

Research Article

MHD Thin Film Flows of a Third Grade Fluid on a Vertical Belt with Slip Boundary Conditions

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Received 23 May 2013; Accepted 14 August 2013

Academic Editor: Oluwale Daniel Makinde

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The problem of heat transfer analysis is considered in electrically conducting thin film flows with slip boundary conditions. The flow is assumed to be obeying the nonlinear rheological constitutive equation of a third grade fluid. We have solved the governing nonlinear equations of present problems using the traditional Adomian decomposition method (ADM). Particular attention is given to the combined effect of heat and MHD on the velocity field. The results include the profile of velocity, volume flux, skin friction, average velocity, and the temperature distribution across the film. The effects of model parameters on velocity, skin friction and temperature variation have been studied. Optimal homotopy asymptotic method (OHAM) is also used for comparison. The numerical results and absolute errors are derived in tables.

1. Introduction

The flow and heat transfer inside thin films are ubiquitous in civil, environmental sciences, mechanical engineering, biological sciences, geophysics, and elsewhere. This is due to their several applications at large scale such as wire and fiber coating, reactor fluidization, paper production, different food stuffs like ketchup, sauce, and honey, transpiration cooling, gaseous diffusion, drilling mud, oil wells, heat pipes, and fluid cells. The problem of chambers for chemical and biological detection systems like fluid of many chemicals was considered by Lavrik et al. [1]. Heat transfer inside thin film flow with variable pressure was discussed by Khaled and Vafai [2]. Thin film unsteady flow with variable viscosity was investigated by Nadeem and Awais [3]. Munson and Young [4] discussed the thin film flow of Newtonian fluids. Alam et al. investigated the thin film flow of Johnson-Segalman fluids for lifting and drainage problems [5]. The MHD thin film flow was discussed by Hameed and Ellahi for a non-Newtonian fluid on a vertical moving belt [6]. Due to complexity of non-Newtonian fluids, it becomes difficult to suggest a single model which exhibits all properties of non-Newtonian fluids; therefore various empirical and semiempirical models have been imposed. The use of heat transfer together with the

MHD and non-Newtonian fluids under the influence of slip boundary conditions is of particular interest in chemical processing. Relevant and interesting work about present work may be found in [7, 8]. One of the established models amongst non-Newtonian fluids is class of third grade fluids which have their constitutive equations based on strong theoretical foundations, where the relation between the stress and strain is not linear. To solve real world problems, different approximate techniques have been used in mathematics, fluid mechanics, and engineering sciences [9]. Some of the common methods are VIM [10–14], DTM [15, 16], HPM [17–19], HAM, and OHAM [20–24]. These methods deal with the nonlinear problems effectively. The work under various configurations on the thin film flows was discussed by Siddiqui et al. [25]. In [26] they examined the thin film flows of Sisko and Oldroyd-6 constant fluids on a moving belt. The heat transfer analysis of thin film is also discussed by Chakraborty and Som [27]. The main aim of the present work is to study heat transfer into a thin film of a third grade fluid on a vertical belt under the influence of transverse magnetic field with slip boundary conditions using ADM. In 1992 Adomian [28, 29] introduced the ADM for the approximate solutions for linear and nonlinear problems. Wazwaz [30, 31] used ADM for the reliable treatment of Bratu-type and Emden-Fowler equations.

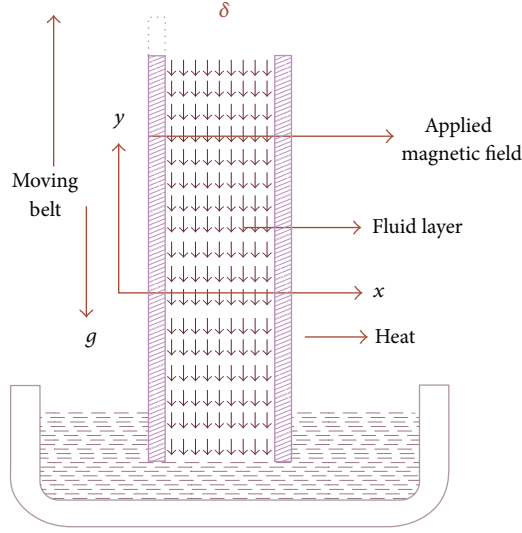


FIGURE 1: Geometry of the lifting problem.

2. Basic Equations

The governing equations of an incompressible isothermal and electrically conducting third grade fluid are

$$\nabla \cdot \mathbf{U} = 0, \quad (1)$$

$$\rho \frac{D\mathbf{U}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{g} + \mathbf{J} \times \mathbf{B}, \quad (2)$$

$$\rho c_p \frac{D\Theta}{Dt} = \kappa \nabla^2 \Theta + \text{tr}(\boldsymbol{\tau} \cdot \mathbf{L}), \quad (3)$$

where ρ is the constant density, \mathbf{g} is body force per unit mass, \mathbf{U} is velocity vector of the fluid, Θ is temperature, κ is thermal conductivity, c_p is specific heat, $\mathbf{L} = \nabla \mathbf{U}$, $D/Dt = \partial/\partial t + (\mathbf{U} \cdot \nabla)$ denote material time derivative, $\boldsymbol{\tau}$ is Cauchy stress tensor, and \mathbf{T} is the shear stress. A uniform magnetic field $\mathbf{B} = (0, B_0, 0)$ is imposed transversely on the belt. The Lorentz force per unit volume is given by

$$\mathbf{J} \times \mathbf{B} = [0, \sigma B_0^2 u(x), 0]. \quad (4)$$

Shear stress tensor \mathbf{T} is given by

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}, \quad (5)$$

where $-p\mathbf{I}$ denotes spherical stress and $\boldsymbol{\tau}$ is defined as

$$\begin{aligned} \boldsymbol{\tau} = & \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2 + \beta_1 \mathbf{A}_3 \\ & + \beta_2 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \beta_3 (\text{tr} \mathbf{A}_1^2) \mathbf{A}_1. \end{aligned} \quad (6)$$

Here α_i and β_j are the material constants, and \mathbf{A}_1 , \mathbf{A}_2 , and \mathbf{A}_3 are the kinematical tensors given by

$$\mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \mathbf{A}_{n-1}, \quad n \geq 1, \quad (7)$$

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}, \quad \beta_3 \geq 0. \quad (8)$$

3. Formulation of the Lift Problem

Consider a wide flat belt moving vertically upward at constant speed U_0 , through a large bath of third grade liquid as shown in Figure 1. The belt carries with it a layer of liquid of constant thickness, δ . For analysis coordinate system is chosen, in which the x -axis is taken parallel to the surface of the belt and y -axis is perpendicular to the belt. Uniform magnetic field is applied transversely to the belt. Assume the flow is steady and laminar after a small distance above the liquid surface layer. The external pressure is atmospheric everywhere.

Velocity and temperature fields are

$$\mathbf{u} = (0, u(x), 0), \quad \Theta = \Theta(x). \quad (9)$$

Boundary conditions are

$$\begin{aligned} u &= U_0 - \gamma T_{xy} \quad \text{at } x = 0, \\ \frac{du}{dx} &= 0, \quad \text{at } x = \delta, \\ \Theta &= \Theta_0, \quad \text{at } x = 0, \\ \Theta &= \Theta_1, \quad \text{at } x = \delta. \end{aligned} \quad (10)$$

Inserting the velocity field given in (9) in (1) and (5)–(7), the continuity equation (1) satisfies identically and (5) gives the following components of stress tensor

$$\begin{aligned} T_{xx} &= -P + (2\alpha_1 + \alpha_2) \left(\frac{du}{dx} \right)^2, \\ T_{xy} &= \mu \frac{du}{dx} + 2(\beta_2 + \beta_3) \left(\frac{du}{dx} \right)^3, \\ T_{yy} &= -P + \alpha_2 \left(\frac{du}{dx} \right), \\ T_{zz} &= -P, \\ T_{xz} &= T_{yz} = 0. \end{aligned} \quad (11)$$

Using (11), the momentum and energy equations reduce to

$$\begin{aligned} 0 &= \mu \frac{d^2 u}{dx^2} + 6(\beta_2 + \beta_3) \left(\frac{du}{dx} \right)^2 \left(\frac{d^2 u}{dx^2} \right) - \rho g - \sigma B_0^2 u(x), \\ 0 &= \kappa \frac{d^2 \Theta}{dx^2} + \mu \left(\frac{du}{dx} \right)^2 + 2(\beta_2 + \beta_3) \left(\frac{du}{dx} \right)^4. \end{aligned} \quad (12)$$

Introducing the following nondimensional variables:

$$\begin{aligned}\bar{u} &= \frac{\delta}{\nu} U, & \bar{x} &= \frac{x}{\delta}, & \bar{\Theta} &= \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0}, \\ \bar{\gamma} &= \frac{\mu\gamma}{\delta}, & \Lambda &= \frac{\gamma\nu}{\delta}, \\ \lambda &= \frac{\mu\nu^2}{k(\Theta_1 - \Theta_0)\delta^2}, & m &= \frac{\delta^3 g}{\nu^2}, \\ M &= \frac{\sigma B_0^2 \delta^2}{\mu}, & \beta &= \frac{(\beta_2 + \beta_3)\nu^2}{\mu\delta^4}, \\ \alpha &= \frac{\delta U_0}{\nu}, & R_e &= \frac{U\delta}{\nu}, & \nu &= \frac{\mu}{\rho},\end{aligned}\quad (13)$$

where m is the gravitational parameter, M is magnetic parameter, β is non-Newtonian effect, Λ is slip parameter, λ is heat dimensionless number, R_e is the local Reynolds number, and α is the nondimensional variable using the above dimensionless variables in (10) and in (12) and dropping bars we obtain

$$\frac{d^2 u}{dx^2} + 6\beta \left(\frac{du}{dx} \right)^2 \left(\frac{d^2 u}{dx^2} \right) - m - Mu(x) = 0, \quad (14)$$

$$\frac{d^2 \Theta}{dx^2} + \lambda \left[\left(\frac{du}{dx} \right)^2 + 2\beta \left(\frac{du}{dx} \right)^4 \right] = 0, \quad (15)$$

$$u_n(0) = \alpha - \Lambda \left(\frac{du_n}{dx} + 2\beta \left(\frac{du_n}{dx} \right)^3 \right), \quad (16)$$

$$\frac{du_n(1)}{dx} = 0, \quad n = 0,$$

$$u_n(0) = -\Lambda \left(\frac{du_n}{dx} + 2\beta \left(\frac{du_n}{dx} \right)^3 \right), \quad (17)$$

$$\frac{du_n(1)}{dx} = 0, \quad n > 0,$$

$$\Theta(0) = 0, \quad \Theta(1) = 1. \quad (18)$$

4. Adomian Decomposition Method (ADM)

A general description of the method is as follows. Begin with an equation $\tilde{F}(u) = \tilde{g}(x)$, where \tilde{F} represents a general nonlinear ordinary differential operator involving both linear and nonlinear terms and \tilde{g} is a source term. The linear term is decomposed into $\tilde{L} + \tilde{R}$, where \tilde{L} is easily invertible and \tilde{R} is the remainder of the linear operator. For convenience, \tilde{L} may be taken as the highest order derivative which avoids difficult integrations which result when complicated Green's functions are involved. Thus the equation may be written as follows:

$$\tilde{L}u + \tilde{R}u + \tilde{N}u = \tilde{g}. \quad (19)$$

Solving for $\tilde{L}u$,

$$\tilde{L}u = \tilde{g} - \tilde{R}u - \tilde{N}u, \quad (20)$$

where \tilde{N} is non-linear operator and \tilde{g} is a source term, since \tilde{L} is easily invertible.

Equation (20) can be written as follows:

$$\tilde{L}^{-1}\tilde{L}u = \tilde{L}^{-1}\tilde{g} - \tilde{L}^{-1}\tilde{R}u - \tilde{L}^{-1}\tilde{N}u. \quad (21)$$

We use \tilde{L}^{-1} depending on the order of the differential equation. If differential equation is an initial-value problem, if it is desired for boundary value problem as well, integrations are used. The constants of integration are evaluated from the given initial and boundary conditions. \tilde{L}^{-1} can also be treated as definite integral from $(\tilde{t}_0$ to \tilde{t}). Solving (21) for u we obtained

$$u = C_1 + C_2 t + \tilde{L}^{-1}\tilde{g} - \tilde{L}^{-1}\tilde{R}u - \tilde{L}^{-1}\tilde{N}u. \quad (22)$$

The non-linear term $\tilde{N}u$, is defined as follows:

$$\tilde{N}u = \sum_{k=0}^{\infty} \tilde{A}_n. \quad (23)$$

Here \tilde{A}_n are special polynomials called Adomian polynomials and \tilde{u} will be equated to $\sum_{k=0}^{\infty} \tilde{u}_n$. The initial velocity is identified as follows:

$$\tilde{u}_0 = \tilde{C}_1 + \tilde{C}_2 t + \tilde{L}^{-1}\tilde{g},$$

$$\sum_{k=0}^{\infty} u_n = u_0 - \tilde{L}^{-1}\tilde{R} \sum_{k=0}^{\infty} u_n - \tilde{L}^{-1} \sum_{k=0}^{\infty} \tilde{A}_n. \quad (24)$$

Comparison for different components of velocity profile is as follows:

$$u_1 = -\tilde{L}^{-1}\tilde{R}u_0 - \tilde{L}^{-1}\tilde{A}_0,$$

$$u_2 = -\tilde{L}^{-1}\tilde{R}u_0 - \tilde{L}^{-1}\tilde{A}_1,$$

$$\vdots$$

$$u_{n+1} = -\tilde{L}^{-1}\tilde{R}u_n - \tilde{L}^{-1}\tilde{A}_n. \quad (25)$$

The Adomian polynomials \tilde{A}_n depend on the velocity components $u_0, u_1, u_2, \dots, u_n$, which play a flourishing role in the rapid convergence of the series. In the above series \tilde{A}_0 depends only on u_0 , \tilde{A}_1 depends on u_0 and u_1 , \tilde{A}_2 depends on u_0, u_1 , and u_2 , and so forth. Relevant discussion about ADM can be seen in [16, 17].

4.1. The ADM Solution of Lifting Problem. Using the inverse operator $\tilde{L}^{-1} = \iint du'$ of the Adomian decomposition method on (14), we obtained

$$u = C_1 x + C_2 + m \frac{x^2}{2} + \tilde{L}^{-1} [Mu] - 6\beta \tilde{L}^{-1} \left[\left(\frac{du}{dx} \right)^2 \frac{d^2 u}{dx^2} \right]. \quad (26)$$

For series solution we use summation on (26)

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= C_1 x + C_2 + m \frac{x^2}{2} \\ &+ M\tilde{L}^{-1} \left(\sum_{n=0}^{\infty} u_n \right) - 6\beta\tilde{L}^{-1} \\ &\times \left[\frac{d}{dx} \left(\sum_{n=0}^{\infty} u_n \right)^2 \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u_n \right) \right]. \end{aligned} \quad (27)$$

Adomian polynomials are defined from (27) as follows:

$$\begin{aligned} \tilde{A}_n &= \left(\frac{d}{dx} \left(\sum_{n=0}^{\infty} u_n \right)^2 \right) \left[\frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u_n \right) \right], \\ n &\geq 0. \end{aligned} \quad (28)$$

From (28) when $n \geq 0$, the Adomian polynomials in components form are

$$\begin{aligned} \tilde{A}_0 &= \left(\frac{du_0}{dx} \right)^2 \frac{d^2 u_0}{dx^2}, \\ \tilde{A}_1 &= \left(\frac{du_0}{dx} \right)^2 \frac{d^2 u_1}{dx^2} + 2 \frac{du_0}{dx} \frac{du_1}{dx}, \\ \tilde{A}_2 &= \left(\frac{du_0}{dx} \right)^2 \frac{d^2 u_2}{dx^2} + \left(\frac{du_1}{dx} \right)^2 \frac{d^2 u_0}{dx^2} \\ &+ 2 \frac{du_0}{dx} \frac{du_2}{dx} \frac{d^2 u_0}{dx^2}. \end{aligned} \quad (29)$$

The series solution becomes

$$\begin{aligned} u_0 + u_1 + u_2 + \dots &= C_1 x + C_2 + m \frac{x^2}{2} \\ &+ ML^{-1} [u_0 + u_1 + u_2 + \dots] \\ &- 6\beta L^{-1} [\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 + \dots]. \end{aligned} \quad (30)$$

The velocity components are obtained by comparing both sides of (33).

4.1.1. Zero Component Problem. Consider

$$u_0(x) = C_1 x + C_2 + m \frac{x^2}{2}. \quad (31)$$

From (16) the boundary condition, for $n = 0$, is

$$u_0(0) = \alpha - \Lambda \left(\frac{du_0}{dx} + 2\beta \left(\frac{du_0}{dx} \right)^3 \right), \quad \frac{du_0(1)}{dx} = 0. \quad (32)$$

By making use of (32) in (31), after simplification we obtain

$$u_0(x) = \alpha + \frac{mx^2}{2} - mx + \Lambda(m + 2\beta m^3). \quad (33)$$

4.1.2. First Component Problem. Consider

$$u_1(x) = ML^{-1}[u_0] - 6\beta L^{-1}[\tilde{A}_0]. \quad (34)$$

From (17) the boundary condition, for $n = 1$, is

$$u_1(0) = -\Lambda \left(\frac{du_1}{dx} + 6\beta \left(\frac{du_0}{dx} \right)^2 \frac{du_1}{dx} \right), \quad \frac{du_1(1)}{dx} = 0. \quad (35)$$

The solution is

$$\begin{aligned} u_1(x) &= \left[(1 + 6m^2\beta)\Lambda \right. \\ &\times \left(\frac{mM}{3} - M\alpha + 2m^3\beta - mM\Lambda - 2m^3M\beta\Lambda \right) \\ &+ \left(\frac{mM}{3} - M\alpha + 2m^3\beta - mM\Lambda - 2m^3M\beta\Lambda \right)x \\ &+ \left(\frac{M\alpha}{2} - 3m^3\beta + \frac{mM\Lambda}{2} + m^3M\beta\Lambda \right)x^2 \\ &\left. + \left(2m^3\beta - \frac{mM}{6} \right)x^3 + \left(\frac{mM}{24} - \frac{m^3\beta}{2} \right)x^4 \right]. \end{aligned} \quad (36)$$

4.1.3. Second Component Problem. Consider

$$u_2(x) = ML^{-1}[u_1] - 6\beta L^{-1}[\tilde{A}_1]. \quad (37)$$

From (17) the boundary condition, for $n = 2$, is

$$\begin{aligned} u_2(0) &= -\Lambda \left(\frac{du_2}{dx} \left(1 + 6\beta \left(\frac{du_0}{dx} \right)^2 \right) + 6\beta \left(\frac{du_1}{dx} \right)^2 \frac{du_0}{dx} \right), \\ &\frac{du_2(1)}{dx} = 0. \end{aligned} \quad (38)$$

The solution is

$$u_2(x) = \Psi_0 - \Psi_1 x + \Psi_2 x^2 - \Psi_3 x^3 + \Psi_4 x^4 - \Psi_5 x^5 + \Psi_6 x^6. \quad (39)$$

The series solution up to the second component is

$$u(x) = u_0(x) + u_1(x) + u_2(x). \quad (40)$$

Using (33), (36), and (39) in (40), we have

$$u(x) = \Phi_0 - \Phi_1 x + \Phi_2 x^2 - \Phi_3 x^3 + \Phi_4 x^4 - \Phi_5 x^5 + \Phi_6 x^6. \quad (41)$$

The constants $(\Psi_0 - \Psi_6)$ and $(\Phi_0 - \Phi_6)$ are given in the appendix.

The dimensionless shear stress is

$$\tau_{xy} = \frac{\mu\nu}{\delta^2} \left[\frac{du}{dx} + 2\beta \left(\frac{du}{dx} \right)^3 \right]. \quad (42)$$

The shear rate becomes

$$\begin{aligned} \tau_{xy}|_{x=0} &= \frac{\mu\nu}{\delta^2} \left[25 \left(90m^2 M\alpha\beta + 90m^2 M^2\alpha\beta\Lambda - 15M\alpha \right. \right. \\ &\quad + 5M^2\alpha + 15M^2\alpha\Lambda - 180m^5\beta^2 \\ &\quad + 180m^2\beta^2 M^2\Lambda^2 - 15m + 5mM \\ &\quad - 2mM^2 - 15\Lambda m M\Lambda + 15mM^2\Lambda^2 \\ &\quad + 30m^3\beta - 36Mm^3\beta + 30\Lambda Mm^3\beta \\ &\quad \left. - 20\Lambda M^2 m^3\beta + 20\Lambda^2 M^2 m^3\beta \right) \\ &\quad + \left(90m^2 M\alpha\beta + 90m^2 M^2\alpha\beta\Lambda - 15M\alpha \right. \\ &\quad + 5M^2\alpha + 15M^2\alpha\Lambda - 180m^5\beta^2 \\ &\quad + 180m^2\beta^2 M^2\Lambda^2 - 15m + 5mM \\ &\quad - 2mM^2 - 15\Lambda m M\Lambda + 15mM^2\Lambda^2 \\ &\quad + 30m^3\beta - 36Mm^3\beta + 30\Lambda Mm^3\beta \\ &\quad \left. \left. - 20\Lambda M^2 m^3\beta + 20\Lambda^2 M^2 m^3\beta \right)^3 \right]. \end{aligned} \quad (43)$$

The coefficient of skin friction is defined as follows:

$$C_f(0) = \frac{\tau_{xy}|_{x=0}}{(1/2)\rho U^2}. \quad (44)$$

By use of (43), we have

$$\begin{aligned} C_f(0) &= \frac{2}{R_e^2} \left[25 \left(90m^2 M\alpha\beta + 90m^2 M^2\alpha\beta\Lambda \right. \right. \\ &\quad - 15M\alpha + 5M^2\alpha + 15M^2\alpha\Lambda \\ &\quad - 180m^5\beta^2 + 180m^2\beta^2 M^2\Lambda^2 - 15m \\ &\quad + 5mM - 2mM^2 - 15\Lambda m M\Lambda + 15mM^2\Lambda^2 \\ &\quad + 30m^3\beta - 36Mm^3\beta + 30\Lambda Mm^3\beta \\ &\quad \left. - 20\Lambda M^2 m^3\beta + 20\Lambda^2 M^2 m^3\beta \right) \\ &\quad + \left(90m^2 M\alpha\beta + 90m^2 M^2\alpha\beta\Lambda \right. \\ &\quad \left. - 15M\alpha + 5M^2\alpha + 15M^2\alpha\Lambda \right. \end{aligned}$$

$$\begin{aligned} &\quad - 180m^5\beta^2 + 180m^2\beta^2 M^2\Lambda^2 \\ &\quad - 15m + 5mM - 2mM^2 \\ &\quad - 15\Lambda m M\Lambda + 15mM^2\Lambda^2 + 30m^3\beta \\ &\quad - 36Mm^3\beta + 30\Lambda Mm^3\beta \\ &\quad \left. - 20\Lambda M^2 m^3\beta + 20\Lambda^2 M^2 m^3\beta \right)^3 \Big]. \end{aligned} \quad (45)$$

5. Volume Flow Rate and Average Velocity

Volume flow rate in nondimensional form is as follows:

$$Q = \int_0^1 u(x) dx. \quad (46)$$

Using velocity field from (41), volume flow rate is obtained as follows.

Average velocity u in dimensionless form is given by

$$u = Q, \quad (47)$$

$$\begin{aligned} Q &= -\frac{m}{3} + \frac{2mM}{15} - \frac{17mM^2}{315} + \alpha - \frac{M\alpha}{3} \\ &\quad + \frac{2M^2\alpha}{15} + \frac{2m^3\beta}{5} - \frac{64}{105}m^3M\beta \\ &\quad + \frac{6}{5}m^2M\alpha\beta - \frac{12m^5\beta^2}{7} + m\Lambda \\ &\quad - \frac{1}{9}mM^2\Lambda - M\alpha\Lambda + \frac{2}{3}M^2\alpha\Lambda \\ &\quad + 4m^3\beta\Lambda - \frac{8}{15}m^3M\beta\Lambda - \frac{33}{20}m^3M^2\beta\Lambda \\ &\quad + \frac{15}{2}m^2M^2\alpha\beta\Lambda - 6mM^2\alpha^2\beta\Lambda \\ &\quad - \frac{143}{5}m^5M\beta^2\Lambda + 78m^4M\alpha\beta^2\Lambda \\ &\quad - 144m^7\beta^3\Lambda - mM\Lambda^2 + \frac{1}{3}mM^2\Lambda^2 \\ &\quad + M^2\alpha\Lambda^2 - 4m^3M\beta\Lambda^2 + \frac{29}{6}m^3M^2\beta\Lambda^2 \\ &\quad + 54m^5M\beta^2\Lambda^2 + 12m^5M^2\beta^2\Lambda^2 \\ &\quad + 12m^4M^2\alpha\beta^2\Lambda^2 + 84m^7M\beta^3\Lambda^2 \\ &\quad + mM^2\Lambda^3 + 8m^3M^2\beta\Lambda^3 \\ &\quad + 36m^5M^2\beta^2\Lambda^3 + 48m^7M^2\beta^3\Lambda^3. \end{aligned} \quad (48)$$

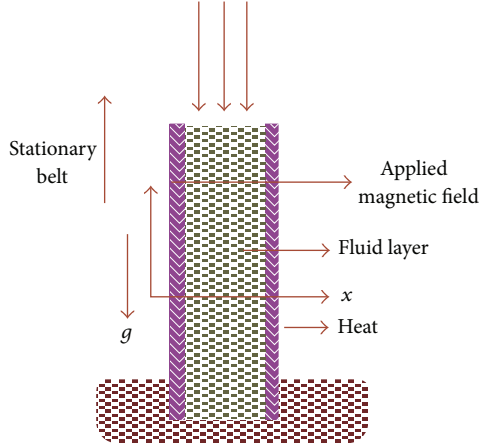


FIGURE 2: Geometry of the drainage problem.

6. Temperature Distribution in Case of Lift Problem

On substituting the series solution for velocity field given in (41) in (15) and solving corresponding to the boundary condition given in (18) for fixed values of $\alpha = 0.1$; $m = 0.5$; $M = 1$; $\Lambda = 0.4$; $\beta = 1$; $\lambda = 100$, we obtained

$$\begin{aligned} \Theta(x) = & 7.1923x - 12.7305x^2 + 9.0310x^3 \\ & - 2.4798x^4 - 0.3026x^5 + 0.4717x^6 \\ & - 0.2248x^7 + 0.03702x^8 + 0.01515x^9 \\ & - 0.0133x^{10} + 0.0044x^{11} - 0.00032x^{12} \\ & - 0.00037x^{13} + 0.000202x^{14} - 0.000048x^{15} \\ & + 0.00000156x^{16} + 0.000003x^{17} - 0.0000014x^{18} \\ & + 3.3326 \times 10^{-7}x^{19} - 5.1691 \times 10^{-8}x^{20} \\ & + 5.0040 \times 10^{-9}x^{21} - 2.2745 \times 10^{-10}x^{22}. \end{aligned} \quad (49)$$

7. Formulation of Drainage Problem

The geometry and assumptions of the problem are the same as those in the previous problem. Consider a film of non-Newtonian liquid draining at volume flow rate Q down the vertical belt, as shown in Figure 2. The belt is stationary and the fluid drain down the belt due to gravity. The coordinate system is selected in the same way as that in the previous case. Assuming the flow is steady, laminar and external pressure is neglected. Consider that fluid shear forces balance gravity and the film thickness remain constant.

Boundary condition for electrically conducting drainage problem is as follows:

$$\begin{aligned} u &= -\gamma T_{xy} \quad \text{at } x = 0, \\ \frac{du}{dx} &= 0, \quad \text{at } x = \delta. \end{aligned} \quad (50)$$

Using nondimensional variables the slip boundary conditions for drainage problem become

$$\begin{aligned} u_n(0) &= -\Lambda \left(\frac{du_n}{dx} + 2\beta \left(\frac{du_n}{dx} \right)^3 \right), \quad \frac{du_n(1)}{dx} = 0, \\ n &\geq 0. \end{aligned} \quad (51)$$

7.1. Solution of the Drainage Problem by ADM. Using ADM on (14), the Adomian polynomials in (29) for both problems are the same. The different velocity components are obtained as follows.

7.1.1. Zero Component Problem. Consider

$$\begin{aligned} u_0(x) &= C_1 + C_2x - m\frac{x^2}{2}, \\ u_0(0) &= -\Lambda \left(\frac{du_0}{dx} + 2\beta \left(\frac{du_0}{dx} \right)^3 \right), \quad \frac{du_0(1)}{dx} = 0. \end{aligned} \quad (52)$$

The solution is

$$u_0(x) = mx - \frac{mx^2}{2} - \Lambda(m + 2\beta m^3). \quad (53)$$

7.1.2. First Component Problem. Consider

$$\begin{aligned} u_1(x) &= ML^{-1}[u_0] - 6\beta L^{-1}[\tilde{A}_0], \\ u_1(0) &= -\Lambda \left(\frac{du_1}{dx} + 6\beta \left(\frac{du_0}{dx} \right)^2 \frac{du_1}{dx} \right), \quad \frac{du_1(1)}{dx} = 0. \end{aligned} \quad (54)$$

For different velocity components, the Adomian polynomials mentioned in (29) are used:

$$\begin{aligned} u_1(x) &= (1 + 6m^2\beta)\Lambda \\ &\times \left(-\frac{mM}{3} - 2m^3\beta + mM\Lambda + 2m^3M\beta\Lambda \right) \\ &+ \left(-\frac{mM}{3} - 2m^3\beta + mM\Lambda + 2m^3M\beta\Lambda \right)x \\ &+ \left(3m^3\beta - \frac{mM\Lambda}{2} - m^3M\beta\Lambda \right)x^2 \\ &+ \left(\frac{mM}{6} - 2m^3\beta \right)x^3 + \left(-\frac{mM}{24} + \frac{m^3\beta}{2} \right)x^4. \end{aligned} \quad (55)$$

7.1.3. *Second Component Problem.* Consider

$$\begin{aligned} u_2(x) &= ML^{-1}[u_1] - 6\beta L^{-1}[\tilde{A}_1], \\ u_2(0) &= -\Lambda \left(\frac{du_2}{dx} \left(1 + 6\beta \left(\frac{du_0}{dx} \right)^2 \right) + 6\beta \left(\frac{du_1}{dx} \right)^2 \frac{du_0}{dx} \right), \\ \frac{du_2(1)}{dx} &= 0. \end{aligned} \quad (56)$$

The solution is

$$u_2(x) = \xi_0 - \xi_1 x + \xi_2 x^2 - \xi_3 x^3 + \xi_4 x^4 - \xi_5 x^5 + \xi_6 x^6. \quad (57)$$

The series solution of the velocity field is

$$u(x) = u_0(x) + u_1(x) + u_2(x). \quad (58)$$

Substituting (53), (55), and (57) in (58),

$$u(x) = \omega_0 - \omega_1 x + \omega_2 x^2 - \omega_3 x^3 + \omega_4 x^4 - \omega_5 x^5 + \omega_6 x^6. \quad (59)$$

The constants $(\xi_0 - \xi_6)$ and $(\omega_0 - \omega_6)$ are defined and listed in the appendix.

Substituting (59) the shear rate is obtained as

$$\begin{aligned} \tau_{xy}|_{x=0} &= \frac{\mu\gamma}{\delta^2} \left[12m^5\beta^2 - 12m^5\beta^2 M^2\Lambda^2 + m - \frac{mM}{3} \right. \\ &\quad + m\Lambda M + \frac{2M^2m}{15} - M^2\Lambda^2m - 2m^3\beta \\ &\quad + \frac{36}{15}m^3M\beta - 2m^3M\beta\Lambda + \frac{4}{3}m^3M^2\beta\Lambda \\ &\quad - 8m^3M^2\beta\Lambda^2 + 2\beta \\ &\quad \times \left(12m^5\beta^2 - 12m^5\beta^2 M^2\Lambda^2 + m - \frac{mM}{3} \right. \\ &\quad + m\Lambda M + \frac{2M^2m}{15} - M^2\Lambda^2m \\ &\quad - 2m^3\beta + \frac{36}{15}m^3M\beta - 2m^3M\beta\Lambda \\ &\quad \left. \left. + \frac{4}{3}m^3M^2\beta\Lambda - 8m^3M^2\beta\Lambda^2 \right)^3 \right]. \end{aligned} \quad (60)$$

8. Volume Flow Rate and Average Velocity of Thin Film Flow

Using (59) in (47) we obtain the following.

The average velocity is defined as \bar{u}/δ

$$\bar{u} = Q,$$

$$Q = \frac{m}{3} - \frac{2mM}{15} + \frac{17mM^2}{315} - \frac{2m^3\beta}{5}$$

$$\begin{aligned} &+ \frac{64}{105}m^3M\beta + \frac{12m^5\beta^2}{7} - m\Lambda \\ &+ \frac{1}{9}mM^2\Lambda - 4m^3\beta\Lambda + \frac{8}{15}m^3M\beta\Lambda \\ &+ \frac{33}{20}m^3M^2\beta\Lambda + \frac{143}{5}m^5M\beta^2\Lambda \\ &+ 144m^7\beta^3\Lambda + mM\Lambda^2 - \frac{1}{3}mM^2\Lambda^2 \\ &+ 4m^3M\beta\Lambda^2 - \frac{29}{6}m^3M^2\beta\Lambda^2 \\ &- 54m^5M\beta^2\Lambda^2 - 3m^5M^2\beta^2\Lambda^2 \\ &- 84m^7M\beta^3\Lambda^2 - mM^2\Lambda^3 - 8m^3M^2\beta\Lambda^3 \\ &- 36m^5M^2\beta^2\Lambda^3 - 48m^7M^2\beta^3\Lambda^3. \end{aligned} \quad (61)$$

9. Heat Distribution for Drainage Problem

Solving (15) with boundary conditions given in (18), after making use of (59) for fixed values of $m = 0.5$, $M = 1$, $\Lambda = 0.4$, $\beta = 1$, and $\lambda = 100$, we obtain

$$\begin{aligned} \Theta(x) &= 7.7356x - 14.3046x^2 + 11.1370x^3 \\ &\quad - 4.2154x^4 + 0.57907x^5 + 0.241003x^6 \\ &\quad - 0.2503x^7 + 0.0947x^8 - 0.01404x^9 \\ &\quad - 0.0068x^{10} + 0.0053x^{11} - 0.0016x^{12} \\ &\quad + 0.000103x^{13} + 0.000127x^{14} \\ &\quad - 0.0000609x^{15} + 0.0000126x^{16} \\ &\quad - 8.11926 \times 10^{-8}x^{17} - 7.8766446 \\ &\quad \times 10^{-7}x^{18} + 2.651929 \times 10^{-7}x^{19} \\ &\quad - 4.828779 \times 10^{-8}x^{20} + 5.004021 \\ &\quad \times 10^{-9}x^{21} - 2.274555 \times 10^{-10}x^{22}. \end{aligned} \quad (62)$$

10. Results and Discussion

The effects of magnetic parameter M , non-Newtonian parameter β , gravitational parameter m , slip parameter Λ , and the dimensionless number λ , for both lift and drainage velocity profiles, are discussed in Figures 5–14. Figures 1 and 2 show the geometry of lift and drainage velocity profiles. Figures 3 and 4 show the comparison of OHAM and ADM for lift and drainage velocity profiles. Numerical results and absolute error for both problems are shown in Tables 1 and 2, respectively. Figures 5 and 10 show that the rise in the non-Newtonian parameter β increases the speed of flow. For

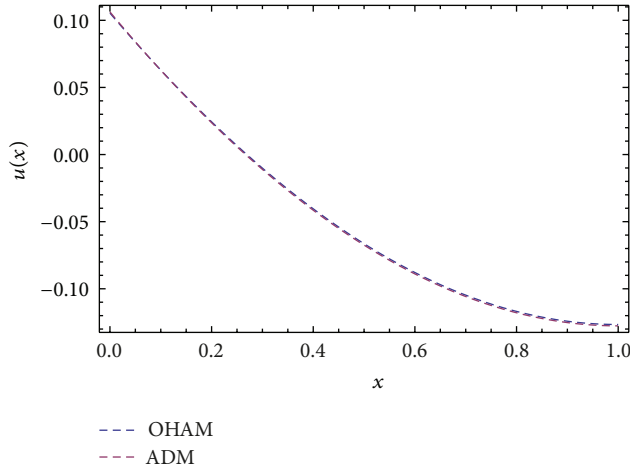


FIGURE 3: Comparison of ADM and OHAM methods for lift velocity profile is shown for the given parameters and auxiliary constants. $m = 0.5$, $M = 0.1$, $\beta = 0.3$, $\Lambda = 0.01$, $\alpha = 0.1$, $C_1 = -0.6221311197$, and $C_2 = -0.0208205048$.

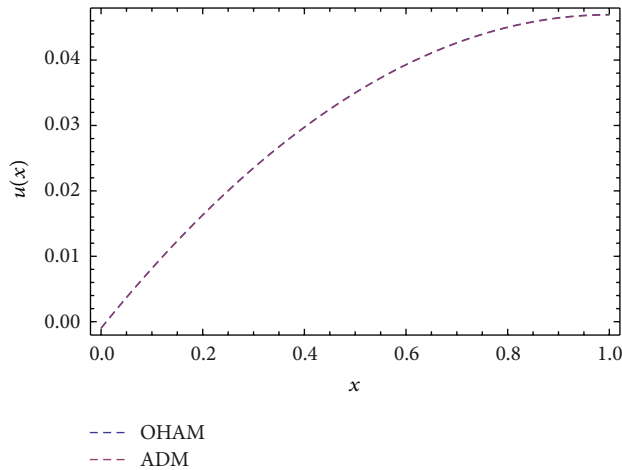


FIGURE 4: Comparison of ADM and OHAM methods for drainage velocity profile is shown for the given parameters and auxiliary constants. $m = 0.1$, $M = 0.1$, $\beta = 0.3$, $\Lambda = 0.01$, $\alpha = 0.1$, $C_1 = -0.93798569$, and $C_2 = -0.000741729$.

small values of β , the velocity profile differs little from the Newtonian one; however when β is increased, these profiles become more flattened showing the shear-thinning effect. Behavior of the velocity field u , for different values of M , by fixing other physical parameters, is shown in Figures 6 and 11 for lifting and drainage of fluid, respectively. Here, it can be seen that the boundary layer thickness is reciprocal to the transverse magnetic field and the velocity decreases as one progresses towards the surface of the fluid. We note that the velocity of fluid is maximum at the surface of the belt and minimum at the surface. Moreover, it is to be noted that for large values of M , the velocity increases rapidly as compared to small values. When the gravitational parameter m increases, the velocity decreases in lifting flow and increases in case of drainage. This can be seen from

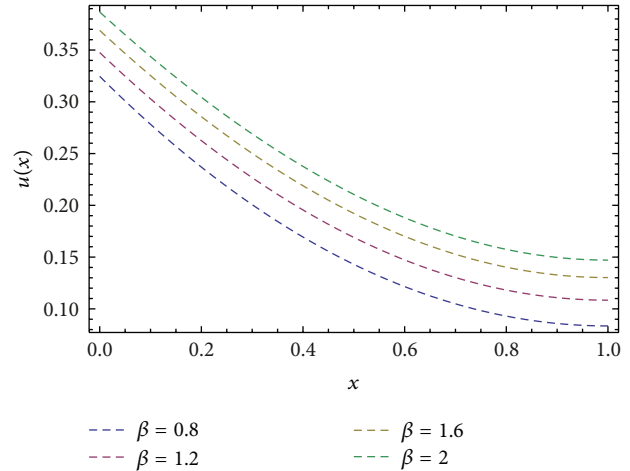


FIGURE 5: The influence of non-Newtonian β on velocity profile for lifting problem keeping $\alpha = 0.1$, $m = 0.5$, $M = 1$, and $\Lambda = 0.4$.

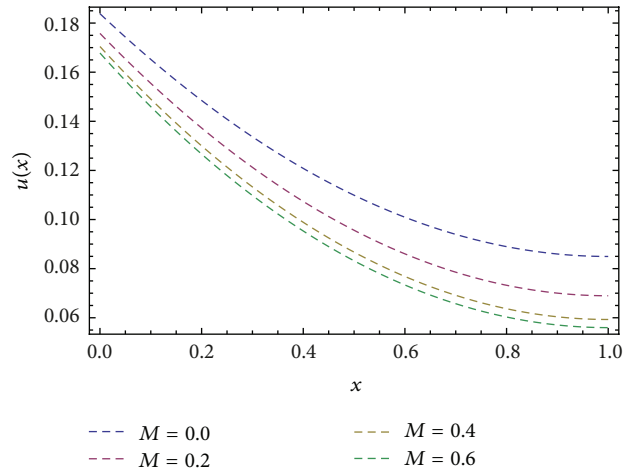


FIGURE 6: The effect of magnetic force “ M ” on the lift velocity profile keeping other parameters fixed. $\alpha = 0.1$, $m = 0.2$, $\beta = 1.2$, and $\Lambda = 0.4$.

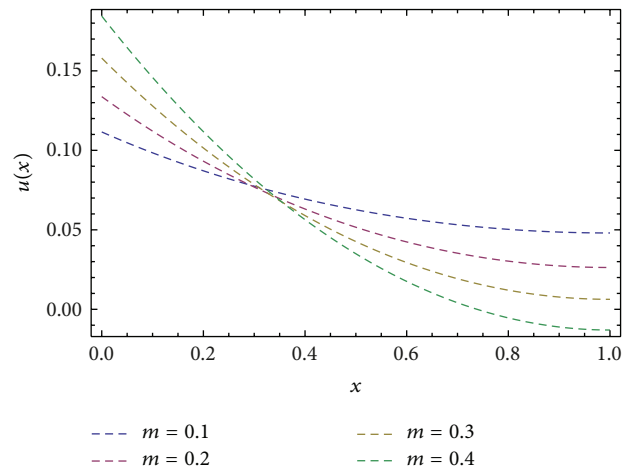


FIGURE 7: The figure shows this gravity effect of “ m ” on velocity profile for lift problem, where $\alpha = 0.1$, $\beta = 1$, $M = 1$, and $\Lambda = 0.2$.

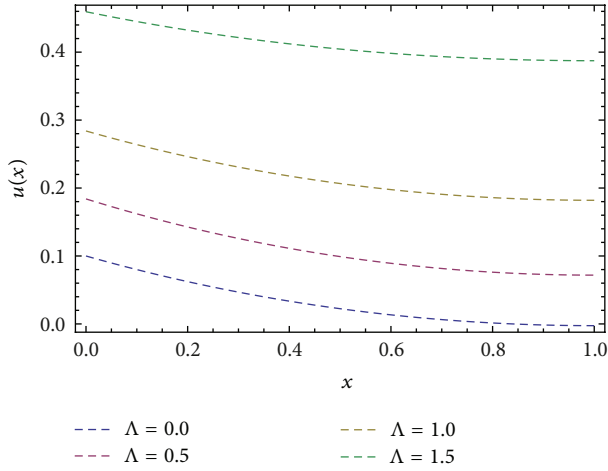


FIGURE 8: Variation of velocity for various values of slip parameter Λ , by fixing $\alpha = 0.1$; $m = 0.2$; $\beta = 1.2$; $M = 0.6$.

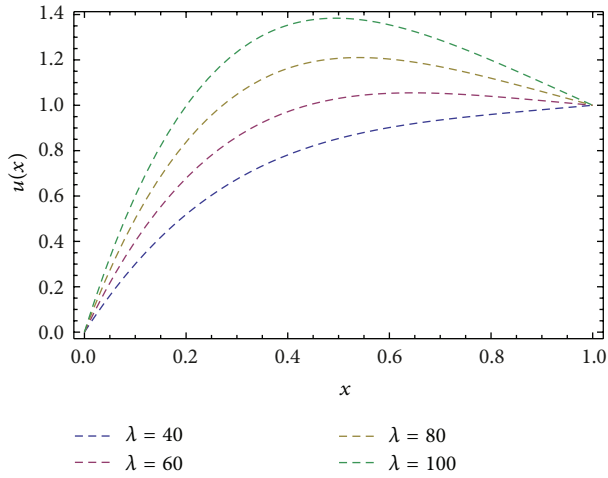


FIGURE 9: The influence of the dimensionless " λ " on the temperature distribution for lift problem keeping, $\alpha = 0.5$, $\beta = 0.2$, $M = 2$, and $m = 1$.

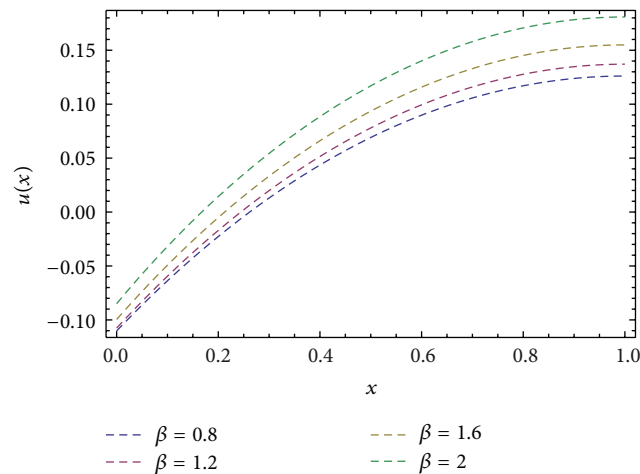


FIGURE 10: The influence of the non-Newtonian effect " β " on the velocity profile $u(x)$ of drainage problem keeping $m = 0.5$, $M = 1$, and $\Lambda = 0.2$.

TABLE 1: Comparison of (OHAM) and (ADM) for lift velocity profile.

x	OHAM	ADM	Absolut error
0.0	0.105386	0.106257	0.870×10^{-3}
0.1	0.062934	0.062767	0.167×10^{-3}
0.2	0.024379	0.023672	0.707×10^{-3}
0.3	-0.010137	-0.011089	0.953×10^{-3}
0.4	-0.0404456	-0.041497	0.105×10^{-2}
0.5	-0.066426	-0.067481	0.106×10^{-2}
0.6	-0.087899	-0.088944	0.105×10^{-2}
0.7	-0.104748	0.105779	0.104×10^{-2}
0.8	-0.116863	-0.117886	0.103×10^{-2}
0.9	-0.124166	-0.125185	0.102×10^{-2}

TABLE 2: Comparison of (OHAM) and (ADM) for drainage velocity profile.

x	OHAM	ADM	Absolut error
0.0	-0.00096249	-0.001043	0.804×10^{-4}
0.1	0.0081824	0.00810405	0.784×10^{-4}
0.2	0.0163486	0.016272	0.767×10^{-4}
0.3	0.0235415	0.0234664	0.752×10^{-4}
0.4	0.0297656	0.0296918	0.739×10^{-4}
0.5	0.0350252	0.0349525	0.728×10^{-4}
0.6	0.0393238	0.0392519	0.719×10^{-4}
0.7	0.0426641	0.0425929	0.713×10^{-4}
0.8	0.0450485	0.0449777	0.708×10^{-4}
0.9	0.0464684	0.0464079	0.705×10^{-4}

Figures 7 and 12, respectively. Due to friction force the gravitational effect seems to be smaller near the belt. It can be seen that there is a point in the domain, where the velocity of the fluid becomes approximately the same for different values of gravitational parameter. The reason is that the friction of the belt becomes negligible at this point. On increasing m after this point in lifting flow, the velocity decreases due to negligible friction but in drainage flow after this point, the velocity of the fluid increases. The effect of slip parameter can be observed from Figures 8 and 13. It is noticed that the speed of the fluid near the belt is greater than the speed at the surface. When we increase the slip parameter, the velocity of the fluid increases and comparatively this increase can be seen more clearly between $\Lambda = 1.0$ and $\Lambda = 1.5$ because the friction goes on decreasing. Figures 9 and 14 indicate the dimensionless temperature distribution for different values of λ . It can be seen that temperature distribution increases as the dimensionless parameter λ increases and becomes more flattened for large values of λ . Figure 15 shows the effect of local Reynolds number versus skin friction. This figure shows that the Reynolds number decreases the skin friction. For large values of Reynolds number, the skin friction vanishes. Figure 16 shows the effect of non-Newtonian parameter β versus skin friction.

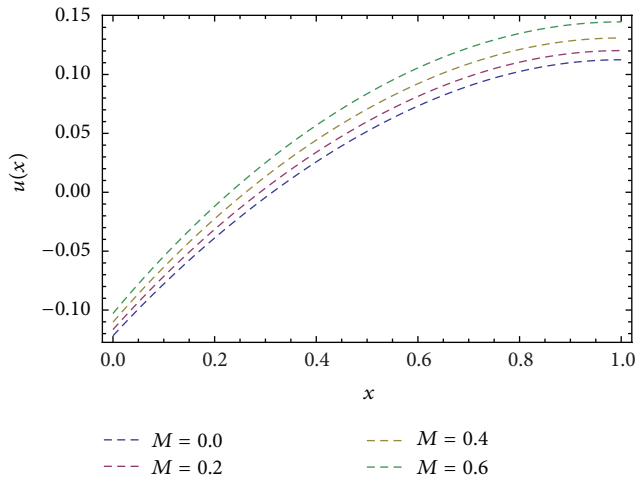


FIGURE 11: The influence of the magnetic force “ M ” on velocity profile shown in this figure for the drainage problem, where $m = 0.5$, $\beta = 1$, and $\Lambda = 0.2$.

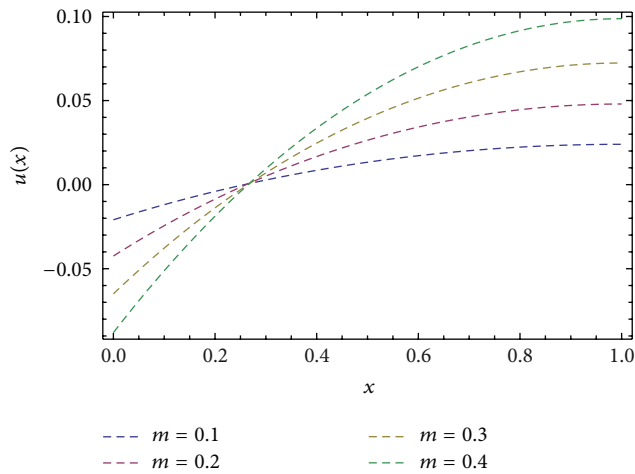


FIGURE 12: This figure shows the influence of the gravitational parameter m on the velocity profile “ $u(x)$ ” for drainage problem keeping $\beta = 1$, $M = 1$, and $\Lambda = 0.2$.

11. Conclusion

The constitutive equation governing the flow of a third grade fluid for lifting and drainage of fluid with slip boundary conditions is solved analytically by using Adomian decomposition method. Expression for velocity field, volume flow rate, skin friction, and temperature distribution is derived and sketched. It is concluded that velocity increases as the gravitational parameter m decreases in lifting case while velocity increases as this parameter increases in drainage. For small values of β , the velocity profile differs little from the Newtonian one; however when β is increased, these profiles become more flattened showing the shear-thinning effect. It can be seen that the boundary layer thickness is reciprocal to the transverse magnetic effect and the velocity decreases as it progresses towards the surface of the fluid. On increasing the slip parameter, the velocity of the fluid increases and

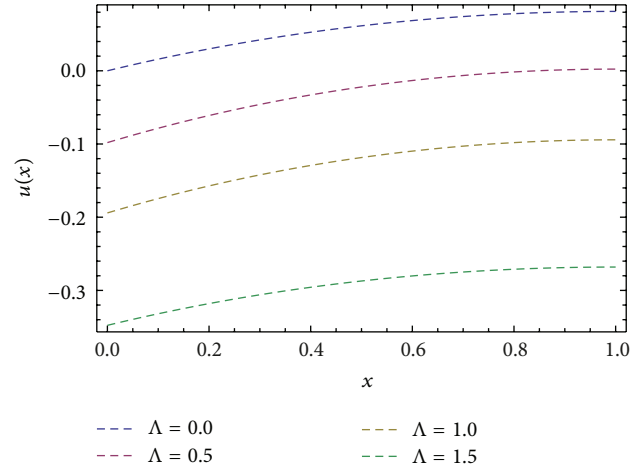


FIGURE 13: Variation of velocity for various values of slip parameter Λ , by fixing $M = 0.6$, $m = 0.2$, and $\beta = 1.2$.

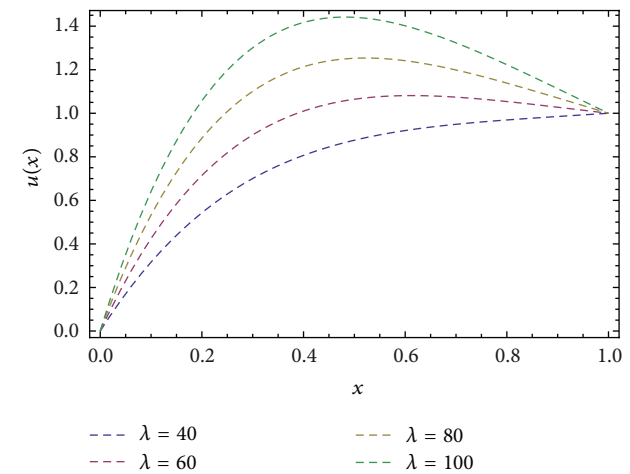


FIGURE 14: The influence of dimensionless number “ λ ” on the temperature distribution “ $u(x)$ ”, for drainage problem keeping $m = 0.5$, $M = 1$, $\Lambda = 0.4$, and $\beta = 1$.

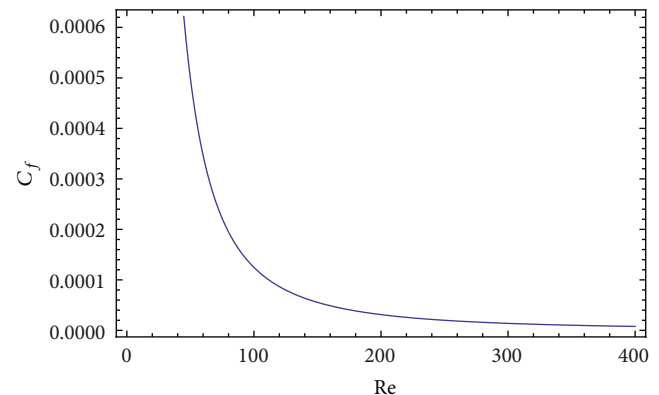


FIGURE 15: Variation of skin friction versus Reynolds number.

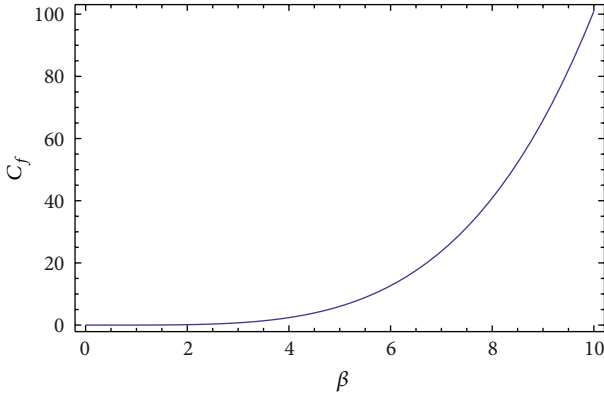


FIGURE 16: Variation of skin friction versus non-Newtonian parameter β .

comparatively this increase can be seen more clearly between $\Lambda = 1.0$ and $\Lambda = 1.5$ because the friction goes on decreasing. According to the best of our knowledge there is no previous literature about discussed problem; this is our first attempt to handle this problem with slip boundary condition. Also this problem is more general when compared to linear viscous model and second grade fluid model.

Appendix

$$\begin{aligned}
 \Psi_0 = & -\frac{2}{15}\Lambda m M^2 + \Lambda^3 m M^2 + \frac{1}{3}\Lambda M^2 \alpha \\
 & + \Lambda^2 M^2 \alpha - \frac{12}{5}\Lambda m^3 M \beta + 4\Lambda^2 m^3 M \beta \\
 & - \frac{5}{4}\Lambda m^3 M^2 \beta + \frac{13}{6}\Lambda^2 m^3 M^2 \beta \\
 & + 8\Lambda^3 m^3 M^2 \beta + 6\Lambda m^2 M \alpha \beta \\
 & + \frac{11}{2}\Lambda m^2 M^2 \alpha \beta - 6\Lambda m M^2 \alpha^2 \beta \\
 & - 12\Lambda m^5 \beta^2 - 27\Lambda m^5 M \beta^2 \\
 & + 66\Lambda^2 m^5 M \beta^2 - \Lambda^2 m^5 M^2 \beta^2 \\
 & + 36\Lambda^3 m^5 M^2 \beta^2 + 78\Lambda m^4 M \alpha \beta^2 \\
 & + 12\Lambda^2 m^4 M^2 \alpha \beta^2 - 144\Lambda m^7 \beta^3 \\
 & + 42\Lambda^2 m^7 M \beta^3 + 48\Lambda^3 m^7 M^2 \beta^3, \\
 \Psi_1 = & \frac{2mM^2}{15} - \Lambda^2 m M^2 - \frac{M^2 \alpha}{3} - \Lambda M^2 \alpha \\
 & + \frac{12}{5}m^3 M \beta - 4\Lambda m^3 M \beta \\
 & + \frac{4}{3}\Lambda m^3 M^2 \beta - 8\Lambda^2 m^3 M^2 \beta
 \end{aligned}$$

$$\begin{aligned}
 & - 6m^2 M \alpha \beta - 6\Lambda m^2 M^2 \alpha \beta \\
 & + 12m^5 \beta^2 - 12\Lambda^2 m^5 M^2 \beta^2, \\
 \Psi_2 = & \frac{1}{6}\Lambda m M^2 - \frac{1}{2}\Lambda^2 m M^2 \\
 & - \frac{1}{2}\Lambda M^2 \alpha + 2m^3 M \beta \\
 & - 8\Lambda m^3 M \beta + \Lambda m^3 M^2 \beta \\
 & - 4\Lambda^2 m^3 M^2 \beta - 9m^2 M \alpha \beta \\
 & - 3\Lambda m^2 M^2 \alpha \beta + 30m^5 \beta^2 \\
 & - 12\Lambda m^5 M \beta^2 - 6\Lambda^2 m^5 M^2 \beta^2,
 \end{aligned}$$

$$\begin{aligned}
 \Psi_3 = & -\frac{mM^2}{18} + \frac{1}{6}\Lambda m M^2 \\
 & + \frac{M^2 \alpha}{6} - \frac{2}{3}m^3 M \beta \\
 & - 6\Lambda m^3 M \beta + \frac{1}{3}\Lambda m^3 M^2 \beta \\
 & - 6m^2 M \alpha \beta + 40m^5 \beta^2 \\
 & - 12\Lambda m^5 M \beta^2,
 \end{aligned}$$

$$\begin{aligned}
 \Psi_4 = & \frac{1}{24}\Lambda m M^2 + \frac{M^2 \alpha}{24} - 2m^3 M \beta \\
 & - \frac{3}{2}\Lambda m^3 M \beta + \frac{1}{12}\Lambda m^3 M^2 \beta \\
 & - \frac{3}{2}m^2 M \alpha \beta + 30m^5 \beta^2 \\
 & - 3\Lambda m^5 M \beta^2,
 \end{aligned}$$

$$\Psi_5 = \frac{mM^2}{120} - \frac{11}{10}m^3 M \beta + 12m^5 \beta^2,$$

$$\Psi_6 = \frac{mM^2}{720} - \frac{11}{60}m^3 M \beta + 2m^5 \beta^2,$$

$$\begin{aligned}
 \Phi_0 = & \Lambda m + \frac{\Lambda m M}{3} - \Lambda^2 m M - \frac{2}{15}\Lambda m M^2 \\
 & + \Lambda^3 m M^2 + \alpha - \Lambda M \alpha + \frac{1}{3}\Lambda M^2 \alpha \\
 & + \Lambda^2 M^2 \alpha + 4\Lambda m^3 \beta - \frac{2}{5}\Lambda m^3 M \beta \\
 & - 4\Lambda^2 m^3 M \beta - \frac{5}{4}\Lambda m^3 M^2 \beta \\
 & + \frac{13}{6}\Lambda^2 m^3 M^2 \beta + 8\Lambda^3 m^3 M^2 \beta \\
 & + \frac{3}{2}\Lambda m^2 M^2 \alpha \beta + 12\Lambda^2 m^2 M^2 \alpha \beta
 \end{aligned}$$

$$\begin{aligned}
& -27\Lambda m^5 M\beta^2 + 54\Lambda^2 m^5 M\beta^2 \\
& -\Lambda^2 m^5 M^2 \beta^2 + 36\Lambda^3 m^5 M^2 \beta^2 \\
& + 8\Lambda m^2 M\alpha\beta^2 - 24\Lambda^2 m^2 M\alpha\beta^2 \\
& + 54\Lambda m^4 M\alpha\beta^2 + 36\Lambda^2 m^4 M^2 \alpha\beta^2 \\
& - 144\Lambda m^7 \beta^3 + 84\Lambda^2 m^7 M\beta^3 \\
& + 48\Lambda^3 m^7 M^2 \beta^3 + 48\Lambda m^4 \alpha\beta^3 \\
& - 48\Lambda^2 m^4 M\alpha\beta^3 - 24\Lambda m\alpha^2 \beta^3, \\
\Phi_1 = & m - \frac{mM}{3} + \Lambda mM + \frac{2mM^2}{15} \\
& - \Lambda^2 mM^2 + M\alpha - \frac{M^2 \alpha}{3} \\
& - \Lambda M^2 \alpha - 2m^3 \beta + \frac{12}{5} m^3 M\beta \\
& - 2\Lambda m^3 M\beta + \frac{4}{3} \Lambda m^3 M^2 \beta \\
& - 8\Lambda^2 m^3 M^2 \beta - 6m^2 M\alpha\beta \\
& - 6\Lambda m^2 M^2 \alpha\beta + 12m^5 \beta^2 \\
& - 12\Lambda^2 m^5 M^2 \beta^2, \\
\Phi_2 = & \frac{m}{2} + \frac{\Lambda mM}{2} + \frac{1}{6} \Lambda mM^2 \\
& - \frac{1}{2} \Lambda^2 mM^2 + \frac{M\alpha}{2} - \frac{1}{2} \Lambda M^2 \alpha \\
& - 3m^3 \beta + 2m^3 M\beta \\
& - 7\Lambda m^3 M\beta + \Lambda m^3 M^2 \beta \\
& - 4\Lambda^2 m^3 M^2 \beta - 9m^2 M\alpha\beta \\
& - 3\Lambda m^2 M^2 \alpha\beta + 30m^5 \beta^2 \\
& - 12\Lambda m^5 M\beta^2 - 3\Lambda^2 m^5 M^2 \beta^2, \\
\Phi_3 = & \frac{mM}{6} - \frac{mM^2}{18} + \frac{1}{6} \Lambda mM^2 \\
& + \frac{M^2 \alpha}{6} - 2m^3 \beta - \frac{2}{3} m^3 M\beta \\
& - 6\Lambda m^3 M\beta + \frac{1}{3} \Lambda m^3 M^2 \beta \\
& - 6m^2 M\alpha\beta + 40m^5 \beta^2 \\
& - 12\Lambda m^5 M\beta^2,
\end{aligned}$$

$$\begin{aligned}
\Phi_4 = & \frac{mM}{24} + \frac{1}{24} \Lambda mM^2 + \frac{M^2 \alpha}{24} \\
& - \frac{m^3 \beta}{2} - 2m^3 M\beta - \frac{3}{2} \Lambda m^3 M\beta \\
& + \frac{1}{12} \Lambda m^3 M^2 \beta - \frac{3}{2} m^2 M\alpha\beta \\
& + 30m^5 \beta^2 - 3\Lambda m^5 M\beta^2, \\
\Phi_5 = & \frac{mM^2}{120} - \frac{11}{10} m^3 M\beta + 12m^5 \beta^2, \\
\Phi_6 = & \frac{mM^2}{720} - \frac{11}{60} m^3 M\beta + 2m^5 \beta^2, \\
\xi_0 = & \frac{2}{15} mM^2 \Lambda + \frac{12}{5} m^3 M\beta \Lambda \\
& + \frac{5}{4} m^3 M^2 \beta \Lambda + 12m^5 \beta^2 \Lambda \\
& + 27m^5 M\beta^2 \Lambda + 144m^7 \beta^3 \Lambda \\
& - 4m^3 M\beta \Lambda^2 - \frac{13}{6} m^3 M^2 \beta \Lambda^2 \\
& - 66m^5 M\beta^2 \Lambda^2 + m^5 M^2 \beta^2 \Lambda^2 \\
& - 84m^7 M\beta^3 \Lambda^2 - mM^2 \Lambda^3 \\
& - 8m^3 M^2 \beta \Lambda^3 - 36m^5 M^2 \beta^2 \Lambda^3 \\
& - 48m^7 M^2 \beta^3 \Lambda^3, \\
\xi_1 = & mM^2 \Lambda^2 - \frac{2mM^2}{15} \\
& - \frac{12}{5} m^3 M\beta - 12m^5 \beta^2 \\
& + 4m^3 M\beta \Lambda - \frac{4}{3} m^3 M^2 \beta \Lambda \\
& + 8m^3 M^2 \beta \Lambda^2 + 12m^5 M^2 \beta^2 \Lambda^2, \\
\xi_2 = & 6m^5 M^2 \beta^2 \Lambda^2 - 2m^3 M\beta \\
& - 30m^5 \beta^2 - \frac{1}{6} mM^2 \Lambda \\
& + 8m^3 M\beta \Lambda - m^3 M^2 \beta \Lambda \\
& + 12m^5 M\beta^2 \Lambda + \frac{1}{2} mM^2 \Lambda^2 \\
& + 4m^3 M^2 \beta \Lambda^2,
\end{aligned}$$

$$\begin{aligned}
\xi_3 &= \frac{mM^2}{18} + \frac{2}{3}m^3M\beta - 40m^5\beta^2 \\
&\quad - \frac{1}{6}mM^2\Lambda + 6m^3M\beta\Lambda \\
&\quad - \frac{1}{3}m^3M^2\beta\Lambda + 12m^5M\beta^2\Lambda, \\
\xi_4 &= m^3M\beta - 30m^5\beta^2 \\
&\quad - \frac{1}{24}mM^2\Lambda + \frac{3}{2}m^3M\beta\Lambda \\
&\quad - \frac{1}{12}m^3M^2\beta\Lambda + 3m^5M\beta^2\Lambda, \\
\xi_5 &= \frac{11}{10}m^3M\beta - 12m^5\beta^2 - \frac{mM^2}{120}, \\
\xi_6 &= -\frac{mM^2}{720} + \frac{11}{120}m^3M\beta - 2m^5\beta^2, \\
\omega_0 &= -m\Lambda - \frac{mM\Lambda}{3} + \frac{2}{15}mM^2\Lambda \\
&\quad - 4m^3\beta\Lambda + \frac{2}{5}m^3M\beta\Lambda + \frac{5}{4}m^3M^2\beta\Lambda \\
&\quad + 27m^5M\beta^2\Lambda + 144m^7\beta^3\Lambda \\
&\quad + mM\Lambda^2 + 4m^3M\beta\Lambda^2 \\
&\quad - \frac{13}{6}m^3M^2\beta\Lambda^2 - 54m^5M\beta^2\Lambda^2 \\
&\quad + m^5M^2\beta^2\Lambda^2 - 84m^7M\beta^3\Lambda^2 \\
&\quad - mM^2\Lambda^3 - 8m^3M^2\beta\Lambda^3 \\
&\quad - 36m^5M^2\beta^2\Lambda^3 - 48m^7M^2\beta^3\Lambda^3, \\
\omega_1 &= \frac{mM}{3} - m - \frac{2mM^2}{15} + 2m^3\beta \\
&\quad - \frac{12}{5}m^3M\beta - 12m^5\beta^2 - mM\Lambda \\
&\quad + 2m^3M\beta\Lambda - \frac{4}{3}m^3M^2\beta\Lambda \\
&\quad + mM^2\Lambda^2 + 8m^3M^2\beta\Lambda^2 \\
&\quad + 12m^5M^2\beta^2\Lambda^2, \\
\omega_2 &= -\frac{m}{2} + 3m^3\beta - 2m^3M\beta \\
&\quad - 30m^5\beta^2 - \frac{mM\Lambda}{2} - \frac{1}{6}mM^2\Lambda \\
&\quad + 7m^3M\beta\Lambda - m^3M^2\beta\Lambda \\
&\quad + 12m^5M\beta^2\Lambda + \frac{1}{2}mM^2\Lambda^2 \\
&\quad + 4m^3M^2\beta\Lambda^2 + 6m^5M^2\beta^2\Lambda^2, \\
\omega_3 &= \frac{mM^2}{18} - \frac{mM}{6} + 2m^3\beta \\
&\quad + \frac{2}{3}m^3M\beta - 40m^5\beta^2 \\
&\quad - \frac{1}{6}mM^2\Lambda + 6m^3M\beta\Lambda \\
&\quad - \frac{1}{3}m^3M^2\beta\Lambda + 12m^5M\beta^2\Lambda, \\
\omega_4 &= -\frac{mM}{24} + \frac{m^3\beta}{2} + 2m^3M\beta \\
&\quad - 30m^5\beta^2 - \frac{1}{24}mM^2\Lambda \\
&\quad + \frac{3}{2}m^3M\beta\Lambda - \frac{1}{12}m^3M^2\beta\Lambda \\
&\quad + 3m^5M\beta^2\Lambda, \\
\omega_5 &= \frac{11}{10}m^3M\beta - \frac{mM^2}{120} - 12m^5\beta^2, \\
\omega_6 &= \frac{11}{60}m^3M\beta - \frac{mM^2}{720} - 2m^5\beta^2.
\end{aligned} \tag{A.1}$$

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