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Research Article

On the Existence of Positive Solutions of Resonant and Nonresonant Multipoint Boundary Value Problems for Third-Order Nonlinear Differential Equations

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Positive solutions for a kind of third-order multipoint boundary value problem under the non-resonant conditions and the resonant conditions are considered. In the nonresonant case, by using Leggett-Williams fixed-point theorem, the existence of at least three positive solutions is obtained. In the resonant case, by using Leggett-Williams norm-type theorem due to O'Regan and Zima, existence result of at least one positive solution is established. The results obtained are valid and new for the problem discussed. Two examples are given to illustrate the main results.

1. Introduction

We consider the third-order m-point boundary value problem given by

$$\begin{gathered} x'''(t) + f(t, x(t)) = 0, \quad t \in [0,1] \\ x''(0) = 0, \qquad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{gathered} \tag{1}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 \le \alpha_i$, $\beta_i \le 1$, $i = 1, 2, \ldots, m-2$, $\sum_{i=1}^{m-2} \alpha_i < 1$, $\sum_{i=1}^{m-2} \beta_i \le 1$, and $f \in C([0,1] \times [0,\infty), R)$. For the convenience of writing later, we denote $\xi_0 = 0, \xi_{m-1} = 1$, and $\alpha_0 = \alpha_{m-1} = \beta_0 = \beta_{m-1} = 0$.

If condition $\sum_{i=1}^{m-2} \beta_i \ne 1$ holds, the problem is nonresonant; that is, the associated linear problem

$$\begin{gathered} x'''(t) = 0, \quad t \in [0,1] \\ x''(0) = 0, \qquad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \\ x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \end{gathered} \tag{2}$$

has only zero solution, and the differential operator with boundary conditions is invertible. Otherwise, the problem is at resonance.

Third-order differential equations arise in many different areas of applied mathematics and physics, as the varying cross-section or deflection of a curved beam having a constant, three-layer beam and so on [1]. Recently, there have been extensive studies on positive solutions for nonresonant two-point or three-point boundary value problems for nonlinear third-order ordinary differential equations. For examples, Anderson [2] established the existence of at least three positive solutions to problem

$$\begin{gathered} -x'''(t) + f(x(t)) = 0, \quad t \in (0,1) \\ x(0) = x'(t_2) = x''(1) = 0, \end{gathered} \tag{3}$$

where $f : R \to [0, +\infty)$ is continuous and $1/2 \le t_2 < 1$.

By using the well-known Guo-Krasnosel'skii fixed-point theorem [3], Palamides and Smyrlis [4] proved that there exists at least one positive solution for the third-order three-point problem:

$$x'''(t) = a(t) f(t, x(t)), \quad t \in (0,1), \\ x''(\eta) = 0, \qquad x(0) = x(1) = 0, \qquad \eta \in (0,1). \tag{4}$$

In another paper [5], Graef and Kong studied the existence of positive solutions for the third-order semipositone boundary value problem:

$$u'''(t) = \lambda f(t, u(t)) + e(t), \quad t \in (0,1), \\ u(0) = u'(p) = \int_q^1 w(s) u''(s)\, ds = 0, \tag{5}$$

where $1/2 < p < q < 1$ are constants and $\lambda > 0$ is a parameter. Also $f : (0,1) \times [0, \infty) \to R, e : (0,1) \to R$, and $e \in L(0,1)$. Moreover $w : [q, 1] \to [0, \infty)$ are continuous functions. By using the Guo-Krasnosel'skii fixed-point theorem, the author established the existence of positive solutions. For more existence results of positive solutions for boundary value problems of third-order ordinary differential equations, one can see [6–12] and references therein.

For boundary value problems of second-order or higher-order differential equations at resonance, many existence results of solutions have been established; see [13–25]. In [25], the authors considered the following problem:

$$x'''(t) = f(t, x, x') + e(t), \quad t \in (0,1), \\ x'(0) = 0, \qquad x(1) = \beta x(\eta), \qquad x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i). \tag{6}$$

By using Mawhin continuation theorem [26], the existence results of solutions were obtained under the resonant conditions $\beta = 1$, $\sum_{i=1}^{m-2} \alpha_i = 1$, and $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$ and $\beta = 1/\eta$, $\sum_{i=1}^{m-2} \alpha_i = 1$, and $\sum_{i=1}^{m-2} \alpha_i \xi_i^2 = 0$, respectively.

It is well known that the problem of existence of positive solution for nonlinear BVP is very difficult when the resonant case is considered. Only few works gave the approach in this area for first- and second-order differential equations [27–34]. To our best knowledge, few paper dealt with the existence result of positive solution for resonant third-order boundary value problems. Motivated by the approach in [28–30, 35], we study the positive solution for problem (1) under nonresonant condition $\sum_{i=1}^{m-2} \beta_i < 1$ and resonant condition $\sum_{i=1}^{m-2} \beta_i = 1$, respectively. By using Leggett-Williams fixed-point theorem and its generalization [27, 29], we establish the existence results of positive solutions. The results obtained in this paper are interesting in the following aspects.

(1) In the nonresonant case, Green's function is established and the results obtained are more general than those of earlier work.

(2) It is the first time that the positive solution is considered for third-order boundary value problem at resonance.

2. Background Definitions and Lemmas

Let X, Y be real Banach spaces. A nonempty closed convex set $C \subset X$ is said to be a cone provided that $ax \in C$, if $x \in C$, $a \geq 0$ and $x, -x \in C$ implies $x = 0$.

Definition 1. The map ψ is a nonnegative continuous concave functional on C if $\psi : C \to +\infty$ is continuous and

$$\psi(tx + (1-t) y) \geq t\psi(x) + (1-t)\psi(y), \\ x, y \in C, \ t \in [0,1]. \tag{7}$$

Definition 2. Let constants $0 < a < b$ be given and let ψ be a nonnegative continuous concave functional on the cone C. Define the convex sets C_r and $C(\psi, a, b)$ as follow:

$$C_r = \{x \in C \mid \|x\| < r\}, \\ C(\psi, a, b) = \{x \in C \mid a \leq \psi(x), \|x\| \leq b\}. \tag{8}$$

Lemma 3 (Leggett-Williams fixed-point theorem [35]). *Let $T : \overline{C}_r \to \overline{C}_r$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on C such that $\psi(x) \leq \|x\|$ for all $x \in \overline{C}_r$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(H_1) *$\{x \in C(\psi, b, d) \mid \psi(x) > b\} \neq \varnothing$ and $\psi(Tx) > b$ for $x \in C(\psi, b, d)$,*

(H_2) *$\|Tx\| < a$ for $\|x\| \leq a$,*

(H_3) *$\psi(Tx) > b$ for $x \in C(\psi, b, c)$ with $\|Tx\| \geq d$.*

Then operator T has at least triple fixed-points x_1, x_2, and x_3 with $\|x_1\| < a, b < \psi(x_2)$, $\|x_3\| > a$, and $\psi(x_3) < b$.

Operator $L : \operatorname{dom} L \subset X \to Y$ is called a Fredholm operator with index zero, which means that $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$, and there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Furthermore, for $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Denote by L_P the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$ and its inverse by K_P, so $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ and the coincidence equation $Lx = Nx$ is equivalent to the operator equation:

$$x = (P + JQN) x + K_P (I - Q) Nx. \tag{9}$$

Let $\gamma : X \to C$ be a retraction, which means a continuous mapping such that $\gamma x = x$ for all $x \in C$ and

$$\Psi := P + JQN + K_P (I - Q) N, \quad \Psi_\gamma := \Psi \circ \gamma. \tag{10}$$

Lemma 4 (Leggett-Williams norm-type theorem [28]). *Assume that C is a cone in X and that Ω_1 and Ω_2 are open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Suppose that $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero and that*

(C1) *$QN : X \to Y$ is bounded and continuous and $K_P(I-Q)N : X \to X$ is compact on every bounded subset of X,*

(C2) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda \in (0,1)$,*

(C3) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*

(C4) *$d_B([I-(P+JQN)\gamma]|_{\ker L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,*

(C5) *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \leq x\}$ for some $\mu > 0$ and $\sigma(u_0)$ is such that $\|x+u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,*

(C6) *$(P+JQN)\gamma(\partial\Omega_2) \subset C$,*

(C7) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main Results for Nonresonant Case

In this section we consider the positive solution for the nonresonant case with the condition $0 < \sum_{i=0}^{m-1} \beta_i < 1$ and we always suppose that $f \in C([0,1] \times [0,\infty), [0,\infty))$.

Firstly, we consider the third-order m-point boundary value problem given by

$$x'''(t) + y(t) = 0, \quad t \in [0,1] \tag{11}$$

$$x''(0) = 0, \qquad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \qquad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i). \tag{12}$$

Lemma 5. *Suppose $y(t) \in C[0,1]$. Then problem* (11) *and* (12) *is equivalent to*

$$x(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{13}$$

where for $i = 1, 2, \ldots, m-1$

$$G(t,s) = \begin{cases} a_1 t + a_0, & t \leq s,\ \xi_{i-1} \leq s \leq \xi_i, \\ -\dfrac{1}{2}t^2 + b_1 t + b_0, & t \geq s,\ \xi_{i-1} \leq s \leq \xi_i, \end{cases}$$

$$\begin{aligned}
a_1 &= \frac{\sum_{k=i}^{m-1} \alpha_k (s-\xi_k)}{1-\sum_{k=0}^{m-1}\alpha_k}, \\
b_1 &= \frac{s - \sum_{k=0}^{i-1}\alpha_k s - \sum_{k=i}^{m-1}\alpha_k \xi_k}{1-\sum_{k=0}^{m-1}\alpha_k} \\
a_0 &= \left(\sum_{k=0}^{i-1} \beta_k \xi_k \frac{\sum_{k=i}^{m-1}\alpha_k (s-\xi_k)}{1-\sum_{k=0}^{m-1}\alpha_k} \right. \\
&\quad + \left(\sum_{k=i}^{m-1} \beta_k \xi_k - 1 \right) \\
&\quad \times \frac{s - \sum_{k=0}^{i-1}\alpha_k s - \sum_{k=i}^{m-1}\alpha_k \xi_k}{1-\sum_{k=0}^{m-1}\alpha_k} \\
&\quad \left. + \frac{1}{2}s^2 + \frac{1}{2} - \frac{1}{2}\sum_{k=i}^{m-1}\beta_k \left(\xi_k^2 + s^2\right) \right) \\
&\quad \times \left(1 - \sum_{k=0}^{m-1}\beta_k \right)^{-1}, \\
b_0 &= \left(\sum_{k=0}^{i-1} \beta_k \xi_k \frac{\sum_{k=i}^{m-1}\alpha_k (s-\xi_k)}{1-\sum_{k=0}^{m-1}\alpha_k} \right. \\
&\quad + \frac{1}{2}\sum_{k=0}^{i-1}\beta_k s^2 - \frac{1}{2}\sum_{k=i}^{m-1}\beta_k \xi_k^2 + \frac{1}{2} \\
&\quad + \left(\sum_{k=i}^{m-1} \beta_k \xi_k - 1 \right) \\
&\quad \left. \times \frac{s - \sum_{k=0}^{i-1}\alpha_k s - \sum_{k=i}^{m-1}\alpha_k \xi_k}{1-\sum_{k=0}^{m-1}\alpha_k} \right) \\
&\quad \times \left(1 - \sum_{k=0}^{m-1}\beta_k \right)^{-1}.
\end{aligned} \tag{14}$$

Proof. Let $G(t,s)$ be Green's function of problem $-x'''(t) = 0$ with boundary condition (12). We can suppose

$$G(t,s) = \begin{cases} a_2 t^2 + a_1 t + a_0 & t \leq s,\ \xi_{i-1} \leq s \leq \xi_i, \\ & i = 1, 2, \ldots, m-1 \\ b_2 t^2 + b_1 t + b_0 & t \geq s,\ \xi_{i-1} \leq s \leq \xi_i, \\ & i = 1, 2, \ldots, m-1, \end{cases} \tag{15}$$

where a_0, a_1, a_2, b_0, b_1, and b_2 are undetermined coefficients.

Considering the properties of Green's function together with boundary condition (12), we have

$$\begin{gathered} a_2 s^2 + a_1 s + a_0 = b_2 s^2 + b_1 s + b_0, \\ 2a_2 s + a_1 = 2b_2 s + b_1, \\ 2a_2 - 2b_2 = 1, \quad a_2 = 0, \\ a_1 = \sum_{k=0}^{i-1} \alpha_k \left(2a_2 \xi_k + a_1\right) + \sum_{k=i}^{m-1} \alpha_k \left(2b_2 \xi_k + b_1\right), \\ b_2 + b_1 + b_0 = \sum_{k=0}^{i-1} \beta_k \left(a_2 \xi_k^2 + a_1 \xi_k + a_0\right), \\ + \sum_{k=i}^{m-1} \beta_k \left(b_2 \xi_k^2 + b_1 \xi_k + b_0\right). \end{gathered} \tag{16}$$

The explicit expression of Green's function is obtained by solving the linear function systems. □

Lemma 6. *Green's function $G(t,s)$ satisfies that $G(t,s) \geq 0, t, s \in [0,1]$.*

Proof. For $\xi_{i-1} \leq s \leq \xi_i$, $i = 1, 2, \ldots, m-1$, and $t \leq s$,

$$\frac{\partial G(t,s)}{\partial t} = a_1 < 0, \tag{17}$$

which means that $G(t,s)$ is decreasing on variable t. Thus

$$G(t,s) \geq G(s,s), \quad 0 \leq t \leq s. \tag{18}$$

On the other hand, for $\xi_{i-1} \leq s \leq \xi_i$, $i = 1, 2, \ldots, m-1$, and $t \geq s$,

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= -t + b_1 \\ &= -t + \frac{s - \sum_{k=0}^{i-1} \alpha_k s - \sum_{k=i}^{m-1} \alpha_k \xi_k}{1 - \sum_{k=0}^{m-1} \alpha_k} \\ &\leq \frac{\sum_{k=i}^{m-1} \alpha_k (s - \xi_k)}{1 - \sum_{k=0}^{i-1} \alpha_k} < 0. \end{aligned} \tag{19}$$

Thus,

$$G(t,s) \geq G(1,s), \quad s \leq t \leq 1. \tag{20}$$

Furthermore,

$$\frac{\partial G(1,s)}{\partial s} \leq -\left(1 - \sum_{k=0}^{i-1} \alpha_k\right) \sum_{k=0}^{m-1} \beta_k \left(1 - \xi_k\right) < 0. \tag{21}$$

Thus, for $t, s \in [0,1]$,

$$\begin{aligned} G(t,s) &\geq G(1,1) \\ &= -\frac{1}{2} + b_1 + b_0 \\ &= -\frac{1}{2} + \frac{1 - \sum_{k=0}^{m-1} \alpha_k}{1 - \sum_{k=0}^{m-1} \alpha_k} \\ &\quad + \left(\left(\frac{1}{2} \sum_{k=0}^{m-1} \beta_k + \frac{1}{2} - \frac{1 - \sum_{k=0}^{m-1} \alpha_k}{1 - \sum_{k=0}^{m-1} \alpha_k} \right) \right. \\ &\quad \left. \times \left(1 - \sum_{k=0}^{m-1} \beta_k\right)^{-1} \right) = 0. \end{aligned} \tag{22}$$

This gives that $G(t,s) \geq 0, t, s \in [0,1]$. □

Lemma 7. *If $y(t) \geq 0$, $t \in [0,1]$, and $x(t)$ is the solution of problem (11) and (12), then*

$$\min_{0 \leq t \leq 1} |x(t)| \geq \delta \max_{0 \leq t \leq 1} |x(t)|, \tag{23}$$

where $\delta = (\sum_{i=1}^{m-2} \beta_i (1 - \xi_i))/(1 - \sum_{i=1}^{m-2} \beta_i \xi_i)$ is a positive constant.

Proof. From

$$x'''(t) = -y(t) \leq 0, \quad t \in [0,1], \tag{24}$$

we see that $x''(t)$ is decreasing on $[0,1]$. Considering $x''(0) = 0$, we have $x''(t) \leq 0$, $t \in (0,1)$. Next we claim that $x'(0) \leq 0$. Suppose that, on the contrary, $x'(0) > 0$. We have

$$\begin{aligned} 0 &= x'(0) - x'(0) = x'(0) - \sum_{i=1}^{m-2} \alpha_i x'(\xi_i) \\ &> \sum_{i=1}^{m-2} \alpha_i \left(x'(0) - x'(\xi_i)\right) \geq 0, \quad \text{a contradiction.} \end{aligned} \tag{25}$$

Thus,

$$\max_{0 \leq t \leq 1} x(t) = x(0), \quad \min_{0 \leq t \leq 1} x(t) = x(1). \tag{26}$$

From the concavity of $x(t)$, we have

$$\xi_i \left(x(1) - x(0)\right) \leq x(\xi_i) - x(0). \tag{27}$$

Multiplying left and right sides by β_i and considering $x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i)$, we have

$$\left(1 - \sum_{i=1}^{m-2} \beta_i \xi_i\right) x(1) \geq \sum_{i=1}^{m-2} \beta_i \left(1 - \xi_i\right) x(0). \tag{28}$$

This completes the proof of Lemma 7. □

Let the Banach space $E = C[0,1]$ be endowed with the maximum norm. We define the cone $C \subset E$ by

$$C = \Big\{ x \in E \mid x(t) \geq 0, x''(0) = 0, \quad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), \quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), x(t) \text{ is concave on } [0,1] \Big\}. \tag{29}$$

Define the continuous nonnegative concave functional $\psi : C \to [0, \infty)$ by

$$\psi(x) = \min_{0 \leq t \leq 1} x(t), \quad x \in C. \tag{30}$$

Define the constants m^*, m_* by

$$m^* = \max_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds, \qquad m_* = \min_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds. \tag{31}$$

Theorem 8. *Suppose that there exist constants $0 < a < b < b/\delta \leq c$ such that*

(A1) $f(t,x) < a/m^*$, $(t,x) \in [0,1] \times [0,a]$,

(A2) $f(t,x) > b/m_*$, $(t,x) \in [0,1] \times [b, b/\delta]$,

(A3) $f(t,x) < c/m^*$, $(t,x) \in [0,1] \times [0,c]$.

Then problem (1) *has at least three positive solutions x_1, x_2, and x_3 satisfying*

$$\|x_1\| \leq a, \qquad b < \min_{0 \leq t \leq 1} x_2, \qquad \|x_3\| > a, \qquad \min_{0 \leq t \leq 1} x_3 < b. \tag{32}$$

Proof. The operator $T : C \to E$ is defined by

$$Tx(t) = \int_0^1 G(t,s) f(s, x(s))\, ds. \tag{33}$$

It is clear that $T : C \to C$ and it is completely continuous.

Next, the conditions of Lemma 3 are checked. If $x \in \overline{C}_c$, then $\|x\| \leq c$ and condition $(A3)$ implies that

$$f(t,x) \leq \frac{c}{m^*}, \quad 0 \leq t \leq 1. \tag{34}$$

Then

$$\begin{aligned} \|T(x)\| &= \max_{0 \leq t \leq 1} \int_0^1 G(t,s) f(s, x(s))\, ds \\ &\leq \max_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds \times \frac{c}{m^*} \leq c. \end{aligned} \tag{35}$$

Thus, $T : \overline{C}_c \to \overline{C}_c$. □

Similar to the proof above, we obtain that $T : \overline{C}_a \to \overline{C}_a$. Hence, condition (H_2) of Lemma 3 is satisfied.

The fact that the constant function $x(t) = b/\delta \in \{P(\psi, b, b/\delta) \mid \psi(x) > b\}$ implies that $\{P(\psi, b, b/\delta) \mid \psi(x) > b\} \neq \emptyset$. If $x \in P(\psi, b, b/\delta)$, from assumption $(A2)$,

$$f(t,x) \geq \frac{b}{m_*}. \tag{36}$$

Thus

$$\begin{aligned} \psi(Tx) &= \min_{0 \leq t \leq 1} \int_0^1 G(t,s) f(s, x(s))\, ds \\ &\geq \min_{0 \leq t \leq 1} \int_0^1 G(t,s)\, ds \times \frac{b}{m_*} = b, \end{aligned} \tag{37}$$

which ensures that condition (H_1) of Lemma 3 is satisfied. Finally we show that condition (H_3) of Lemma 3 also holds. Suppose that $x \in P(\psi, b, c)$ with $\|Tx\| > b/\delta$. Then

$$\psi(Tx) = \min_{0 \leq t \leq 1} Tx(t) \geq \delta \times \|Tx\| > \delta \times \frac{b}{\delta} = b. \tag{38}$$

So, condition (H_3) of Lemma 3 is satisfied. Thus, an application of Lemma 3 implies that the nonresonant third-order boundary value problem (1) has at least three positive solutions x_1, x_2, and x_3 satisfying

$$\|x_1\| \leq a, \qquad b < \min_{0 \leq t \leq 1} x_2, \qquad \|x_3\| > a, \qquad \min_{0 \leq t \leq 1} x_3 < b. \tag{39}$$

Here an example is given to illustrate the main results of this section. We consider the following nonresonant three-point boundary value problem:

$$\begin{gathered} x''(t) + f(t,x) = 0, \quad t \in (0,1) \\ x''(0) = 0, \qquad x'(0) = \frac{1}{2} x'\left(\frac{1}{3}\right), \qquad x(1) = \frac{1}{2} x\left(\frac{1}{3}\right), \end{gathered} \tag{40}$$

where

$$f(t,x) = \begin{cases} \dfrac{1}{10} e^t + \dfrac{x^3 + 5}{\pi}, & 0 < x < 6 \\ \dfrac{1}{10} e^t + \dfrac{221}{\pi}, & x \geq 6. \end{cases} \tag{41}$$

Here $\alpha_1 = 1/2$, $\beta_1 = 1/2$, $\xi_1 = 1/3$, and

$$G(t,s) = \begin{cases} \frac{1}{2}s^2 + st - \frac{10}{3}s \\ \quad -\frac{1}{3}t + \frac{3}{2}, & 0 \le s \le \frac{1}{3},\ t \le s, \\ -\frac{1}{2}t^2 + 2st - \frac{1}{3}t \\ \quad -\frac{10}{3}s + \frac{3}{2}, & 0 \le s \le \frac{1}{3},\ t \ge s, \\ (s-1)^2, & \frac{1}{3} \le s \le 1,\ t \le s, \\ -\frac{1}{2}t^2 + \frac{1}{2}s^2 + st \\ \quad +1 - 2s, & \frac{1}{3} \le s \le 1,\ t \ge s. \end{cases} \tag{42}$$

By a simple computation, we can get that

$$m^* = \max_{0\le t\le 1} \int_0^1 G(t,s)\,ds = \frac{32}{81}, \qquad m_* = \min_{0\le t\le 1} \int_0^1 G(t,s)\,ds = \frac{16}{81}. \tag{43}$$

We choose $a = 1,\ b = 4$, and $c = 32$. It is easy to check that

(1) $f(t,x) < 81/32$, $[t,x] \in [0,1] \times [0,1]$,

(2) $f(t,x) > 81/4$, $[t,x] \in [0,1] \times [4,10]$,

(3) $f(t,x) < 81$, $[t,x] \in [0,1] \times [0,32]$.

Thus all conditions of Theorem 8 hold. This ensures that problem (40) has at least three positive solutions $x_1,\ x_2$, and x_3 satisfying

$$\max_{0\le t\le 1} x_1 \le 1, \qquad \min_{0\le t\le 1} x_2 > 4, \qquad \max_{0\le t\le 1} x_3 > 1, \qquad \min_{0\le t\le 1} x_3 < 4. \tag{44}$$

4. Main Results for Resonant Case

In this section the condition $\sum_{i=1}^{m-2} \beta_i = 1$ is considered. Obviously, problem (1) is at resonance under this condition. The norm-type Leggett-Williams fixed-point theorem will be used to establish the existence results of positive solution.

We define the spaces $X = Y = C[0,1]$ endowed with the maximum norm. It is well known that X and Y are the Banach spaces.

Define the linear operator $L : \operatorname{dom} L \subset X \to Y$ and

$$(Lx)(t) = -x'''(t), \quad t \in [0,1], \tag{45}$$

where

$$\operatorname{dom} L = \Big\{ x \in X \mid x''' \in C[0,1], x''(0) = 0, \quad x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i) \Big\} \tag{46}$$

and the nonlinear operator $N : X \to Y$ with

$$(Nx)(t) = f(t, x(t)), \quad t \in [0,1]. \tag{47}$$

It is obvious that $\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) \equiv c, t \in [0,1]\}$. Denote the function $G(s)$, $s \in [0,1]$ as follow:

$$G(s) = (1-s)^2 - \sum_{k=i}^{m-2} \beta_k (\xi_k - s)^2 + 2\frac{1 - \sum_{k=i}^{m-2} \beta_k \xi_k}{1 - \sum_{k=1}^{m-2} \alpha_k} \sum_{k=i}^{m-2} \alpha_k (\xi_k - s), \quad \xi_{i-1} \le s \le \xi_i, \quad i = 1, 2, \ldots, m-2. \tag{48}$$

Define the function $k(t,s)$ as follow:

$$k(t,s) = \begin{cases} \frac{1}{6}(1-s)^3 - \frac{1}{2}(t-s)^2 - st + \frac{1}{2}s - k_0 \\ \quad - \frac{\sum_{k=0}^{i-1} \beta_k \left((1/2)s^2 - \xi_k s + (1/2)\xi_k^2\right)(1/2 - t)}{1 - \sum_{i=0}^{m-1} \beta_i \xi_i}, & t \ge s,\ \xi_{i-1} \le s \le \xi_i, \\ \frac{1}{6}(1-s)^3 - st + \frac{1}{2}s - k_0 \\ \quad - \frac{\sum_{k=0}^{i-1} \beta_k \left((1/2)s^2 - \xi_k s + (1/2)\xi_k^2\right)(1/2 - t)}{1 - \sum_{i=0}^{m-1} \beta_i \xi_i}, & t \le s,\ \xi_{i-1} \le s \le \xi_i, \end{cases} \tag{49}$$

for $i = 1, 2, \ldots, m-1$ and $k_0 = (1/2 - t)[1 - (1/2)(1 - \sum_{i=0}^{m-1} \beta_i \xi_i)]$.

The function $U(t,s)$ and positive number κ are given by

$$U(t,s) = k(t,s) + \frac{G(s)}{\int_0^1 G(s)\,ds}\left(1 - \int_0^1 k(t,s)\,ds\right), \quad t,s \in [0,1],$$
$$\kappa := \min\left\{1, \min_{s\in[0,1]} \frac{\int_0^1 G(s)\,ds}{G(s)}, \min_{t,s\in[0,1]} \frac{1}{U(t,s)}\right\},$$
$$\sigma = 1 + \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i^2}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{50}$$

Theorem 9. *Suppose that there exists positive constant $R \in (0,\infty)$ such that $f : [0,1] \times [0,R] \to (-\infty, +\infty)$ is continuous and satisfies the following conditions:*

(S1) *$f(t,x) \geq -\kappa x$, for $(t,x) \in [0,1] \times [0,R]$,*

(S2) *$f(t,x) < 0$ for $[t,x] \in [0,1] \times [(1 - (\kappa\sigma/2))R, R]$,*

(S3) *there exist $r \in (0,R)$, $M \in (0,1)$, $t_0 \in [0,1]$, $a \in (0,1]$, and continuous functions*

$$g : [0,1] \longrightarrow [0,+\infty), \qquad h : (0,r] \longrightarrow [0,+\infty) \tag{51}$$

such that

$$f(t,x) \geq g(t)h(x), \quad [t,x] \in [0,1] \times (0,r] \tag{52}$$

and $h(x)/x^a$ is nonincreasing on $(0,r]$ with

$$\frac{h(r)}{r^a}\int_0^1 U(t_0,s)\,g(s)\,ds \geq \frac{1-M}{M^a}. \tag{53}$$

Then resonant problem (1) *has at least one positive solution.*

Proof. Firstly we prove that

$$\operatorname{Im} L = \left\{y \in Y \mid \int_0^1 G(s)\,y(s)\,ds = 0\right\}. \tag{54}$$

Indeed, for each $y \in \{y \in Y \mid \int_0^1 G(s)y(s)ds = 0\}$, we choose

$$x(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{t}{1 - \sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} (\xi_i - s)\,y(s)\,ds. \tag{55}$$

We can check that

$$-x'''(t) = y(t), \qquad x''(0) = 0, \qquad x''(0) = 0,$$
$$x'(0) = \sum_{i=1}^{m-2}\alpha_i x'(\xi_i), \qquad x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i), \tag{56}$$

which means $x(t) \in \operatorname{dom} L$. Thus

$$\left\{y \in Y \mid \int_0^1 G(s)\,y(s)\,ds = 0\right\} \subset \operatorname{Im} L. \tag{57}$$

On the other hand, for each $y(t) \in \operatorname{Im} L$, there exists $x(t) \in \operatorname{dom} L$ such that

$$-x'''(t) = y(t), \qquad x''(0) = 0, \qquad x''(0) = 0,$$
$$x'(0) = \sum_{i=1}^{m-2}\alpha_i x'(\xi_i), \qquad x(1) = \sum_{i=1}^{m-2}\beta_i x(\xi_i). \tag{58}$$

Integrating both sides on $[0,t]$, we have

$$x(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{1}{2}x''(0)\,t^2 + x'(0)\,t + x(0). \tag{59}$$

Considering the boundary condition together with the resonant condition $\sum_{i=0}^{m-1}\beta_i = 1$, we have

$$\int_0^1 (1-s)^2 y(s)\,ds - \sum_{i=1}^{m-2}\beta_i \int_0^{\xi_i} (\xi_i - s)^2 y(s)\,ds + 2\frac{1 - \sum_{i=1}^{m-2}\beta_i\xi_i}{1 - \sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}(\xi_i - s)\,y(s)\,ds = 0. \tag{60}$$

Thus,

$$\operatorname{Im} L = \left\{y \in Y \mid \int_0^1 G(s)\,y(s)\,ds = 0\right\}. \tag{61}$$

It is obvious that $\dim \operatorname{Ker} L = 1$ and $\operatorname{Im} L$ is closed.

Secondly we see $Y = Y_1 \oplus \operatorname{Im} L$, where

$$Y_1 = \left\{y_1 \mid y_1 = \frac{1}{\int_0^1 G(s)\,ds}\int_0^1 G(s)\,y(s)\,ds, y \in Y\right\}. \tag{62}$$

In fact, for each $y(t) \in Y$, we have

$$\int_0^1 G(s)\left[y(s) - y_1\right]ds = 0. \tag{63}$$

This induces that $y - y_1 \in \operatorname{Im} L$. Since $Y_1 \cap \operatorname{Im} L = \{0\}$, we have $Y = Y_1 \bigoplus \operatorname{Im} L$. Thus L is a Fredholm operator with index zero.

Define two projections $P : X \to X$ and $Q : Y \to Y$ by

$$Px = \int_0^1 x(s)\,ds,$$
$$Qy = \frac{1}{\int_0^1 G(s)\,ds}\int_0^1 G(s)\,y(s)\,ds. \tag{64}$$

Clearly, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Note that, for $y \in \operatorname{Im} L$, the inverse K_P of L_P is given by

$$(K_P)\,y = \int_0^t k(t,s)\,y(s)\,ds. \tag{65}$$

In fact, It is easy to check that

$$L(K_P)(y) = \left(-\int_0^1 k(t,s)\,y(s)\,ds\right)''' = y(t)$$
$$K_P(L)(x) = \int_0^1 k(t,s)\left(-x'''(s)\right)ds = x(t). \tag{66}$$

Next we will check that every condition of Lemma 4 is fulfilled. Remark that f can be extended continuously on $[0,1]\times(-\infty,+\infty)$ and condition $(C1)$ of Lemma 4 is fulfilled.

Define the set of nonnegative functions C and subsets of X Ω_1, Ω_2 by

$$\begin{aligned} C &= \{x \in X : x(t) \geq 0, t \in [0,1]\},\\ \Omega_1 &= \{x \in X : r > |x(t)| > M\|x\|, t \in [0,1]\},\\ \Omega_2 &= \{x \in X : \|x(t)\| < R, t \in [0,1]\}. \end{aligned} \tag{67}$$

Remark that Ω_1 and Ω_2 are open and bounded sets. Furthermore

$$\begin{aligned} \overline{\Omega}_1 &= \{x \in X : r \geq |x(t)| \geq M\|x\|, t \in [0,1]\}\\ &\subset \Omega_2,\ C \cap \overline{\Omega}_2 \setminus \Omega_1 \neq \emptyset. \end{aligned} \tag{68}$$

Let the isomorphism $J = I$ and $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which ensures that condition $(C3)$ of Lemma 4 is fulfilled.

Then we prove that $(C2)$ of Lemma 4 is fulfilled. For this purpose, suppose that there exist $x_0 \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$. Then

$$-x_0'''(t) = \lambda_0 f(t, x_0) \tag{69}$$

for all $t \in [0,1]$. Thus

$$x_0''(t) = -\lambda_0 \int_0^t f(s, x_0(s))\,ds, \tag{70}$$

$$\begin{aligned} x_0'(t) = &-\lambda_0 \int_0^t (t-s) f(s, x_0(s))\,ds\\ &-\lambda_0 \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}(\xi_i - s) f(s, x_0(s))\,ds. \end{aligned} \tag{71}$$

Let $x_0(t_0) = \|x_0\| = R$. The proof is divided into three cases.

(1) We show that $t_0 \neq 1$. Suppose, on the contrary, that $x_0(t)$ achieves maximum value R only at $t_0 = 1$. Then the boundary condition $x(1) = \sum_{i=1}^{m-2}\beta_i x_0(\xi_i)$ in combination with the resonant condition $\sum_{i=1}^{m-2}\beta_i = 1$ yields that $\max_{1\leq i\leq m-2} x_0(\xi_i) \geq R$, which is a contradiction.

(2) We claim that $t_0 \neq 0$. Suppose, on the contrary, that $x_0(t)$ achieves maximum value R at $t_0 = 0$. From condition $(S2)$, there exists $t_1 > 0$ near to zero such that

$$f(t, x_0(t)) < 0, \quad t \in [0, t_1]. \tag{72}$$

Then

$$\begin{aligned} x_0'(t) = &-\lambda_0 \int_0^t (t-s) f(s, x_0(s))\,ds\\ &-\lambda_0 \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}(\xi_i - s) f(s, x)\,ds > 0,\\ &t \in [0, t_1], \end{aligned} \tag{73}$$

which contradicts the fact that $x_0(t)$ achieves maximum value at $t_0 = 0$.

(3) Thus there exists $t_0 \in (0,1)$ such that $x_0(t_0) = R = \max_{0\leq t\leq 1} x_0(t)$. We may choose $\eta < t_0$ nearest to t_0 with $x_0''(\eta) = 0$. From the mean value theory, we claim that there exists $\xi \in (\eta, t_0)$ such that

$$x_0(\eta) = x_0(t_0) - x_0'(\xi)(t_0 - \eta). \tag{74}$$

However,

$$\begin{aligned} x_0'(\xi) = &-\lambda_0 \int_0^{\xi} (\xi - s) f(s, x_0)\,ds\\ &-\lambda_0 \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}(\xi_i - s) f(s, x)\,ds\\ \leq{} & \lambda_0\kappa \int_0^{\xi}(\xi - s) x_0(s)\,ds\\ &+\lambda_0\kappa \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}(\xi_i - s) x_0\,ds\\ \leq{} & \lambda_0\kappa R \int_0^{\xi}(\xi - s)\,ds\\ &+\lambda_0\kappa R \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}(\xi_i - s)\,ds\\ ={} & \frac{1}{2}\lambda_0\kappa R\left(\xi^2 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i^2}{1-\sum_{i=1}^{m-2}\alpha_i}\right). \end{aligned} \tag{75}$$

Thus

$$\begin{aligned} x_0(\eta) &= x_0(t_0) - x_0'(\xi)(t_0 - \eta)\\ &\geq R - \frac{1}{2}\lambda_0\kappa\left(\xi^2 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i^2}{1-\sum_{i=1}^{m-2}\alpha_i}\right)(t_0 - \eta) R\\ &\geq \left[1 - \frac{\kappa}{2}\left(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i^2}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\right] R\\ &= \left(1 - \frac{\kappa\sigma}{2}\right) R. \end{aligned} \tag{76}$$

Then, considering assumption ($S2$), we have

$$0 \geq x_0''(t_0) - x_0''(\eta) = -\lambda_0 \int_\eta^{t_0} f(s, x_0(s))\, ds > 0, \quad (77)$$

which is a contradiction. Thus ($C2$) of Lemma 4 is fulfilled. □

Remark 10. The sign of third-order derivative of a function $h(t)$ at point t_0 cannot be confirmed when t_0 is a maximal value of $h(t)$. Thus the methods in [28] are not applicable directly to this problem.

For $x \in \operatorname{Ker} L \cap \Omega_2$, define the projection $H(x, \lambda)$ as follows:

$$H(x, \lambda) = x - \lambda |x| - \frac{\lambda}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |x|)\, ds, \quad (78)$$

where $\lambda \in [0, 1]$ and $x \in \operatorname{Ker} L \cap \Omega_2$. Suppose $H(x, \lambda) = 0$. In view of ($S1$) we obtain

$$\begin{aligned} c &= \lambda |c| + \frac{\lambda}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |c|)\, ds \\ &\geq \lambda |c| - \frac{\lambda}{\int_0^1 G(s)\, ds} \int_0^1 G(s) \kappa |c|\, ds \\ &= \lambda |c| (1 - \kappa) \geq 0. \end{aligned} \quad (79)$$

Hence $H(x, \lambda) = 0$ implies $c \geq 0$. Hence, if $H(R, \lambda) = 0$, we get

$$0 \leq R(1 - \lambda) \int_0^1 G(s)\, ds = \lambda \int_0^1 G(s) f(s, R)\, ds, \quad (80)$$

contradicting ($S2$). Thus $H(x, \lambda) \neq 0$ for $x \in \partial\Omega_2$ and $\lambda \in [0, 1]$. Therefore

$$\begin{aligned} d_B(H(x, 0), \operatorname{Ker} L \cap \Omega_2, 0) &= d_B(H(x, 1), \operatorname{Ker} L \cap \Omega_2, 0) \\ &= d_B(I, \operatorname{Ker} L \cap \Omega_2, 0) = 1. \end{aligned} \quad (81)$$

This ensures

$$\begin{aligned} &d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \\ &\quad = d_B(H(x, 1), \operatorname{Ker} L \cap \Omega_2, 0) \neq 0. \end{aligned} \quad (82)$$

Let $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0, 1]$. From condition ($S1$), we see

$$\begin{aligned} (\Psi_\gamma x)(t) &= \int_0^1 |x(t)|\, dt + \frac{1}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |x(s)|)\, ds \\ &\quad + \int_0^1 k(t, s) \\ &\quad \times \left[f(s, |x(s)|) - \frac{1}{\int_0^1 G(s)\, ds} \right. \\ &\quad \left. \times \int_0^1 G(\tau) f(\tau, |x(\tau)|)\, d\tau \right] ds \\ &= \int_0^1 |x(t)|\, dt + \int_0^1 U(t, s) f(s, |x(s)|)\, ds \\ &\geq \int_0^1 |x(s)|\, ds - \kappa \int_0^1 U(t, s) |x(s)|\, ds \\ &= \int_0^1 (1 - \kappa U(t, s)) |x(s)|\, ds \geq 0. \end{aligned} \quad (83)$$

Hence $\Psi_\gamma(\overline{\Omega}_2) \setminus \Omega_1 \subset C$. Moreover, for $x \in \partial\Omega_2$, we have

$$\begin{aligned} (P + JQN)\gamma x &= \int_0^1 |x(s)|\, ds \\ &\quad + \frac{1}{\int_0^1 G(s)\, ds} \int_0^1 G(s) f(s, |x(s)|)\, ds \\ &\geq \int_0^1 \left(1 - \frac{\kappa}{\int_0^1 G(s)\, ds} G(s) \right) |x(s)|\, ds \geq 0, \end{aligned} \quad (84)$$

which means $(P + JQN)\gamma(\partial\Omega_2) \subset C$. These ensure that ($C6$) and ($C7$) of Lemma 4 hold.

At last, we confirm that ($C5$) is satisfied. Taking $u_0(t) \equiv 1$ on $[0, 1]$, we see

$$u_0 \in C \setminus \{0\}, \qquad C(u_0) = \{x \in C \mid x(t) > 0 \text{ on } [0, 1]\} \quad (85)$$

and we can choose $\sigma(u_0) = 1$. For $x \in C(u_0) \cap \partial\Omega_1$, we have

$$\begin{gathered} x(t) > 0, \quad t \in [0, 1], \; 0 < \|x\| \leq r, \\ x(t) \geq M \|x\| \quad \text{on } [0, 1]. \end{gathered} \quad (86)$$

Therefore, in view of ($S3$), we obtain, for all $x \in C(u_0) \cap \partial\Omega_1$,

$$\begin{aligned}(\Psi x)(t_0) &= \int_0^1 x(s)\,ds + \int_0^1 U(t_0,s) f(s,x(s))\,ds \\ &\geq M\|x\| + \int_0^1 U(t_0,s) g(s) h(x(s))\,ds \\ &= M\|x\| + \int_0^1 U(t_0,s) g(s) \frac{h(x(s))}{x^a(s)} x^a(s)\,ds \\ &\geq M\|x\| + \frac{h(r)}{r^a}\int_0^1 U(t_0,s) g(s) M^a \|x\|^a ds \\ &\geq M\|x\| + (1-M)\|x\| = \|x\|.\end{aligned} \tag{87}$$

So $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for all $x \in C(u_0) \cap \partial\Omega_1$, which means ($C5$) of Lemma 4 holds.

Thus with the application of Lemma 4, we confirm that the equation $Lx = Nx$ has a solution $x \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which implies that the resonant problem (1) has at least one positive solution.

Finally an example is given to illustrate the main results of the resonance case. We investigate the resonant third-order three-point boundary value problem:

$$\begin{gathered}x'''(t) + \left(-\frac{1}{2}t^2 + \frac{1}{2}t + \frac{5}{16}\right)\left(x^2 - 4x + \frac{11}{5}\right) \\ \times \sqrt{x^2 - 6x + 10} = 0, \quad t \in [0,1] \\ x''(0) = 0, \qquad x'(0) = \frac{1}{2}x'\left(\frac{2}{3}\right), \qquad x(0) = x\left(\frac{2}{3}\right),\end{gathered} \tag{88}$$

where $\alpha_1 = 1/2$, $\beta = 1$, $\xi = 2/3$, and

$$\begin{aligned}f(t,x) = {} & \left(-\frac{1}{2}t^2 + \frac{1}{2}t + \frac{5}{16}\right)\left(x^2 - 4x + \frac{11}{5}\right) \\ & \times \sqrt{x^2 - 6x + 10}.\end{aligned} \tag{89}$$

Here

$$G(s) = \begin{cases} 1 - \dfrac{4}{3}s, & 0 \leq s \leq \dfrac{2}{3} \\ (1-s)^2, & \dfrac{2}{3} \leq s \leq 1, \end{cases}$$

$$k(t,s) = \begin{cases} \dfrac{1}{6}(1-s)^3 - \dfrac{1}{2}(t-s)^2 \\ \quad -st + \dfrac{1}{2}s - \dfrac{5}{6}\left(\dfrac{1}{2} - t\right), & t \geq s,\ 0 \leq s \leq \dfrac{2}{3} \\ \dfrac{1}{6}(1-s)^3 - st \\ \quad + \dfrac{1}{2}s - \dfrac{5}{6}\left(\dfrac{1}{2} - t\right), & t \leq s,\ 0 \leq s \leq \dfrac{2}{3}. \\ \dfrac{1}{6}(1-s)^3 - \dfrac{1}{2}(t-s)^2 \\ \quad -st + \dfrac{1}{2}s - \left(\dfrac{1}{2} - t\right) \\ \quad \times\left(\dfrac{3}{2}s^2 - 2s + \dfrac{3}{2}\right), & t \geq s,\ \dfrac{2}{3} \leq s \leq 1 \\ \dfrac{1}{6}(1-s)^3 - st + \dfrac{1}{2}s \\ \quad -\left(\dfrac{1}{2} - t\right) \\ \quad \times\left(\dfrac{3}{2}s^2 - 2s + \dfrac{3}{2}\right), & t \leq s, \dfrac{2}{3} \leq s \leq 1. \end{cases} \tag{90}$$

By a simple computation, we have

$$\begin{gathered}\int_0^1 G(s)\,ds = \frac{31}{81}, \qquad \sigma = \frac{13}{9}, \\ \kappa = \frac{31}{81}, \qquad \int_0^1 U(0,s)\,ds = 1.\end{gathered} \tag{91}$$

Choose $R = 1$, $r = 1/4$, $t_0 = 0$, $a = 1$, and $M = 1/2$.

We take

$$\begin{gathered}g(t) = -\frac{1}{2}t^2 + \frac{1}{3}t + \frac{5}{16}, \quad t \in [0,1], \\ h(x) = \sqrt{x^2 - 6x + 10}, \quad x \in \left[0, \frac{1}{4}\right].\end{gathered} \tag{92}$$

Then,

$$\begin{gathered}\frac{7}{48} \leq g(t) \leq \frac{53}{144} < \frac{31}{81}, \quad t \in [0,1], \\ x^2 - 4x + \frac{11}{5} \geq -x, \quad x \in [0,1].\end{gathered} \tag{93}$$

It is easy to check that

(1) $f(t,x) > -(31/81)x$, for all $(t,x) \in [0,1] \times [0,1]$,

(2) $f(t,x) < 0$, for all $(t,x) \in [0,1] \times [1055/1458, 1]$,

(3) $f(t,x) \geq (101/80)(-(1/2)t^2 + (1/3)t + (5/16))\sqrt{x^2 - 6x + 10} \geq g(t)h(x), [t,x] \in [0,1]\times (0,1/4]$, and $h(x)/x = \sqrt{x^2 - 6x + 10}/x$ is nonincreasing on $(0,1/4]$ with

$$\frac{h(r)}{r^a}\int_0^1 U(0,s) g(s)\,ds > \frac{7\sqrt{137}}{48} > 1 = \frac{1-M}{M^a}. \tag{94}$$

Then all conditions of Theorem 9 are satisfied. This ensures that the resonant problem has at least one solution, positive on $[0,1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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