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Research Article

A Generalization of the SMW Formula of Operator $A + YGZ^*$ to the $\{2\}$-Inverse Case

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The classical Sherman-Morrison-Woodbury (for short SMW) formula $(A + YGZ^*)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}$ is generalized to the $\{2\}$-inverse case. Some sufficient conditions under which the SMW formula can be represented as $(A + YGZ^*)^{-} = A^{-} - A^{-}Y(G^{-} + Z^*A^{-}Y)^{-}Z^*A^{-}$ are obtained.

1. Introduction

There are numerous applications of the SMW formula in various fields (see [1–6, 12]). An excellent review by Hager [3] described some of the applications to statistics networks, structural analysis, asymptotic analysis, optimization, and partial differential equations. In this note, we consider the SMW formula in which the inverse is replaced by the $\{2\}$-inverse. As we know, the inverse, the group inverse, the Moore-Penrose inverse, and the Drazin inverse all belong to the $\{2\}$-inverse. Hence, the classical SMW formula is generalized.

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$ by $\mathscr{B}(\mathscr{H}, \mathscr{K})$ and by $\mathscr{B}(\mathscr{H})$ when $\mathscr{H} = \mathscr{K}$. For $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, let T^*, let $\mathscr{R}(T)$, and let $\mathscr{N}(T)$ be the adjoint, the range, and the null space of T, respectively. If $A \in \mathscr{B}(\mathscr{H})$ and $G \in \mathscr{B}(\mathscr{K})$ both are invertible, and $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, then $A + YGZ^*$ is invertible if and only if $G^{-1} + Z^*A^{-1}Y$ is invertible. In this case,

$$\left(A + YGZ^*\right)^{-1} = A^{-1} - A^{-1}Y\left(G^{-1} + Z^*A^{-1}Y\right)^{-1}Z^*A^{-1}. \tag{1}$$

The original SMW [1, 5, 6] formula (1) is only valid when A is invertible. In particular, the SMW formula (1) implies that $(I - A^*A)^{-1} = I + A^*(I - AA^*)^{-1}A$. An operator T is called generalized invertible if there is an operator S such that (I) $TST = T$. The operator S is not unique in general. In order to force its uniqueness, further conditions have to be imposed. The most likely convenient additional conditions are

$$\text{(II)}\ STS = S, \qquad \text{(III)}\ (TS)^* = TS, \qquad \text{(IV)}\ (ST)^* = ST, \qquad \text{(V)}\ TS = ST. \tag{2}$$

One also considers (I_k) $T^kST = T^k$ with some $k \in \mathbf{Z}^+$. Clearly, (I) = (I_1). Elements $S \in \mathscr{B}(\mathscr{H})$ satisfying (II) are called $\{2\}$-inverse of T, denoted by $S = T^{-}$. Similarly, (I, II, and V)-inverses are called group inverses, denoted by $S = T^{\#}$. (I, II, III, and IV)-inverses are Moore-Penrose inverses (for short MP inverses), denoted by $S = T^{+}$. It is well known that T has the MP inverse if and only if $\mathscr{R}(T)$ is closed. And $(\mathrm{I}_k$, II, and V)-inverses are called Drazin inverses, denoted by $S = T^d$ (see [7]), where k is the Drazin index of T. In general, the $\{2\}$-inverse of T is not unique. It is clear that TT^{-}, $T^{-}T$ are idempotents and $T^{+} = T^{d} = T^{\#} = T^{-} = T^{-1}$ if $T \in \mathscr{B}(\mathscr{H})$ is invertible (see [8, 9]).

2. Main Results

The following lemmas are used to prove our main results.

Lemma 1. *If $A \in \mathscr{B}(\mathscr{H})$ and $P = P^2 \in \mathscr{B}(\mathscr{H})$, then*

(i) $PA = A \Leftrightarrow \mathscr{R}(A) \subset \mathscr{R}(P)$,

(ii) $AP = A \Leftrightarrow \mathscr{N}(P) \subset \mathscr{N}(A)$.

It is well known that $\mathscr{R}(A) \subset \mathscr{R}(B)$ if and only if there exists an operator C such that $A = BC$ (see [10, 11]). So, we have the following result.

Lemma 2. *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$. Then, $\mathscr{R}(A^- A) = \mathscr{R}(A^-)$, and $\mathscr{N}(AA^-) = \mathscr{N}(A^-)$.*

Proof. Since $A^- = A^- AA^-$,

$$\mathscr{R}(A^-) = \mathscr{R}(A^- AA^-) \subset \mathscr{R}(A^- A) \subset \mathscr{R}(A^-). \quad (3)$$

So, $\mathscr{R}(A^-) = \mathscr{R}(A^- A)$. Similarly, from

$$\mathscr{N}(A^-) \subset \mathscr{N}(AA^-) \subset \mathscr{N}(A^- AA^-) = \mathscr{N}(A^-), \quad (4)$$

there comes $\mathscr{N}(AA^-) = \mathscr{N}(A^-)$. □

Now, the first main result of this paper is given as follows, which generalizes the main results [2, Theorems 2.1–2.4] and [12, Theorem 1] with very short proof.

Theorem 3. *Let $A \in \mathscr{B}(\mathscr{H})$, let $G \in \mathscr{B}(\mathscr{K})$, and let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. Let also $B = A + YGZ^*$, and let $T = G^- + Z^* A^- Y$. If*

$$\begin{gathered} \mathscr{R}(A^-) \subset \mathscr{R}(B^-), \qquad \mathscr{N}(A^-) \subset \mathscr{N}(B^-), \\ \mathscr{N}(G^-) \subset \mathscr{N}(Y), \qquad \mathscr{N}(T^-) \subset \mathscr{N}(G), \end{gathered} \quad (5)$$

then

$$(A + YGZ^*)^- = A^- - A^- Y(G^- + Z^* A^- Y)^- Z^* A^-. \quad (6)$$

Proof. Let the conditions in (5) hold. By Lemma 2, we have

$$\begin{gathered} \mathscr{R}(A^-) \subset \mathscr{R}(B^-) = \mathscr{R}(B^- B), \\ \mathscr{N}(AA^-) = \mathscr{N}(A^-) \subset \mathscr{N}(B^-). \end{gathered} \quad (7)$$

By Lemma 1, we get that $B^- BA^- = A^-$ and $B^- AA^- = B^-$. Hence, $B^- Y + B^-(B - A)A^- Y = A^- Y$. Similarly, from $\mathscr{N}(G^-) \subset \mathscr{N}(Y)$, we get that $YGG^- = Y$ and

$$\begin{aligned} B^- YGT &= B^- YGG^- + B^- YGZ^* A^- Y \\ &= B^- Y + B^- (B - A) A^- Y \\ &= A^- Y. \end{aligned} \quad (8)$$

So, the condition $\mathscr{N}(T^-) \subset \mathscr{N}(G)$ implies that $B^- YG = B^- YGTT^- = A^- YT^-$. From $B = A + YGZ^*$, we deduce that $B^- BA^- = B^- AA^- + B^- YGZ^* A^-$. Hence,

$$\begin{aligned} B^- &= B^- AA^- \\ &= B^- BA^- - B^- YGZ^* A^- \\ &= A^- - A^- YT^- Z^* A^-. \end{aligned} \quad (9)$$

□

For $T \in \mathscr{B}(\mathscr{H})$, let $T^{\odot}$ denote any kind of standard inverse T^{-1}, group inverse $T^{\#}$, MP inverse T^{+}, and Drazin inverse, respectively. Since $T^{\odot}$ belongs to {2}-inverse, we get the following corollary.

Corollary 4. *Let $A \in \mathscr{B}(\mathscr{H})$, let $G \in \mathscr{B}(\mathscr{K})$, and let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ such that $A^{\odot}$ and $G^{\odot}$ exist. Let also $B = A + YGZ^*$, and let $T = G^{\odot} + Z^* A^{\odot} Y$ such that $B^{\odot}$ and $T^{\odot}$ exist. If*

$$\begin{gathered} \mathscr{R}(A^{\odot}) \subset \mathscr{R}(B^{\odot}), \qquad \mathscr{N}(A^{\odot}) \subset \mathscr{N}(B^{\odot}), \\ \mathscr{N}(G^{\odot}) \subset \mathscr{N}(Y), \qquad \mathscr{N}(T^{\odot}) \subset \mathscr{N}(G), \end{gathered} \quad (10)$$

then

$$(A + YGZ^*)^{\odot} = A^{\odot} - A^{\odot} Y(G^{\odot} + Z^* A^{\odot} Y)^{\odot} Z^* A^{\odot}. \quad (11)$$

The following is our second main result.

Theorem 5. *Let $A \in \mathscr{B}(\mathscr{H})$, let $G \in \mathscr{B}(\mathscr{K})$, and let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. Let also $B = A + YGZ^*$, and let $T = G^- + Z^* A^- Y$. If any of the following items holds:*

$$\begin{gathered} (i) \ \ \mathscr{N}(A^- A) \subset \mathscr{N}(Z^*), \qquad \mathscr{R}(Z^*) \subset \mathscr{R}(G^-), \\ \mathscr{N}(T^- T) \subset \mathscr{N}(Y), \\ (ii) \ \ \mathscr{N}(G^-) \subset \mathscr{N}(Y), \qquad \mathscr{R}(Z^*) \subset \mathscr{R}(TT^-), \\ \mathscr{R}(Y) \subset \mathscr{R}(AA^-), \end{gathered} \quad (12)$$

then $(A + YGZ^*)^- = A^- - A^- Y(G^- + Z^* A^- Y)^- Z^* A^-$.

Proof. Define $X = A^- - A^- YT^- Z^* A^-$. We will prove that $XBX = X$.

(i) Since $\mathscr{N}(A^- A) \subset \mathscr{N}(Z^*)$, $\mathscr{N}(T^- T) \subset \mathscr{N}(Y)$, and $\mathscr{R}(Z^*) \subset \mathscr{R}(G^-)$, by Lemma 1, we have $Z^*(I - A^- A) = 0$, $Y(I - T^- T) = 0$, and $(I - G^- G)Z^* = 0$. Hence,

$$\begin{aligned} XB &= (A^- - A^- YT^- Z^* A^-)(A + YGZ^*) \\ &= A^- A + A^- YGZ^* - A^- YT^- Z^* A^- A \\ &\quad - A^- YT^- Z^* A^- YGZ^* \\ &= A^- A + A^- YGZ^* - A^- YT^- Z^* \\ &\quad - A^- YT^- (T - G^-) GZ^* \\ &= A^- A + A^- Y (I - T^- T) GZ^* \\ &\quad - A^- YT^- (I - G^- G) Z^* \\ &= A^- A, \end{aligned} \quad (13)$$

and $XBX = A^- AX = X$.

(ii) If $\mathcal{N}(G^-) \subset \mathcal{N}(Y)$, $\mathcal{R}(Z^*) \subset \mathcal{R}(TT^-)$, and $\mathcal{R}(Y) \subset \mathcal{R}(AA^-)$, then $Y(I-GG^-)=0$, $(I-TT^-)Z^*=0$, and $(I-AA^-)Y=0$. We have

$$\begin{aligned} BX &= (A+YGZ^*)(A^- - A^-YT^-Z^*A^-) \\ &= AA^- + YGZ^*A^- - AA^-YT^-Z^*A^- \\ &\quad - YGZ^*A^-YT^-Z^*A^- \\ &= AA^- + YGZ^*A^- - YT^-Z^*A^- \\ &\quad - YG(T-G^-)T^-Z^*A^- \\ &= AA^- + YG(I-TT^-)Z^*A^- \\ &\quad - Y(I-GG^-)T^-Z^*A^- \\ &= AA^- \end{aligned} \tag{14}$$

and $XBX = XAA^- = X$.

□

From Theorem 5, we get the following corollary which generalizes some recent results (e.g., see [2, Theorem 2.1] and [12, Theorem 1]).

Corollary 6. *Let $A \in \mathcal{B}(\mathcal{H})$, let $G \in \mathcal{B}(\mathcal{K})$, and let $Y, Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $A^{\odot}$ and $G^{\odot}$ exist. Let also $B = A + YGZ^*$, and let $T = G^{\odot} + Z^*A^{\odot}Y$ such that $B^{\odot}$ and $T^{\odot}$ exist. If*

$$\begin{aligned} &\mathcal{N}(A^{\odot}A) \subset \mathcal{N}(Z^*), \qquad \mathcal{R}(Z^*) \subset \mathcal{R}(G^{\odot}), \\ &\mathcal{N}(T^{\odot}T) \subset \mathcal{N}(Y), \qquad \mathcal{N}(G^{\odot}) \subset \mathcal{N}(Y), \\ &\mathcal{R}(Z^*) \subset \mathcal{R}(TT^{\odot}), \qquad \mathcal{R}(Y) \subset \mathcal{R}(AA^{\odot}), \end{aligned} \tag{15}$$

then

$$(A+YGZ^*)^{\odot} = A^{\odot} - A^{\odot}Y\left(G^{\odot} + Z^*A^{\odot}Y\right)^{\odot}Z^*A^{\odot}. \tag{16}$$

Proof. Firstly, if $\odot$ denotes the standard inverse, then (15) holds automatically, and the result (SMW formula) follows immediately by the proof of Theorem 5.

Secondly, if $\odot$ denotes the MP inverse, by the proof of Theorem 5, $X = A^+ - A^+YT^+Z^*A^+$ satisfies $XB = A^+A$ and $BX = AA^+$. Thus, $(XB)^* = XB$, and $(BX)^* = BX$. Moreover, we have $BXB = AA^+(A+YGZ^*) = A + AA^+YGZ^* = B$ and $XBX = A^+A(A^+ - A^+YT^+Z^*A^+) = X$. By the definition of MP inverse, we know that $(A+YGZ^*)^+ = A^+ - A^+Y(G^+ + Z^*A^+Y)^+Z^*A^+$.

Lastly, if $\odot$ denotes the Drazin inverse (resp., group inverse), by the proof of Theorem 5, $X = A^d - A^dYT^dZ^*A^d$ satisfies $XB = BX = AA^d = A^dA$. Moreover, $XBX = A^dAX = A^dA(A^d - A^dYT^dZ^*A^d) = X$, and $B - B^2X = B(I - BX) = (A+YGZ^*)(I-AA^d) = A(I-AA^d)$ is quasi-nilpotent (resp., $B - B^2X = 0$ for the group inverse case). So, by the definition of Drazin inverse, we have $(A+YGZ^*)^d = A^d - A^dYT^dZ^*A^d$ (resp. $(A+YGZ^*)^{\#} = A^{\#} - A^{\#}YT^{\#}Z^*A^{\#}$). □

3. Concluding Remark

In this note, we mainly extend the SMW formula to the form $(A+YGZ^*)^- = A^- - A^-Y(G^- + Z^*A^-Y)^-Z^*A^-$ under some sufficient conditions. If A, G, T, and B in Theorem 3 are invertible, then (5) and (6) hold automatically. Hence, Theorem 3 generalizes the classical SMW formula. In [2, Theorem 2.1], when A is MP-invertible and $G^{-1} + Z^*A^+Y$ is invertible, the sufficient conditions under which the SMW formula can be represented as

$$(A+YGZ^*)^+ = A^+ - A^+Y\left(G^{-1} + Z^*A^-Y\right)^{-1}Z^*A^+ \tag{17}$$

are given. It is obvious that Theorem 3 also generalizes [2, Theorem 2.3] greatly.

Under weaker assumptions than those used in the literature, our results are new and robust to the classical SMW formula even for the finite dimensional case. It is natural to ask if we can extend our results for the various inverses of $A + YGZ^*$ in some weaker assumptions, which will be our future research topic.

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