable.

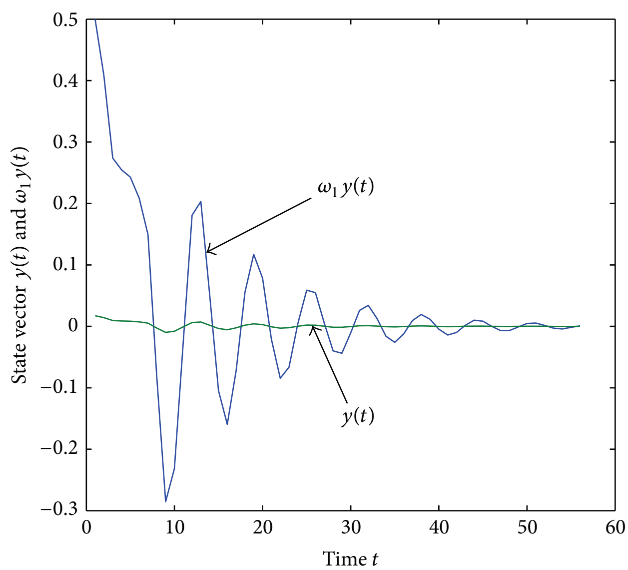

Figure 2: The state variables of system (55) in Example 2.

5. Conclusions

Combined with delay partitioning technique, by using the convex combination between decomposed time delay and positive matrix, this paper researches the stability problem of a class of delayed neural networks with interval time-varying delays. The benefit of the method used in this paper is that it can utilize more information on the slope of activations and time delays. Illustrative examples show that the new criteria derived in this paper are less conservative than some previous results obtained in the references cited therein.

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