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Research Article

On the Fine Spectrum of the Operator Defined by the Lambda Matrix over the Spaces of Null and Convergent Sequences

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The main purpose of this paper is to determine the fine spectrum with respect to Goldberg's classification of the operator defined by the lambda matrix over the sequence spaces c_0 and c. As a new development, we give the approximate point spectrum, defect spectrum, and compression spectrum of the matrix operator Λ on the sequence spaces c_0 and c. Finally, we present a Mercerian theorem. Since the matrix Λ is reduced to a regular matrix depending on the choice of the sequence (λ_k) having certain properties and its spectrum is firstly investigated, our work is new and the results are comprehensive.

1. Introduction

Let X and Y be Banach spaces, and let $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T; that is,

$$R(T) = \{y \in Y : y = Tx, x \in X\}. \tag{1}$$

By $B(X)$, we also denote the *set of all bounded linear operators* on X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a nontrivial complex normed space and $T : D(T) \to X$ a linear operator defined on a subspace $D(T) \subseteq X$. We do not assume that $D(T)$ is dense in X or that T has closed graph $\{(x, Tx) : x \in D(T)\} \subseteq X \times X$. By the statement "*T is invertible*," it is meant that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$, such that $S = T^{-1}$ is necessarily uniquely determined and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $M > 0$ for which $\|Tx\| \geq M\|x\|$ for all $x \in D(T)$. Associated with each complex number, α is the perturbed operator

$$T_\alpha = \alpha I - T \tag{2}$$

defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T, X)$ consists of those $\alpha \in \mathbb{C}$, the complex field, for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

2. The Subdivisions of Spectrum

In this section, we define the parts of spectrum called point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum, and compression spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1. The Point Spectrum, Continuous Spectrum, and Residual Spectrum. The name resolvent is appropriate since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided that T_α^{-1} exists. More importantly, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and the spectral theory is concerned with those properties.

For instance, we are interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We will also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and is bounded and whose domain is dense in X. For our investigation of T, T_α, and T_α^{-1}, we need some basic concepts in the spectral theory which are given, as follows (see [1, pages 370-371]).

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values α of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into the following three disjoint sets.

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_α^{-1} exists and is unbounded, and the domain of T_α^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

Therefore, these three subspectra form a disjoint subdivision

$$\sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X). \tag{3}$$

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we will have to discuss. Indeed, it is well known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite-dimensional case.

2.2. The Approximate Point Spectrum, Defect Spectrum, and Compression Spectrum. In this subsection, following Appell et al. [2], we define three more subdivisions of the spectrum called *approximate point spectrum*, *defect spectrum*, and *compression spectrum*.

Given a bounded linear operator T in a Banach space X, we call a sequence (x_k) in X a *Weyl sequence* for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $k \to \infty$. Then, the *approximate point spectrum* $\sigma_{\mathrm{ap}}(T, X)$ *of* T is defined by

$$\sigma_{\mathrm{ap}}(T, X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - T\}. \tag{4}$$

Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective}\} \tag{5}$$

is called the *defect spectrum* of T.

The two subspectra given by (4) and (5) form a (not necessarily disjoint) subdivision

$$\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X) \cup \sigma_\delta(T, X) \tag{6}$$

of the spectrum. There is another subspectrum,

$$\sigma_{\mathrm{co}}(T, X) = \{\alpha \in \mathbb{C} : \overline{R(\alpha I - T)} \neq X\} \tag{7}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X) \cup \sigma_{\mathrm{co}}(T, X) \tag{8}$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{\mathrm{ap}}(T, X)$ and $\sigma_{\mathrm{co}}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (3), we note that

$$\begin{aligned} \sigma_r(T, X) &= \sigma_{\mathrm{co}}(T, X) \setminus \sigma_p(T, X), \\ \sigma_c(T, X) &= \sigma(T, X) \setminus \left[\sigma_p(T, X) \cup \sigma_{\mathrm{co}}(T, X)\right]. \end{aligned} \tag{9}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness, results for linear operator equations in Banach spaces and their adjoints are also useful.

Proposition 1 (see [2, Proposition 1.3, page 28]). *Spectrum and subspectrum of an operator* $T \in B(X)$ *and its adjoint* $T^* \in B(X^*)$ *are related by the following relations:*

(a) $\sigma(T^*, X^*) = \sigma(T, X)$,

(b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$,

(c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$,

(d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$,

(e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$,

(f) $\sigma_{\mathrm{co}}(T^*, X^*) \supseteq \sigma_p(T, X)$,

(g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)–(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum, and the point spectrum is dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{\mathrm{ap}}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on the Hilbert spaces are most similar to matrices in finite dimensional spaces (see [2]).

2.3. Goldberg's Classification of Spectrum. If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

(A) $R(T) = X$,

(B) $R(T) \neq \overline{R(T)} = X$,

(C) $\overline{R(T)} \neq X$,

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$, and C_3. If an operator is in state C_2, for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [3]). Figure 1 due to Wenger [4] may be useful for the readers.

If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then $\alpha \in \rho(T, X)$. All scalar values of α not in $\rho(T, X)$ comprise the spectrum of T. The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T. That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, and $C_3\sigma(T, X)$. For example, if T_α is in a given state, C_2 (say), then we write $\alpha \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate subdivision (3) in Table 1.

Observe that the case in the first row and the second column cannot occur in a Banach space X, by the closed graph theorem. If we are not in the third column, that is, if α is not an eigenvalue of T, we may always consider the resolvent operator T_α^{-1} (on a possibly "thin" domain of definition) as "algebraic inverse" of $\alpha I - T$.

By a *sequence space*, we understand a linear subspace of the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contain ϕ, the set of all finitely nonzero sequences, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, c_0, and bv for the spaces of all bounded, convergent, null, and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty}|x_k - x_{k+1}|$, respectively, while ϕ is not a Banach space with respect to any norm. Also by ℓ_p, we denote the space of all p-absolutely summable sequences which is a Banach space with the norm $\|x\|_p = (\sum_{k=0}^{\infty}|x_k|^p)^{1/p}$, where $1 \le p < \infty$.

Let μ and ν be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix transformation from μ into ν, and we denote it by writing $A : \mu \to \nu$ if for every sequence $x = (x_k) \in \mu$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in ν, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \quad \text{for each } n \in \mathbb{N}. \tag{10}$$

By $(\mu : \nu)$, we denote the class of all matrices A such that $A : \mu \to \nu$. Thus, $A \in (\mu : \nu)$ if and only if the series on the right side of (10) converges for each $n \in \mathbb{N}$ and each $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \nu$ for all $x \in \mu$.

Throughout this paper, let $\lambda = (\lambda_k)$ be a strictly increasing sequence of positive reals tending to infinity; that is,

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \qquad \lim_{k\to\infty}\lambda_k = \infty. \tag{11}$$

Following Mursaleen and Noman [20], we define the matrix $\Lambda = (\lambda_{nk})$ of weighted mean relative to the sequence λ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{12}$$

for all $k, n \in \mathbb{N}$. It is easy to show that the matrix Λ is regular and is reduced, in the special case $\lambda_k = k + 1$ for all $k \in \mathbb{N}$, to

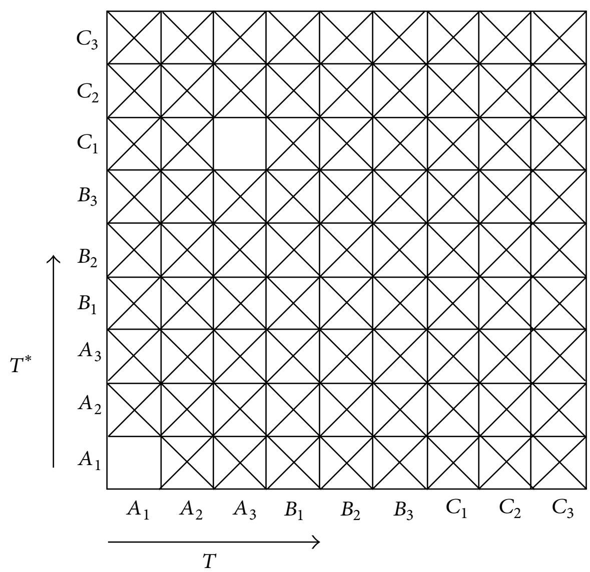

Figure 1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X.

the matrix C_1 of Cesàro mean of order one. Introducing the concept of Λ-*strong convergence*, several results on Λ-strong convergence of numerical sequences and Fourier series were given by Móricz [21]. Since we have

$$Q_n = \sum_{k=0}^{n} q_k = \lambda_n, \qquad r_{nk} = \frac{q_k}{Q_n} = \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} = \lambda_{nk} \tag{13}$$

in the special case $q_k = \lambda_k - \lambda_{k-1}$ for all $k \in \mathbb{N}$, the matrix Λ is also reduced to the Riesz means $R^q = (r_{nk})$ with respect to the sequence $q = (q_k)$. We say that a sequence $x = (x_k) \in \omega$ is λ-convergent if $\Lambda x \in c$. In particular, we say that x is a λ-null sequence if $\Lambda x \in c_0$ and we say that x is λ-bounded if $\Lambda x \in \ell_\infty$.

Lemma 2 (see [22, Theorem 1.3.6, page 6]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if*

(1) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded;*

(2) *the columns of A are in c;*

(3) *the sequence of row sums of A is in c.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 3. *$\Lambda : c \to c$ is a bounded linear operator with the norm $\|\Lambda\|_{(c:c)} = 1$.*

Lemma 4 (see [22, Example 8.4.5.A, page 129]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*

(2) *the columns of A are in c_0.*

Table 1: Subdivision of spectrum of a linear operator.

		1 T_α^{-1} exists and is bounded	2 T_α^{-1} exists and is unbounded	3 T_α^{-1} does not exist
A	$R(\alpha I - T) = X$	$\alpha \in \rho(T, X)$	—	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$
B	$\overline{R(\alpha I - T)} = X$	$\alpha \in \rho(T, X)$	$\alpha \in \sigma_c(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$
C	$\overline{R(\alpha I - T)} \neq X$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 5. *$\Lambda : c_0 \to c_0$ is a bounded linear operator with the norm* $\|\Lambda\|_{(c_0:c_0)} = 1$.

We give a short survey concerned with the spectrum of the linear operators defined by some triangle matrices over certain sequence spaces. Wenger [4] examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [5] generalized this result to the weighted mean methods. The fine spectrum of the operator on the sequence space ℓ_p was studied by González [23], where $1 < p < \infty$. The spectrum of the Cesàro operator on the sequence spaces c_0 and bv were also investigated by Reade [6], Akhmedov and Başar [7], and Okutoyi [8], respectively. The fine spectrum of the Rhaly operators on the sequence spaces c_0 and c were examined by Yıldırım [9]. Furthermore, Coşkun [10] has studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. Besides, de Malafosse [11] and Altay and Başar [12], respectively, studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k\in\mathbb{N}}|x_k|/r^k$, $(r > 0)$. Altay and Karakuş [24] determined the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv. In 2010, Srivastava and Kumar [16] determined the spectra and the fine spectra of the double sequential band matrix Δ_ν on ℓ_1, where Δ_ν is defined by $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_k)$ and they have just generalized these results by the double sequential band matrix Δ_{uv} defined by $\Delta_{uv}x = (u_n x_n + v_{n-1}x_{n-1})_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ (see [18]). Altun [25] studied the fine spectra of the Toeplitz operators, which are represented by upper and lower triangular n-band infinite matrices, over the sequence spaces c_0 and c. Later, Karakaya and Altun determined the fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c, in [26]. Quite recently, Akhmedov and El-Shabrawy [15] obtained the fine spectrum of the double sequential band matrix $\Delta_{a,b}$, defined as a double-band matrix with the convergent sequences $\widetilde{a} = (a_k)$ and $\widetilde{b} = (b_k)$ having certain properties, over the sequence space c. The fine spectrum with respect to Goldberg's classification of the operator $B(r, s, t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p with $1 < p < \infty$ has recently been studied by Furkan et al. [14]. Quite recently, Karaisa and Başar [19] have determined the fine spectrum of the upper triangular triple band matrix $B'(r, s, t)$ over the sequence space ℓ_p, where $0 < p < \infty$. At this stage, Table 2 may be useful.

In this work, our purpose is to determine the fine spectrum of the operator Λ over the sequence spaces c_0 and c with respect to Goldberg's classification. Additionally, we give the approximate point spectrum, defect spectrum, and compression spectrum of the matrix operator Λ over the spaces c_0 and c. Finally, we state and prove a Mercerian theorem.

3. The Fine Spectrum of the Operator Λ on the Sequence Space c_0

In this section, we examine the spectrum, the point spectrum, the continuous spectrum, the residual spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum of the operator Λ on the sequence space c_0. For simplicity in the notation, we write throughout that $c_n = (\lambda_n - \lambda_{n-1})/\lambda_n$ for all $n \in \mathbb{N}$ and we use this abbreviation with other letters.

Theorem 6. $\sigma(\Lambda, c_0) \subseteq \{\alpha \in \mathbb{C} : |2\alpha - 1| \leq 1\}$.

Proof. Let $|2\alpha - 1| > 1$. Since $\Lambda - \alpha I$ is triangle, $(\Lambda - \alpha I)^{-1}$ exists, and solving the matrix equation $(\Lambda - \alpha I)x = y$ for x in terms of y gives the matrix $(\Lambda - \alpha I)^{-1} = B = (b_{nk})$, where

$$b_{nk} = \begin{cases} \dfrac{(-1)^{n-k}\left(\lambda_k - \lambda_{k-1}\right)}{\lambda_n \alpha^2 \prod_{j=k}^{n}\left(c_j/\alpha - 1\right)}, & 0 \leq k \leq n-1, \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1} - \alpha\lambda_n}, & k = n, \\ 0, & k > n, \end{cases} \tag{14}$$

Table 2: Spectrum and fine spectrum of some triangle matrices in certain sequence spaces.

$\sigma(A,\lambda)$	$\sigma_p(A,\lambda)$	$\sigma_c(A,\lambda)$	$\sigma_r(A,\lambda)$	Refer to
$\sigma(C_1^p,c)$	—	—	—	[4]
$\sigma(W,c)$	—	—	—	[5]
$\sigma(C_1,c_0)$	—	—	—	[6]
$\sigma(C_1,c_0)$	$\sigma_p(C_1,c_0)$	$\sigma_c(C_1,c_0)$	$\sigma_r(C_1,c_0)$	[7]
$\sigma(C_1,bv)$	—	—	—	[8]
$\sigma(R,c_0)$	$\sigma_p(R,c_0)$	$\sigma_c(R,c_0)$	$\sigma_r(R,c_0)$	[9]
$\sigma(R,c)$	$\sigma_p(R,c)$	$\sigma_c(R,c)$	$\sigma_r(R,c)$	[9]
$\sigma(C_1^p,c_0)$	—	—	—	[10]
$\sigma(\Delta,s_r)$	—	—	—	[11]
$\sigma(\Delta,c_0)$	—	—	—	[11]
$\sigma(\Delta,c)$	—	—	—	[11]
$\sigma(\Delta^{(1)},c)$	$\sigma_p(\Delta^{(1)},c)$	$\sigma_c(\Delta^{(1)},c)$	$\sigma_r(\Delta^{(1)},c)$	[12]
$\sigma(\Delta^{(1)},c_0)$	$\sigma_p(\Delta^{(1)},c_0)$	$\sigma_c(\Delta^{(1)},c_0)$	$\sigma_r(\Delta^{(1)},c_0)$	[12]
$\sigma(B(r,s),\ell_p)$	$\sigma_p(B(r,s),\ell_p)$	$\sigma_c(B(r,s),\ell_p)$	$\sigma_r(B(r,s),\ell_p)$	[13]
$\sigma(B(r,s),bv_p)$	$\sigma_p(B(r,s),bv_p)$	$\sigma_c(B(r,s),bv_p)$	$\sigma_r(B(r,s),bv_p)$	[13]
$\sigma(B(r,s,t),\ell_p)$	$\sigma_p(B(r,s,t),\ell_p)$	$\sigma_c(B(r,s,t),\ell_p)$	$\sigma_r(B(r,s,t),\ell_p)$	[14]
$\sigma(B(r,s,t),bv_p)$	$\sigma_p(B(r,s,t),bv_p)$	$\sigma_c(B(r,s,t),bv_p)$	$\sigma_r(B(r,s,t),bv_p)$	[14]
$\sigma(\Delta_{a,b},c)$	$\sigma_p(\Delta_{a,b},c)$	$\sigma_c(\Delta_{a,b},c)$	$\sigma_r(\Delta_{a,b},c)$	[15]
$\sigma(\Delta_\nu,\ell_1)$	$\sigma_p(\Delta_\nu,\ell_1)$	$\sigma_c(\Delta_\nu,\ell_1)$	$\sigma_r(\Delta_\nu,\ell_1)$	[16]
$\sigma(\Delta_{uv}^2,c_0)$	$\sigma_p(\Delta_{uv}^2,c_0)$	$\sigma_c(\Delta_{uv}^2,c_0)$	$\sigma_r(\Delta_{uv}^2,c_0)$	[17]
$\sigma(\Delta_{uv},\ell_1)$	$\sigma_p(\Delta_{uv},\ell_1)$	$\sigma_c(\Delta_{uv},\ell_1)$	$\sigma_r(\Delta_{uv},\ell_1)$	[18]
$\sigma(B'(r,s,t),\ell_p)$	$\sigma_p(B'(r,s,t),\ell_p)$	$\sigma_c(B'(r,s,t),\ell_p)$	$\sigma_r(B'(r,s,t),\ell_p)$	[19]

for all $k,n\in\mathbb{N}$. Thus, we observe that

$$\left\|(\Lambda-\alpha I)^{-1}\right\|_{(c_0:c_0)}=\sup_{n\in\mathbb{N}}\sum_{k=0}^{\infty}\left|b_{nk}\right|. \tag{15}$$

The inequality $|2\alpha-1|>1$ is equivalent to $\gamma>-1$, where $-(1/\alpha)=\gamma+i\beta$. For all $\alpha\in\mathbb{C}$,

$$\left|1-\frac{c_j}{\alpha}\right|=\left|1+(\gamma+i\beta)c_j\right|\geq 1+\gamma c_j \tag{16}$$

holds for all $j\in\mathbb{N}$. So, $1/|1-(c_j/\alpha)|\leq 1/(1+\gamma c_j)$.

Firstly we take $-1<\gamma<0$. Since $0<c_j\leq 1$, we have $1+\gamma\leq 1+\gamma c_j<1$. Therefore $1/(1+\gamma c_j)<1/(1+\gamma)$ and $1<1/(1+\gamma)<\infty$ for $0<1+\gamma<1$.

$$\sum_{k=0}^{\infty}\left|b_{nk}\right|=\sum_{k=0}^{n}\left|b_{nk}\right|<\frac{\lambda_{n-1}}{\lambda_n|\alpha|^2(1+\gamma)^{n+1}}+\frac{1}{|\alpha|(1+\gamma)}<\infty. \tag{17}$$

Secondly we get $0\leq\gamma$. Since $1<1+\gamma c_j\leq 1+\gamma$, $1/(1+\gamma c_j)<1$. So,

$$\sum_{k=0}^{\infty}\left|b_{nk}\right|=\sum_{k=0}^{n}\left|b_{nk}\right|<\frac{\lambda_{n-1}}{\lambda_n|\alpha|^2}+\frac{1}{|\alpha|}<\infty. \tag{18}$$

Therefore, we have

$$\left\|(\Lambda-\alpha I)^{-1}\right\|_{(c_0:c_0)}=\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|b_{nk}\right|<\infty, \tag{19}$$

that is, $(\Lambda-\alpha I)^{-1}\in(c_0:c_0)$. But for $|2\alpha-1|\leq 1$,

$$\left\|(\Lambda-\alpha I)^{-1}\right\|_{(c_0:c_0)}=\infty, \tag{20}$$

that is, $(\Lambda-\alpha I)^{-1}$ is not in $B(c_0)$. This completes the proof. □

Theorem 7. *Define μ and η by $\mu=\limsup_{j\to\infty}c_j$ and $\eta=\liminf_{j\to\infty}c_j$. Then,*

$$\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{2-\mu}\right|\leq\frac{1-\mu}{2-\mu}\right\}\cup S\subseteq\sigma(\Lambda,c_0),\quad \text{where } S=\overline{\{c_j:j\in\mathbb{N}\}}. \tag{21}$$

Proof. Let $|\alpha-1/(2-\mu)|<(1-\mu)/(2-\mu)$ and $\alpha\neq c_j$ for any $j\in\mathbb{N}$. Then,

$$\begin{aligned}1-\frac{c_j}{\alpha}&=\frac{\lambda_j\lambda_{j-1}}{\lambda_j\lambda_{j-1}}-\frac{c_j}{\alpha}\\&=\frac{\lambda_{j-1}}{\lambda_j}\left[\frac{\lambda_j}{\lambda_{j-1}}-\frac{\lambda_j-\lambda_{j-1}}{\lambda_{j-1}}+\left(1-\frac{1}{\alpha}\right)\frac{\lambda_j-\lambda_{j-1}}{\lambda_{j-1}}\right]\\&=\frac{\lambda_{j-1}}{\lambda_j}\left[1+\left(1-\frac{1}{\alpha}\right)\frac{\lambda_j-\lambda_{j-1}}{\lambda_{j-1}}\right].\end{aligned} \tag{22}$$

So we have,

$$|b_{nk}| = \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k-1}|\alpha|^2 \prod_{j=k}^{n} \left|1 + (1 - 1/\alpha)\left(\left(\lambda_j - \lambda_{j-1}\right)/\lambda_{j-1}\right)\right|}. \tag{23}$$

Note that $|1 + (1 - \alpha^{-1})(\lambda_j - \lambda_{j-1})/\lambda_{j-1}| \leq 1$ if and only if

$$\left[1 + (1 + \gamma)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right]^2 + \left(\beta\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right)^2 \leq 1, \tag{24}$$

where $-\alpha^{-1} = \gamma + i\beta$. So, one can see that

$$2(1+\gamma)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}} + \left[(1+\gamma)^2 + \beta^2\right]\left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right)^2 \leq 0, \tag{25}$$

which is equivalent to the inequality

$$2(1+\gamma) + \left[(1+\gamma)^2 + \beta^2\right]\left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right) \leq 0. \tag{26}$$

For inequality (26) to be true for all sufficiently large j, it is sufficient to have

$$\limsup_{j\to\infty}\left[2(1+\gamma) + \left[(1+\gamma)^2 + \beta^2\right]\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right] < 0. \tag{27}$$

We can write $(\lambda_j - \lambda_{j-1})/\lambda_{j-1} = \lambda_j(\lambda_j - \lambda_{j-1})/(\lambda_j\lambda_{j-1})$ and $\lambda_j/\lambda_{j-1} = 1/(1 - c_j)$. Therefore,

$$\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}} = \frac{c_j}{1 - c_j}, \tag{28}$$

$$\limsup_{j\to\infty}\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}} = \frac{\mu}{1-\mu}, \tag{29}$$

since the function g defined by $g(x) = x/(1-x)$ is monotone increasing in x for $0 < x < 1$.

For (27) to be true for all sufficiently large j, it is sufficient to have μ satisfying

$$2(1+\gamma) + \left[(1+\gamma)^2 + \beta^2\right]\frac{\mu}{1-\mu} < 0 \tag{30}$$

which is equivalent to

$$\left|\alpha - \frac{1}{2-\mu}\right| < \frac{1-\mu}{2-\mu}. \tag{31}$$

Therefore, for all $n \geq N$, for some fixed N,

$$\begin{aligned}\sum_{k=N}^{n-1}|b_{nk}| &= \sum_{k=N}^{n-1}\frac{\lambda_k - \lambda_{k-1}}{\lambda_{k-1}|\alpha|^2\prod_{j=k}^{n}\left|1+(1-1/\alpha)\left(\left(\lambda_j - \lambda_{j-1}\right)/\lambda_{j-1}\right)\right|} \\ &\geq \frac{1}{|\alpha|^2}\sum_{k=N}^{n-1}\frac{\lambda_k - \lambda_{k-1}}{\lambda_{k-1}}\end{aligned} \tag{32}$$

which diverges in the light of (29).

If $\alpha = c_j$ for any $j \in \mathbb{N}$, then clearly α lies in the spectrum of Λ. This completes the proof. □

Theorem 8. *$\sigma(\Lambda, c_0) \subseteq \{\alpha \in \mathbb{C} : |\alpha - 1/(2-\eta)| \leq (1-\eta)/(2-\eta)\} \cup S$.*

Proof. Let α be fixed and satisfy the inequality

$$\left|\alpha - \frac{1}{2-\eta}\right| > \frac{1-\eta}{2-\eta}, \tag{33}$$

and $\alpha \neq c_j$ for any $j \in \mathbb{N}$. We will show that $\alpha \in \rho(\Lambda, c_0)$. From Theorem 6, we need consider only those values of α satisfying $|2\alpha - 1| > 1$; that is, $\gamma > -1$. Under the assumption on α, we wish to verify that

$$\left|1 + \left(1 - \frac{1}{\alpha}\right)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right| > 1 \tag{34}$$

for all sufficiently large j. It will be sufficient to show that

$$\liminf_{j\to\infty}\left\{2(1+\gamma) + \left[(1+\gamma)^2 + \beta^2\right]\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right\} > 0, \tag{35}$$

that is,

$$2(1+\gamma) + \left[(1+\gamma)^2 + \beta^2\right]\frac{\eta}{1-\eta} > 0 \tag{36}$$

which is equivalent to (33).

Define the function f by $f(t) = 1 + 2(1+\gamma)t + [(1+\gamma)^2 + \beta^2]t^2$. f has a minumum at $t_0 = -(1+\gamma)/[(1+\gamma)^2 + \beta^2]$. The above inequality is equivalent to $\eta(\gamma^2 + \beta^2) + 2\gamma > \eta - 2$ and is also equivalent to

$$\frac{\eta}{2(1-\eta)} > -\frac{1+\gamma}{(1+\gamma)^2 + \beta^2} = t_0. \tag{37}$$

Therefore, for those values of η satisfying (37), f is monotone increasing. Let $\epsilon > 0$ and small. Then $f(\eta/(1-\eta) - \epsilon) = f(\eta/(1-\eta)) - (2\epsilon)g(\epsilon)$, where $g(\epsilon) = 1 + \gamma + [(1+\gamma)^2 + \beta^2][\eta/(1-\eta) - \epsilon/2]$. Note that $g(\epsilon) > 0$ for small ϵ, since f is monotone increasing for $t > \eta/[2(1-\eta)]$, we will now show that $f(\eta/(1-\eta)) > 1$. From (37),

$$\gamma^2 + \beta^2 + \frac{2\gamma}{\eta} > \frac{\eta - 2}{\eta} \tag{38}$$

which is equivalent to

$$\left|\frac{1}{1-\eta} - \frac{\eta}{\alpha(1-\eta)}\right| > 1. \tag{39}$$

But $1/(1-\eta) = 1 + \eta/(1-\eta)$, so we have $f(\eta/(1-\eta)) = |1 + (1 - \alpha^{-1})\eta/(1-\eta)|^2 > 1$. Now choose $\epsilon > 0$ and so small that $f(\eta/(1-\eta) - \epsilon) = f(\eta/(1-\eta)) - 2\epsilon g(\epsilon) = m^2 > 1$. Then, by the definition of η, there exists an N such that $n > N$ implies

$(\lambda_{n+1}-\lambda_n)/\lambda_n > \eta/(1-\eta)-\epsilon$, so that $f((\lambda_{n+1}-\lambda_n)/\lambda_n) > f(\eta/(1-\eta)-\epsilon) = m^2$. Using (23),

$$\begin{aligned}
&\frac{|b_{nk}|}{|b_{n+1,k}|} \\
&= \frac{(\lambda_k-\lambda_{k-1})/\lambda_{k-1}|\alpha|^2\prod_{j=k}^{n}\left|1+(1-1/\alpha)\left(\left(\lambda_j-\lambda_{j-1}\right)/\lambda_{j-1}\right)\right|}{(\lambda_k-\lambda_{k-1})/\lambda_{k-1}|\alpha|^2\prod_{j=k}^{n+1}\left|1+(1-1/\alpha)\left(\left(\lambda_j-\lambda_{j-1}\right)/\lambda_{j-1}\right)\right|} \\
&= \left|1+\left(1-\frac{1}{\alpha}\right)\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right| \\
&= f\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_n}\right) > m^2 > 1,
\end{aligned} \tag{40}$$

for all $n \geq N$. Therefore $\{|b_{nk}|\}$ is monotone decreasing in n for each $k, n > N$, so that B has bounded columns. It remains to show that B has finite norm.

For ϵ being used, from (29), we can enlarge N, if necessary, to ensure that $(\lambda_n-\lambda_{n-1})/\lambda_{n-1} < \mu/(1-\mu)+1$ for $n \geq N$. From (23),

$$\begin{aligned}
&\sum_{k=N}^{n-1}|b_{nk}| \\
&= \sum_{k=N}^{n-1}\frac{\lambda_k-\lambda_{k-1}}{\lambda_{k-1}|\alpha|^2\prod_{j=k}^{n}\left|1+(1-1/\alpha)\left(\left(\lambda_j-\lambda_{j-1}\right)/\lambda_{j-1}\right)\right|} \\
&\leq \frac{1}{|\alpha|^2}\left(\frac{\mu}{1-\mu}+1\right)\sum_{k=N}^{n-1}\frac{1}{\prod_{j=k}^{n}\left|1+(1-1/\alpha)\left(\left(\lambda_j-\lambda_{j-1}\right)/\lambda_{j-1}\right)\right|} \\
&\leq \frac{1}{|\alpha|^2}\left(\frac{\mu}{1-\mu}+1\right)\sum_{k=N}^{n-1}m^{-n+k-1} < H,
\end{aligned} \tag{41}$$

where H is a constant independent of n. Further

$$\begin{aligned}
|b_{nn}| &= \frac{\lambda_n}{|\alpha|\left|\lambda_n-\left(\left(\lambda_n-\lambda_{n-1}\right)/\alpha\right)\right|} \\
&= \frac{\lambda_n}{|\alpha|\left|\lambda_{n-1}+(1-1/\alpha)\left(\lambda_n-\lambda_{n-1}\right)\right|} \\
&= \frac{\lambda_n/\lambda_{n-1}}{|\alpha|\left|1+(1-1/\alpha)\left(\left(\lambda_n-\lambda_{n-1}\right)/\lambda_{n-1}\right)\right|} \\
&= \frac{1+\left(\lambda_n-\lambda_{n-1}\right)/\lambda_{n-1}}{|\alpha|\left|1+(1-1/\alpha)\left(\left(\lambda_n-\lambda_{n-1}\right)/\lambda_{n-1}\right)\right|} \\
&< \frac{1+\mu/(1-\mu)+1}{|\alpha|m}.
\end{aligned} \tag{42}$$

Hence, B has a finite norm. □

Corollary 9. *Let $\delta = \lim_{j\to\infty} c_j$ exist. Then,*

$$\sigma(\Lambda, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\} \cup S. \tag{43}$$

If $T \in B(c_0)$ with the matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is dened by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $\|x\| = \sum_{k=0}^{\infty}|x_k|$.

Theorem 10. *Let δ be defined as in Corollary 9. Then, $\sigma_p(\Lambda^*, c_0^*) = \{\alpha \in \mathbb{C} : |\alpha - 1/(2-\delta)| < (1-\delta)/(2-\delta)\} \cup S$.*

Proof. Suppose that $\Lambda^* x = \alpha x$ for $x \neq \theta$ in $c_0^* \cong \ell_1$. Then, by solving the system of linear equations

$$\begin{aligned}
x_1 &= \frac{\lambda_1-\lambda_0}{\lambda_0}\left(1-\frac{1}{\alpha}\right)x_0, \\
x_2 &= \frac{\lambda_2-\lambda_1}{\lambda_0}\left(1-\frac{c_1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)x_0 \\
&\vdots \\
x_n &= \frac{\lambda_n-\lambda_{n-1}}{\lambda_0}\left(1-\frac{1}{\alpha}\right)x_0\prod_{j=1}^{n-1}\left(1-\frac{c_j}{\alpha}\right) \\
&\vdots
\end{aligned} \tag{44}$$

we can write $x_n = (\lambda_n-\lambda_{n-1})/\lambda_{n-1}(1-\alpha^{-1})x_0\prod_{j=1}^{n-1}[1+(1-\alpha^{-1})(\lambda_j-\lambda_{j-1})/\lambda_{j-1}]$. Let $|\alpha - 1/(2-\delta)| < (1-\delta)/(2-\delta)$ or $\alpha \in S$ and $u_n = \prod_{j=1}^{n-1}[1+(1-\alpha^{-1})(\lambda_j-\lambda_{j-1})/\lambda_{j-1}]$. One can see that $|1+(1-\alpha^{-1})(\lambda_j-\lambda_{j-1})/\lambda_{j-1}| < 1$ for all sufficiently large j if and only if

$$\left[1+(1+\gamma)\frac{\lambda_j-\lambda_{j-1}}{\lambda_{j-1}}\right]^2 + \left(\beta\frac{\lambda_j-\lambda_{j-1}}{\lambda_{j-1}}\right)^2 < 1, \quad \text{where } -\frac{1}{\alpha} = \gamma + i\beta. \tag{45}$$

Then, we have, from the discussion in Theorem 7 and the hypothesis on α,

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|1+\left(1-\frac{1}{\alpha}\right)\frac{\lambda_n-\lambda_{n-1}}{\lambda_{n-1}}\right| < 1 \tag{46}$$

for all sufficiently large n, so $\sum_{n=0}^{\infty}|u_n|$ is convergent. Since $|(1-\alpha^{-1})(\lambda_n-\lambda_{n-1})/\lambda_{n-1}x_0|$ is bounded, it follows that $\sum_{n=0}^{\infty}|x_n|$ is convergent, so that $\Lambda^* x = \alpha x$ has nonzero solutions. Therefore, the proof is completed. □

Theorem 11. *Let δ be defined as in Corollary 9. Then*

$$\sigma_p(\Lambda, c_0) = \left\{\alpha = c_n \in \mathbb{C} : 0 \leq \alpha \leq \frac{\delta}{2-\delta}\right\} \cup \{1\}. \tag{47}$$

Proof. Let c_k be any diagonal entry satisfying $0 < c_k \leq \delta/(2-\delta)$. Let j be the smallest integer such that $c_j = c_k$. By setting $x_n = 0$ for $n > j+1$, $x_0 = 0$, the system $(\Lambda^* - c_j I)x = \theta$ reduces to a homogeneous linear system of j equations in

$j+1$ unknowns, so that nontrivial solutions exist. Therefore $\Lambda - c_j I \in 3$.

$\Lambda - \alpha I$ is not one to one for $\alpha = 0, 1$ and so $\Lambda - \alpha I \in 3$. This step concludes the proof. □

Lemma 12 (see [3, page 59]). *T has a dense range if and only if T^* is one to one.*

Theorem 13. $\sigma_r(\Lambda, c_0) = \sigma_p(\Lambda^*, c_0^*) \setminus \sigma_p(\Lambda, c_0)$.

Proof. For $\alpha \in \sigma_p(\Lambda^*, c_0^*) \setminus \sigma_p(\Lambda, c_0)$, the operator $\Lambda - \alpha I$ is triangle, so has an inverse. But $\Lambda^* - \alpha I$ is not one to one by Theorem 10. Therefore by Lemma 12, $\overline{R(\Lambda - \alpha I)} \neq c_0$, and this step concludes the proof. □

Theorem 14. *Let δ be defined as in Corollary 9 and $c_n \geq \delta$ for all sufficiently large n. Then,*

$$\sigma_c(\Lambda, c_0) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\delta}\right| = \frac{1-\delta}{2-\delta}, \alpha \neq 1, \frac{\delta}{2-\delta}\right\}. \tag{48}$$

Proof. Fix $\alpha \neq 1, \delta/(2-\delta)$, and satisfying $|\alpha - 1/(2-\delta)| = (1-\delta)/(2-\delta)$. Since the operator $\Lambda - \alpha I$ is triangle, it has an inverse. Consider the adjoint operator $\Lambda^* - \alpha I$. As in Theorem 11, x_0 is arbitrary and

$$x_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\left(1 - \frac{1}{\alpha}\right) x_0 \prod_{j=k}^{n}\left[1 + \left(1 - \frac{1}{\alpha}\right)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right] \tag{49}$$

for all positive n. From the hypothesis, there exists a positive integer N such that $n \geq N$ implies $c_n \geq \delta$. This fact, together with the condition on α, implies that $|1 + (1-\alpha^{-1})(\lambda_n - \lambda_{n-1})/\lambda_{n-1}| \geq 1$ for $n \geq N$. Thus, $|x_n| = C(\lambda_n - \lambda_{n-1})/\lambda_{n-1}$ for $n \geq N$, where C is a positive constant independent of n. We can write

$$\begin{aligned}\frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}} &= c_n\left(1 + \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}}\right) \\ &\geq c_n.\end{aligned} \tag{50}$$

Therefore $(x_n) \in \ell_1 \Leftrightarrow x_0 = 0$; that is, $\Lambda^* - \alpha I$ is one to one. From Lemma 12, the range of $\Lambda - \alpha I$ is dense in c_0. This completes the proof. □

Lemma 15 (see [3, page 60]). *T has a bounded inverse if and only if T^* is onto.*

Theorem 16. *Let δ be defined as in Corollary 9 and less than 1. If α satisfies $|\alpha - 1/(2-\delta)| < (1-\delta)/(2-\delta)$ and $\alpha \notin S$, then $\alpha \in C_1\sigma(\Lambda, c_0)$.*

Proof. First of all $\Lambda - \alpha I$ is a triangle; hence $1-1$. Therefore $\Lambda - \alpha I \in 1 \cup 2$. To verify that $\Lambda - \alpha I \in C_1\sigma(\Lambda, c_0)$ it is sufficient to show that $\Lambda^* - \alpha I$ is onto by Lemma 15.

Suppose $y = (\Lambda^* - \alpha I)x$, where $x, y \in \ell_1$. Then, $x_0 = 1/(1-\alpha)y_0 - \lambda_0/[(\lambda_1 - \lambda_0)(1-\alpha)]y_1$ and

$$(c_n - \alpha) x_n + (\lambda_n - \lambda_{n-1}) \sum_{k=n+1}^{\infty} \frac{x_k}{\lambda_k} = y_n, \quad n > 0. \tag{51}$$

Choose $x_1 = 0$ and solve (51) for x in terms of y to get

$$(\lambda_1 - \lambda_0) \sum_{k=2}^{\infty} \frac{x_k}{\lambda_k} = y_1, \tag{52}$$

$$(c_n - \alpha) x_n = y_n - (\lambda_n - \lambda_{n-1}) \sum_{k=n+1}^{\infty} \frac{x_k}{\lambda_k}. \tag{53}$$

For example, substituting (52) into (53), with $n = 2$, yields

$$(c_2 - \alpha) x_2 = y_2 - (\lambda_2 - \lambda_1) \sum_{k=3}^{\infty} \frac{x_k}{\lambda_k}, \tag{54}$$

so that $x_2 = (\lambda_2 - \lambda_1)/[\alpha(\lambda_1 - \lambda_0)]y_1 - (1/\alpha)y_2$. For $n = 3$,

$$x_3 = -\left(\frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_0}\right)(c_2 - \alpha)\frac{1}{\alpha^2} y_1 + \left(\frac{\lambda_3 - \lambda_2}{\lambda_2}\right)\frac{1}{\alpha^2} y_2 - \frac{1}{\alpha} y_3. \tag{55}$$

Continuing this process, the entries of the matrix $B = (b_{nk})$ such that $By = x$ are calculated as

$$\begin{gathered} b_{00} = \frac{1}{1-\alpha}, \qquad b_{01} = -\frac{\lambda_0}{(\lambda_1 - \lambda_0)(1-\alpha)} \\ b_{21} = \frac{\lambda_2 - \lambda_1}{(\lambda_1 - \lambda_0)\alpha}, \qquad b_{nn} = -\frac{1}{\alpha}, \quad n > 1, \\ b_{n,n-1} = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n-1}\alpha^2}, \quad n > 2 \\ b_{n1} = \frac{\lambda_n - \lambda_{n-1}}{(\lambda_1 - \lambda_0)\alpha} \prod_{j=2}^{n-1}\left(1 - \frac{c_j}{\alpha}\right), \quad n > 2, \\ b_{nk} = \frac{\lambda_n - \lambda_{n-1}}{\lambda_k \alpha^2} \prod_{j=k+1}^{n-1}\left(1 - \frac{c_j}{\alpha}\right), \quad 1 < k < n-1, \end{gathered} \tag{56}$$

and $b_{nk} = 0$ otherwise.

To show that $B \in B(\ell_1)$, it is sufficient to establish that $\sum_{n=0}^{\infty} |b_{nk}|$ is finite independent of k. $\sum_{n=0}^{\infty} |b_{n0}| = 1/|1-\alpha|$. We may write $1 - (c_j/\alpha) = (\lambda_{j-1}/\lambda_j)[1 + (1-\alpha^{-1})(\lambda_j - \lambda_{j-1})/\lambda_{j-1}]$. Also, $\sup_{n\in\mathbb{N}}|(\lambda_n - \lambda_{n-1})/\lambda_{n-1}| \leq M < \infty$. Therefore,

$$\sum_{n=0}^{\infty} |b_{n1}| \leq \frac{1}{|\alpha|}\left[M + M \sum_{n=3}^{\infty} \prod_{j=2}^{n-1}\left|1 + \left(1 - \frac{1}{\alpha}\right)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right|\right] \tag{57}$$

and, for $k > 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} |b_{nk}| \leq{}& \frac{1}{|\alpha|} + \frac{M}{|\alpha|^2} \\ &+ \frac{M}{|\alpha|^2} \sum_{n=k+2}^{\infty} \prod_{j=k+1}^{n-1}\left|1 + \left(1 - \frac{1}{\alpha}\right)\frac{\lambda_j - \lambda_{j-1}}{\lambda_{j-1}}\right|.\end{aligned} \tag{58}$$

Since $k > 1$, the series in inequality (24) is absolutely convergent from Theorem 7. Therefore, $\|B\|_{(\ell_1:\ell_1)}$ is finite.

Because of $(\Lambda - \alpha I)^{-1}$ is bounded, it is continuous, and $\alpha \in C_1\sigma(\Lambda, c_0)$. This completes the proof. □

Theorem 17. *Let δ be defined as in Corollary 9 and $\delta < 1$. If $\alpha = \delta$ or $\alpha = c_n$ for all $n \in \mathbb{N}$ and $\delta/(2-\delta) < \alpha < 1$, then $\alpha \in C_1\sigma(\Lambda, c_0)$.*

Proof. First assume that Λ has distinct diagonal entries and fix $j \geq 1$. Then the system $(\Lambda - c_j I)x = \theta$ implies that $x_n = 0$ for $n = 0, 1, \ldots, j-1$, and for $n \geq j$

$$\left(c_j - c_n\right) x_n - \sum_{k=0}^{n-1} \lambda_{nk} x_k = 0. \tag{59}$$

The system (59) yields the following recursion relation:

$$x_{n+1} = \frac{\lambda_n c_j x_n}{\lambda_{n+1}\left(c_j - c_{n+1}\right)} \tag{60}$$

which can be solved for x_n to yield

$$\begin{aligned} x_{j+m} &= \frac{\lambda_j x_j c_j^m}{\lambda_{j+m} \prod_{i=1}^m \left(c_j - c_{j+i}\right)} = x_j \prod_{i=1}^m \frac{\lambda_{j+i-1}}{\lambda_{j+i}\left(1 - c_{j+i}/c_j\right)} \\ &= x_j \left\{ \prod_{i=1}^m \frac{\lambda_{j+i}}{\lambda_{j+i-1}} \left[1 - \frac{\lambda_j \left(\lambda_{j+i} - \lambda_{j+i-1}\right)}{\lambda_{j+i}\left(\lambda_j - \lambda_{j-1}\right)} \right] \right\}^{-1} \\ &= x_j \left\{ \prod_{i=1}^m \left[\frac{\lambda_{j+i}}{\lambda_{j+i-1}} - \frac{\lambda_j \left(\lambda_{j+i} - \lambda_{j+i-1}\right)}{\lambda_{j+i-1}\left(\lambda_j - \lambda_{j-1}\right)} \right] \right\}^{-1} \\ &= x_j \left\{ \prod_{i=1}^m \left[\frac{\lambda_{j+i}}{\lambda_{j+i-1}} - \frac{\lambda_{j+i} - \lambda_{j+i-1}}{\lambda_{j+i-1}} + \left(1 - \frac{1}{c_j}\right) \frac{\lambda_{j+i} - \lambda_{j+i-1}}{\lambda_{j+i-1}} \right] \right\}^{-1} \\ &= x_j \left\{ \prod_{i=1}^m \left[1 + \left(1 - \frac{1}{c_j}\right) \frac{\lambda_{j+i} - \lambda_{j+i-1}}{\lambda_{j+i-1}} \right] \right\}^{-1}. \end{aligned} \tag{61}$$

Since $0 < c_j < 1$, the argument of Theorem 7 applies and (24) is true. Therefore $x \in c_0$ implies $x = \theta$ and $\Lambda - c_j I$ is $1-1$, so that $\Lambda - c_j I \in 1 \cup 2$.

Clearly $\Lambda - c_j I \in C$. It remains to show that $\Lambda^* - c_j I$ is onto.

Suppose that $(\Lambda^* - c_j I)x = y$, where $x, y \in \ell_1$. By choosing $x_{j+1} = 0$ we can solve for $x_0, x_1, \ldots, x_j$ in terms of $y_0, y_1, \ldots, y_{j+1}$. As in Theorem 16, the remaining equations can be written in the form $x = By$, where the nonzero entries of $B = (b_{nk})$ are as follows

$$\begin{aligned} b_{j+m,j+m} &= -\frac{1}{c_j}; \\ b_{j+2,j+1} &= \frac{\lambda_{j+2} - \lambda_{j+1}}{c_j\left(\lambda_{j+1} - \lambda_j\right)}; \\ b_{j+m,j+m-1} &= \frac{\lambda_{j+m} - \lambda_{j+m-1}}{c_j^2 \lambda_{j+m-1}}, \quad m > 2; \\ b_{j+m,j+k} &= \frac{\lambda_{j+m} - \lambda_{j+m-1}}{c_j^2 \lambda_{j+k}} \prod_{i=j+k+1}^{j+m-1} \left(1 - \frac{c_i}{c_j}\right), \\ &\qquad 1 < k < m-1, \ m > 3; \\ b_{j+m,j+1} &= \frac{\lambda_{j+m} - \lambda_{j+m-1}}{c_j\left(\lambda_{j+1} - \lambda_j\right)} \prod_{i=j+2}^{j+m-1} \left(1 - \frac{c_i}{c_j}\right), \quad m > 2. \end{aligned} \tag{62}$$

From (62), we have

$$\begin{aligned} \sum_{n=j+1}^{\infty} \left|b_{n,j+1}\right| &= \frac{\lambda_{j+2} - \lambda_{j+1}}{c_j\left(\lambda_{j+1} - \lambda_j\right)} + \frac{1}{c_j\left(\lambda_{j+1} - \lambda_j\right)} \\ &\quad \times \sum_{n=j+3}^{\infty} \left(\lambda_n - \lambda_{n-1}\right) \prod_{i=j+2}^{n-1} \left|1 - \frac{c_i}{c_j}\right|. \end{aligned} \tag{63}$$

For $m > 1$,

$$\begin{aligned} \sum_{n=m+j}^{\infty} \left|b_{n,m+j}\right| &= \frac{1}{c_j} + \frac{\lambda_{j+m+1} - \lambda_{j+m}}{c_j^2 \lambda_{j+m}} \\ &\quad + \frac{1}{c_j^2} \sum_{n=j+m+2}^{\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{j+m}} \prod_{i=j+m+1}^{n-1} \left|1 - \frac{c_i}{c_j}\right|. \end{aligned} \tag{64}$$

Using $1 - (c_j/\alpha) = (\lambda_{j-1}/\lambda_j)[1 + (1 - \alpha^{-1})(\lambda_j - \lambda_{j-1})/\lambda_{j-1}]$, one can convert (63) and (64) the similar expressions to (57) and (58), and therefore $\|B\|_{(\ell_1:\ell_1)}$ is finite.

Suppose that Λ does not have distinct diagonal entries. The restriction on α guarantees that no zero diagonal entries are being considered. Let $c_j \neq 0$ be any diagonal entry which occurs more than once, and let k, r denote, respectively, the smallest and largest integers for which $c_j = c_k = c_r$. From (61) it follows that $x_n = 0$ for $n \geq r$. Also, $x_n = 0$ for $0 \leq n < k$. Therefore the system $(\Lambda - c_j I)x = \theta$ becomes

$$\left(c_j - c_n\right) x_n - \sum_{i=j}^{n-1} \lambda_{ni} x_i = 0, \quad k < n \leq r. \tag{65}$$

Case 1. Let $r = k+1$. Then (65) reduces to the single equation

$$\left(c_j - c_{k+1}\right) x_{j+1} - \left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1}}\right) x_k = 0 \tag{66}$$

which implies that $x_k = 0$, since $c_j = c_r = c_{k+1}$ and $c_j \neq 0$. Therefore $x = \theta$.

Case 2. Let $r > k+1$. From (65) one can obtain the recursion formula $x_n = \lambda_{n+1}(c_j - c_{n+1})x_{n+1}/(\lambda_n c_j)$ with $k < n < r$. Since $x_r = 0$ it then follows that $x_n = 0$ for $k < n < r$. Using (65) with $n = k+1$ yields $x_k = 0$ and so again $x = \theta$.

To show that $\Lambda^* - c_j I$ is onto, suppose $(\Lambda^* - c_j I)x = y$, where $x, y \in \ell_1$. By choosing $x_{j+1} = 0$ one can solve for $x_0, x_1, \ldots, x_j$ in terms of $y_0, y_1, \ldots, y_{j+1}$. As in Theorem 16, the remaining equations can be written in the form $x = By$, where the nonzero entries of B are as in (62) with the other entries of B clearly zero.

Since $k \leq j \leq r$, there are two cases to consider.

Case 1. If $j = r$, then the proof proceeds exactly as in the argument following (62).

Case 2. If $j < r$, then from (62), $b_{j+m,j+k} = b_{j+m,j+1} = 0$ at least for $m \geq r - j + 2$. If there are other values of n with $j < n < r$ for which $c_n - c_j$, then additional entries of B will be zero. These zero entries do not affect the validity of the argument showing that (63) converges.

If $\delta = 0$, then 0 does not lie inside the disc, and so it is not considered in this theorem.

Let $\alpha = \delta > 0$. If $\lambda_{nn} \leq \delta$ for each $n \geq 1$, all i sufficiently large, then the argument of Theorem 16 applies and $\Lambda - \delta I \in C_1$. If $\lambda_{nn} = \delta$ for some n, then the proof of Theorem 17 applies with c_j replaced by δ and again, $\Lambda - \delta I \in C_1$.

Therefore, in all cases, $\Lambda - c_j I \in 1 \cup 2$. □

Theorem 18. *If $\alpha \in \sigma_p(\Lambda, c_0), \alpha \in C_3\sigma(\Lambda, c_0)$.*

Proof. For $\alpha \in \sigma_p(\Lambda, c_0), \Lambda - \alpha I \in 3$ and $\Lambda^* - \alpha I$ is not one to one. Therefore $\overline{R(\Lambda - \alpha I)} \neq c_0$ by Lemma 12. This concludes the proof. □

Theorem 19. *The statement $A_3\sigma(\Lambda, c_0) = C_2\sigma(\Lambda, c_0) = \emptyset$ holds.*

Proof. Let δ be defined as in Corollary 9 and $c_n \geq \delta$ for all sufficiently large n, then $A_3\sigma(\Lambda, c_0) = \emptyset$ and $C_2\sigma(\Lambda, c_0) = \emptyset$ follow from Corollary 9, Theorems 14, and 16–18. □

Define the set E by

$$E = \overline{\left\{c_j : c_j \leq \frac{\eta}{2 - \eta}\right\}}, \tag{67}$$

where η is as in Theorem 7.

We will consider $\delta = \eta$, that is, for which the main diagonal entries converge, where δ as in Corollary 9.

Theorem 20. *The following results hold:*

(a) $\sigma_{ap}(\Lambda, c_0) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)\} \cup E$,

(b) $\sigma_\delta(\Lambda, c_0) = \sigma(\Lambda, c_0)$,

(c) $\sigma_{co}(\Lambda, c_0) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$.

Proof. (a) Since the relation

$$\begin{aligned} C_1\sigma(\Lambda, c_0) = &\left\{\left\{\alpha : \left|\alpha - \frac{1}{2-\delta}\right| < \frac{1-\delta}{2-\delta}\right\} \setminus S\right\} \\ &\bigcup \left\{\alpha = \lambda_{nn} : \frac{\delta}{2-\delta} < \alpha < 1\right\} \end{aligned} \tag{68}$$

holds by Theorems 16 and 17 and from Table 1, $\sigma_{\text{ap}}(\Lambda, c_0) = \sigma(\Lambda, c_0) \setminus C_1\sigma(\Lambda, c_0)$. Therefore, we have $\sigma_{\text{ap}}(\Lambda, c_0) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)\} \cup E$.

(b) Since $\sigma_\delta(\Lambda, c_0) = \sigma(\Lambda, c_0) \setminus A_3\sigma(\Lambda, c_0)$ from Table 1 and $A_3\sigma(\Lambda, c_0) = \emptyset$ by Theorem 19, we have $\sigma_\delta(\Lambda, c_0) = \sigma(\Lambda, c_0)$.

(c) Since the equality $\sigma_{\text{co}}(\Lambda, c_0) = C_1\sigma(\Lambda, c_0) \cup C_2\sigma(\Lambda, c_0) \cup C_3\sigma(\Lambda, c_0)$ holds from Table 1, we have $\sigma_{\text{co}}(\Lambda, c_0) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$ by Theorems 16–19. □

The next corollary can be obtained from Proposition 1.

Corollary 21. *The following results hold:*

(a) $\sigma_{ap}(\Lambda^*, \ell_1) = \sigma(\Lambda, c_0)$,

(b) $\sigma_\delta(\Lambda^*, \ell_1) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)(2-\delta)\} \cup E$,

(c) $\sigma_p(\Lambda^*, \ell_1) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$.

4. The Fine Spectrum of the Operator Λ on the Sequence Space c

In this section, we investigate the fine spectrum of the operator Λ over the sequence space c.

Theorem 22. $\sigma(\Lambda, c) \subseteq \{\alpha \in \mathbb{C} : |2\alpha - 1| \leq 1\}$.

Proof. This is obtained in a similar way to that used in the proof of Theorem 6. □

Theorem 23. *Suppose that μ, η and S be defined as in Theorem 7. Then,*

$$\begin{aligned} &\left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\mu}\right| \leq \frac{1-\mu}{2-\mu}\right\} \cup S \\ &\subseteq \sigma(\Lambda, c) \\ &\subseteq \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\eta}\right| \leq \frac{1-\eta}{2-\eta}\right\} \cup S. \end{aligned} \tag{69}$$

Proof. This is similar to the proof of Theorems 7 and 8. To avoid the repetition of the similar statements, we omit the detail. □

Corollary 24. *Let δ be defined as in Corollary 9. Then,*

$$\sigma(\Lambda, c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\delta}\right| \leq \frac{1-\delta}{2-\delta}\right\} \cup S. \tag{70}$$

If $T : c \to c$ is a bounded linear operator with the matrix A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix}, \tag{71}$$

where χ is the limit of the sequence of row sums of A minus the sum of the limit of the columns of A and b is the column vector whose kth entry is the limit of the kth column of A for each $k \in \mathbb{N}$. For $\Lambda : c \to c$, the matrix $\Lambda^* \in B(\ell_1)$ is of the form

$$\Lambda^* = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda^t \end{bmatrix}. \tag{72}$$

Theorem 25. *Let δ be defined as in Corollary 9. Then,*

$$\sigma_p(\Lambda^*, c^*) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\delta}\right| < \frac{1-\delta}{2-\delta}\right\} \cup S. \quad (73)$$

Proof. Suppose that $\Lambda^* x = \alpha x$ for $x \neq \theta$ in $c^* \cong \ell_1$. Then, by solving the system of linear equations

$$\begin{aligned} (1-\alpha) x_0 &= 0, \\ x_2 &= \frac{\lambda_1 - \lambda_0}{\lambda_0}\left(1 - \frac{1}{\alpha}\right) x_1, \\ x_3 &= \frac{\lambda_2 - \lambda_1}{\lambda_0}\left(1 - \frac{c_1}{\alpha}\right)\left(1 - \frac{1}{\alpha}\right) x_1 \\ &\vdots \\ x_n &= \frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_0}\left(1 - \frac{1}{\alpha}\right) x_1 \prod_{j=1}^{n-1}\left(1 - \frac{c_j}{\alpha}\right) \\ &\vdots \end{aligned} \quad (74)$$

we get by assumption $(1-\alpha)x_0 = 0$ with $\alpha = 1$ that $x = (x_0, x_1, 0, 0, \ldots) \in c$. If $\alpha \neq 1$, we have $x_0 = 0$ and $(x_n) \in \ell_1$ if and only if $|1 + (1-\alpha^{-1})(\lambda_j - \lambda_{j-1})/\lambda_{j-1}| < 1$, by Theorem 11. This completes the proof. □

Theorem 26. *Let δ be defined as in Corollary 9. Then,*

$$\sigma_p(\Lambda, c) = \left\{\alpha = c_n \in \mathbb{C} : 0 \leq \alpha \leq \frac{\delta}{2-\delta}\right\} \cup \{1\}. \quad (75)$$

Proof. The proof is identical to the proof of Theorem 11. □

Theorem 27. $\sigma_r(\Lambda, c) = \sigma_p(\Lambda^*, c^*) \setminus \sigma_p(\Lambda, c)$.

Proof. For $\alpha \in \sigma_p(\Lambda^*, c^*) \setminus \sigma_p(\Lambda, c)$, the operator $\Lambda - \alpha I$ is triangle, so has an inverse. But $\Lambda^* - \alpha I$ is not one to one by Theorem 26. Therefore by Lemma 12, $\overline{R(\Lambda - \alpha I)} \neq c$ and this concludes the proof. □

Since Theorems 28–31 can be proved in a similar way to that used in the proof of Theorems 14 and 16–18; respectively, to avoid the repetition of the similar statements we omit the detailed proof and give them without proof.

Theorem 28. *Let δ be defined as in Corollary 9 and $c_n \geq \delta$ for all sufficiently large n. Then,*

$$\sigma_c(\Lambda, c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{2-\delta}\right| = \frac{1-\delta}{2-\delta}, \alpha \neq 1, \frac{\delta}{2-\delta}\right\}. \quad (76)$$

Theorem 29. *Let δ be defined as in Corollary 9 and less than 1. If α satisfies $|\alpha - 1/(2-\delta)| < (1-\delta)/(2-\delta)$ and $\alpha \notin S$, then $\alpha \in C_1\sigma(\Lambda, c)$.*

Theorem 30. *Let δ be defined as in Corollary 9 and $\delta < 1$. If $\alpha = \delta$ or $\alpha = c_n$ for all $n \in \mathbb{N}$ and $\delta/(2-\delta) < \alpha < 1$, then $\alpha \in C_1\sigma(\Lambda, c)$.*

Theorem 31. *If $\alpha \in \sigma_p(\Lambda, c)$, $\alpha \in C_3\sigma(\Lambda, c)$.*

Theorem 32. *The following statement holds: $A_3\sigma(\Lambda, c) = C_2\sigma(\Lambda, c) = \emptyset$.*

Proof. Let δ be defined as in Corollary 9 and $c_n \geq \delta$ for all sufficiently large n, then $A_3\sigma(\Lambda, c) = \emptyset$ and $C_2\sigma(\Lambda, c) = \emptyset$ follow from Corollary 24 and Theorems 28–31. □

Theorem 33. *The following results hold:*

(a) $\sigma_{ap}(\Lambda, c) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)\} \cup E$,

(b) $\sigma_\delta(\Lambda, c) = \sigma(\Lambda, c)$,

(c) $\sigma_{co}(\Lambda, c) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$.

Proof. (a) Since the relation $C_1\sigma(\Lambda, c) = \{\{\alpha : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \setminus S\} \bigcup \{\alpha = \lambda_{nn} : \delta/(2-\delta) < \alpha < 1\}$ holds by Theorems 29 and 30 and from Table 1, $\sigma_{\text{ap}}(\Lambda, c) = \sigma(\Lambda, c) \setminus C_1\sigma(\Lambda, c)$. Therefore, we have $\sigma_{\text{ap}}(\Lambda, c) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)\} \cup E$.

(b) Since $\sigma_\delta(\Lambda, c) = \sigma(\Lambda, c) \setminus A_3\sigma(\Lambda, c)$ from Table 1 and $A_3\sigma(\Lambda, c) = \emptyset$ by Theorem 32, we have $\sigma_\delta(\Lambda, c) = \sigma(\Lambda, c)$.

(c) Since the equality $\sigma_{\text{co}}(\Lambda, c) = C_1\sigma(\Lambda, c) \cup C_2\sigma(\Lambda, c) \cup C_3\sigma(\Lambda, c)$ holds from Table 1, we have $\sigma_{\text{co}}(\Lambda, c) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$ by Theorems 29–32. □

The next corollary can be obtained from Proposition 1.

Corollary 34. *The following results hold:*

(a) $\sigma_{ap}(\Lambda^*, \ell_1) = \sigma(\Lambda, c)$,

(b) $\sigma_\delta(\Lambda^*, \ell_1) = \{\alpha : |\alpha - (2-\delta)^{-1}| = (1-\delta)/(2-\delta)\} \cup E$,

(c) $\sigma_p(\Lambda^*, \ell_1) = \{\alpha \in \mathbb{C} : |\alpha - (2-\delta)^{-1}| < (1-\delta)/(2-\delta)\} \cup S$.

Let A be an infinite matrix and let the set c_A denote the convergence domain of that matrix A, a theorem which proves that $c_A = c$ is called a *Mercerian theorem*, after Mercer, who proved a significant theorem of this type [28, page 186].

Now, we may give our final theorem.

Theorem 35. *Suppose that $|\alpha + 1| > |\alpha - 1|$. Then the convergence field of $A = \alpha I + (1-\alpha)\Lambda$ is c.*

Proof. By Theorem 22, $\Lambda - [\alpha/(\alpha-1)]I$ has an inverse in $B(c)$. That is to say that

$$A^{-1} = \frac{1}{1-\alpha}\left(\Lambda - \frac{\alpha}{\alpha - 1} I\right)^{-1} \in B(c). \quad (77)$$

Since A is a triangle and is in $B(c)$, A^{-1} is also conservative which implies that $c_A = c$ [22, page 12]. □

5. Conclusion

Although the matrix Λ is used for obtaining some new sequence spaces by its domain from the classical sequence spaces, it is not considered for determining the spectrum or fine spectrum acting as a linear operator on any of the classical sequence spaces c_0, c, or ℓ_p. Following Altay and Başar [12] and Karakaya and Altun [26], we determine the fine spectrum with respect to Goldberg's classification of the operator defined by the triangle matrix Λ over the sequence spaces c_0 and c which reduces to a new regular triangle matrix depending on choosing the strictly increasing sequence $\lambda = (\lambda_k)$ of positive real numbers tending to infinity. Additionally, we give the approximate point spectrum, the defect spectrum, and the compression spectrum of the matrix operator Λ over the spaces c_0 and c. Since the present paper is devoted to the fine spectrum of the operator defined by the lambda matrix over the sequence spaces c_0 and c with new subdivision of spectrum, this makes it significant. We should note that the main results of the present paper are given as an extended abstract without proof by Yeşilkayagil and Başar [29].

The generalized weighted means $G(u, v) = (g_{nk})$ is defined by

$$g_{nk} = \begin{cases} u_n v_k, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{78}$$

for all $k, n \in \mathbb{N}$, where u_n depends only on n and v_k only on k such that $u_n, v_k \neq 0$. It is immediate that in the case $u_n = 1/\lambda_n$ and $v_k = \lambda_k - \lambda_{k-1}$, the generalized weighted means $G(u, v)$ corresponds to the matrix Λ. Although the Riesz means R^q, the generalized weighted means $G(u, v)$, and the matrix Λ were used for different purposes, their fine spectrum over the classical sequence spaces was not studied. As a beginning, the present work has an advantage.

Finally, we record from now on that our next paper will be devoted to the investigation of the fine spectrum of the matrix operator Λ on the spaces ℓ_p and bv_p in the cases $0 < p < 1$ and $1 \le p < \infty$, where bv_p denotes the space of all sequences whose Δ-transforms are in the space ℓ_p and was studied in the case $0 < p < 1$ by Altay and Başar [30] and in the case $1 \le p \le \infty$ by Başar and Altay [31].

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References

[1] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, NY, USA, 1978.

[2] J. Appell, E. De Pascale, and A. Vignoli, *Nonlinear Spectral Theory*, vol. 10 of *de Gruyter Series in Nonlinear Analysis and Applications*, Walter de Gruyter, Berlin, Germany, 2004.

[3] S. Goldberg, *Unbounded Linear Operators*, Dover Publications Inc., New York, NY, USA, 1985.

[4] R. B. Wenger, "The fine spectra of the Hölder summability operators," *Indian Journal of Pure and Applied Mathematics*, vol. 6, no. 6, pp. 695–712, 1975.

[5] B. E. Rhoades, "The fine spectra for weighted mean operators," *Pacific Journal of Mathematics*, vol. 104, no. 1, pp. 219–230, 1983.

[6] J. B. Reade, "On the spectrum of the Cesàro operator," *The Bulletin of the London Mathematical Society*, vol. 17, no. 3, pp. 263–267, 1985.

[7] A. M. Akhmedov and F. Başar, "On the fine spectrum of the Cesàro operator in c_0," *Mathematical Journal of Ibaraki University*, vol. 36, pp. 25–32, 2004.

[8] J. T. Okutoyi, "On the spectrum of C_1 as an operator on bv," *Communications, Faculty of Science, University of Ankara. Series A1*, vol. 41, no. 1-2, pp. 197–207, 1992.

[9] M. Yıldırım, "On the spectrum and fine spectrum of the compact Rhaly operators," *Indian Journal of Pure and Applied Mathematics*, vol. 27, no. 8, pp. 779–784, 1996.

[10] C. Coşkun, "The spectra and fine spectra for p-Cesàro operators," *Turkish Journal of Mathematics*, vol. 21, no. 2, pp. 207–212, 1997.

[11] B. de Malafosse, "Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ," *Hokkaido Mathematical Journal*, vol. 31, no. 2, pp. 283–299, 2002.

[12] B. Altay and F. Başar, "On the fine spectrum of the difference operator Δ on c_0 and c," *Information Sciences*, vol. 168, no. 1–4, pp. 217–224, 2004.

[13] H. Bilgiç and H. Furkan, "On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces l_1 and bv," *Mathematical and Computer Modelling*, vol. 45, no. 7-8, pp. 883–891, 2007.

[14] H. Furkan, H. Bilgiç, and F. Başar, "On the fine spectrum of the operator $B(r, s, t)$ over the sequence spaces ℓ_p and bv_p, $(1 < p < \infty)$," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 2141–2152, 2010.

[15] A. M. Akhmedov and S. R. El-Shabrawy, "On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 2994–3002, 2011.

[16] P. D. Srivastava and S. Kumar, "Fine spectrum of the generalized difference operator Δ_v on sequence space l_1," *Thai Journal of Mathematics*, vol. 8, no. 2, pp. 221–233, 2010.

[17] B. L. Panigrahi and P. D. Srivastava, "Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0," *Thai Journal of Mathematics*, vol. 9, no. 1, pp. 57–74, 2011.

[18] P. D. Srivastava and S. Kumar, "Fine spectrum of the generalized difference operator Δ_{uv} on sequence space l_1," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6407–6414, 2012.

[19] A. Karaisa and F. Başar, "Fine spectra of upper triangular triple-band matrix over the sequence space ℓ_p, $(0 < p < \infty)$," *Abstract and Applied Analysis*. In press.

[20] M. Mursaleen and A. K. Noman, "On the spaces of λ-convergent and bounded sequences," *Thai Journal of Mathematics*, vol. 8, no. 2, pp. 311–329, 2010.

[21] F. Móricz, "On Λ-strong convergence of numerical sequences and Fourier series," *Acta Mathematica Hungarica*, vol. 54, no. 3-4, pp. 319–327, 1989.

[22] A. Wilansky, *Summability through Functional Analysis*, vol. 85 of *North-Holland Mathematics Studies*, North-Holland Publishing, Amsterdam, The Netherlands, 1984.

[23] M. González, "The fine spectrum of the Cesàro operator in $\ell_p\,(1 < p < \infty)$," *Archiv der Mathematik*, vol. 44, no. 4, pp. 355–358, 1985.

[24] B. Altay and M. Karakuş, "On the spectrum and the fine spectrum of the Zweier matrix as an operator on some sequence spaces," *Thai Journal of Mathematics*, vol. 3, no. 2, pp. 153–162, 2005.

[25] M. Altun, "On the fine spectra of triangular Toeplitz operators," *Applied Mathematics and Computation*, vol. 217, no. 20, pp. 8044–8051, 2011.

[26] V. Karakaya and M. Altun, "Fine spectra of upper triangular double-band matrices," *Journal of Computational and Applied Mathematics*, vol. 234, no. 5, pp. 1387–1394, 2010.

[27] F. P. Cass and B. E. Rhoades, "Mercerian theorems via spectral theory," *Pacific Journal of Mathematics*, vol. 73, no. 1, pp. 63–71, 1977.

[28] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London, UK, 1970.

[29] M. Yeşilkayagil and F. Başar, "On the fine spectrum of the operator defined by a lambda matrix over the sequence space c_0 and c," in *Proceedings of the 1st International Conference on Analysis and Applied Mathematics (ICAAM '12)*, vol. 1470 of *AIP Conference Proceedings*, pp. 199–202, October 2012.

[30] B. Altay and F. Başar, "The fine spectrum and the matrix domain of the difference operator Δ on the sequence space ℓ_p, $(0 < p < 1)$," *Communications in Mathematical Analysis*, vol. 2, no. 2, pp. 1–11, 2007.

[31] F. Başar and B. Altay, "On the space of sequences of p-bounded variation and related matrix mappings," *Ukrainian Mathematical Journal*, vol. 55, no. 1, pp. 136–147, 2003.