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Research Article

Complexity of Products of Some Complete and Complete Bipartite Graphs

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The number of spanning trees in graphs (networks) is an important invariant; it is also an important measure of reliability of a network. In this paper, we derive simple formulas of the complexity, number of spanning trees, of products of some complete and complete bipartite graphs such as cartesian product, normal product, composition product, tensor product, and symmetric product, using linear algebra and matrix analysis techniques.

1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff [1], can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \ldots, v_n\}$; then the Kirchhoff matrix H defined as $n \times n$, characteristic matrix, $H = D - A$, where D is the diagonal matrix whose elements are the degrees of the vertices of G. While A is the adjacency matrix of G, $H = [a_{ij}]$ is defined as follows:

(i) $a_{ij} = -1 v_i$ and v_j are adjacent and $i \neq j$,

(ii) a_{ij} equals the degree of vertex v_i if $i = j$,

(iii) $a_{ij} = 0$ otherwise.

All of the cofactors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_1 \geq \cdots \geq \mu_p$ denote the eigenvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [2] have shown that $\tau(G) = (1/p) \prod_{k=1}^{p-1} \mu_k$. The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = (1/p) \prod_{k=1}^{p-1} (d - \lambda_k)$, where $\lambda_0 = d, \lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first results is due to Cayley [3] who showed that the complete graph on n vertices, K_n has n^{n-2} spanning trees, $n \geq 2$. Another result is that $\tau(K_{p,q}) = p^{q-1} q^{p-1}$, $p, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in, for example, [4, 5]. Another result is due to Sedláček [6] who derived a formula for the wheel on $n+1$ vertices, W_{n+1}; he showed that $\tau(W_{n+1}) = ((3+\sqrt{5})/2)^n + ((3-\sqrt{5})/2)^n - 2$, for $n \geq 3$. Sedlacek [7] also later derived a formula for the number of spanning trees in a Mobius ladder, M_n, $\tau(M_n) = (n/2)[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2]$ for $n \geq 2$. Another class of graphs by Boesch et al., for which an explicit formula has been derived, is based on a prism [8, 9].

Now, we can introduce the following lemmas.

Lemma 1 (see [10]). *Consider* $\tau(G) = (1/n^2) \det(nI - \overline{D} + \overline{A})$ *where* $\overline{A}$ *and* $\overline{D}$ *are the adjacency and degree matrices of* $\overline{G}$ *and the complement of G, respectively, and I is the* $n \times n$ *unit matrix.*

Lemma 2. *Let $E_n(x)$ be $n \times n$ matrix, $x \geq 2$ such that*

$$E_n(x) = \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix}. \quad (1)$$

Then,

$$\det(E_n) = (x + n - 1)(x - 1)^{n-1}. \quad (2)$$

Proof. From the definition of the circulant determinants, we have

$$\begin{aligned} \det(E_n(x)) &= \det \begin{pmatrix} x & 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & x \end{pmatrix} \\ &= \prod_{j=1}^{n} \left(x + \omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}\right) \\ &= (x + 1 + 1 + \cdots + 1) \\ &\quad \times \prod_{j=1,\omega_j \neq 1}^{n} \left(x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + \cdots + \omega_j^{n-1}}_{=-1} \right) \\ &= (x + n - 1) \times (x - 1)^{n-1}. \end{aligned} \quad (3)$$

□

We can generalize the previous lemma as follows.

Lemma 3. *Let $A, B \in F^{n\times n}$ and $F \in F^{kn\times kn}$ such that*

$$F = \begin{pmatrix} A & B & \cdots & \cdots & \cdots & B \\ B & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & B \\ B & \cdots & \cdots & \cdots & B & A \end{pmatrix}. \quad (4)$$

Then,

$$\det F = [\det(A - B)]^{k-1} \det[A + (k-1)B]. \quad (5)$$

Lemma 4 (see [11]). *Let $A \in F^{n\times n}$, let $B \in F^{n\times m}$, let $C \in F^{m\times n}$, and let $D \in F^{m\times m}$; assume that A, D are nonsingular matrices. Then*

$$\begin{aligned} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= (-1)^{nm} \det\left(A - BD^{-1}C\right) \det D \\ &= (-1)^{nm} \det A \det\left(D - CA^{-1}B\right). \end{aligned} \quad (6)$$

Formulas in Lemmas 2, 3, and 4 give some sort of symmetry in some matrices which facilitates our calculation of determinants.

2. Number of Spanning Trees of Cartesian Product of Graphs

The Cartesian product, $G_1 \times G_2$, is the simple graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_2)]$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$ [12].

Theorem 5. *For $n, m \geq 1$, we have*

$$\begin{aligned} \tau(K_2 \times K_{m,n}) &= m^{n-1} n^{m-1} (m+2)^{n-1} \\ &\quad \times (n+2)^{m-1} (n + m + 2). \end{aligned} \quad (7)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} &\tau(K_2 \times K_{m,n}) \\ &= \frac{1}{(2(m+n))^2} \det\left(2(m+n) I - \overline{D} + \overline{A}\right) \\ &= \frac{1}{4(m+n)^2} \end{aligned}$$

$$
\times \det \begin{pmatrix}
n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & n+2 & 0 & \cdots & \cdots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & m+2 & 1 & \cdots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+2 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & n+2 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & 0 & m+2 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+2
\end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix}
n+2 & 2 & \cdots & 2 & 1 & \cdots & \cdots & 1 \\
2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\
2 & \cdots & 2 & n+2 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & m+2 & 2 & \cdots & 2 \\
\vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\
1 & \cdots & \cdots & 1 & 2 & \cdots & 2 & m+2
\end{pmatrix}
$$

$$
\times \det \begin{pmatrix}
n+2 & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & n+2 & -1 & \cdots & \cdots & -1 \\
-1 & \cdots & \cdots & -1 & m+2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
-1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+2
\end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \times \det A \det \left(C - B^T A^{-1} B \right) \times \det D \det \left(F - E^T D^{-1} E \right)
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix}
n+2 & 2 & \cdots & 2 \\
2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 \\
2 & \cdots & 2 & n+2
\end{pmatrix}_{m \times m}
$$

$$\times \det\begin{pmatrix} \frac{n(m+2)+m(2m+3)}{n+2m} & \frac{2n+3m}{n+2m} & \cdots & \frac{2n+3m}{n+2m} \\ \frac{2n+3m}{n+2m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{2n+3m}{n+2m} \\ \frac{2n+3m}{n+2m} & \cdots & \frac{2n+3m}{n+2m} & \frac{n(m+2)+m(2m+3)}{n+2m} \end{pmatrix}_{n\times n}$$

$$\times \det\begin{pmatrix} n+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+2 \end{pmatrix}_{m\times m}$$

$$\times \det\begin{pmatrix} \frac{n(m+2)+(m+4)}{n+2} & \frac{-m}{n+2} & \cdots & \frac{-m}{n+2} \\ \frac{-m}{n+2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+2} \\ \frac{-m}{n+2} & \cdots & \frac{-m}{n+2} & \frac{n(m+2)+(m+4)}{n+2} \end{pmatrix}_{n\times n}$$

$$= \frac{1}{4(m+n)^2} \times 2^m \det\begin{pmatrix} \frac{n+2}{2} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+2}{2} \end{pmatrix}_{m\times m}$$

$$\times \left(\frac{2n+3m}{n+2m}\right)^n \det\begin{pmatrix} \frac{n(m+2)+m(2m+3)}{2n+3m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+2)+m(2m+3)}{2n+3m} \end{pmatrix}_{n\times n}$$

$$\times \det\begin{pmatrix} n+2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+2 \end{pmatrix}_{m\times m}$$

$$\times \left(\frac{-m}{n+2}\right)^n \det\begin{pmatrix} \frac{n(m+2)+(m+4)}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(m+2)+(m+4)}{-m} \end{pmatrix}_{n\times n}$$

$$= \frac{1}{4(m+n)^2} \times 2^m \times \left(\frac{n+2}{2}+m-1\right)\left(\frac{n+2}{2}-1\right)^{m-1}$$

$$\times \left(\frac{2n+3m}{n+2m}\right)^n \times \left(\frac{n(m+2)+m(2m+3)}{2n+3m}+n-1\right)$$

$$\times \left(\frac{n(m+2) + m(2m+3)}{2n+3m} - 1 \right)^{n-1} \times (n+2)^m \times \left(-\frac{m}{n+2} \right)^n$$
$$\times \left(-\frac{n(m+2) + (m+4)}{m} + n - 1 \right) \times \left(-\frac{n(m+2) + (m+4)}{m} - 1 \right)^{n-1}. \tag{8}$$

Thus,

$$\tau(K_2 \times K_{m,n}) = m^{n-1} n^{m-1} (m+2)^{m-1} (n+2)^{m-1} \times (n+m+2). \tag{9}$$

In particular,

$$\tau(K_2 \times K_{n,n}) = 2n^{2n-2}(n+1)(n+2)^{2n-2}; \quad n \geq 1. \tag{10}$$

□

Theorem 6. *For $m, n \geq 1$, we have*

$$\tau(K_3 \times K_{m,n}) = 3n^{m-1} m^{n-1} (m+3)^{2n-2} (n+3)^{2m-2} \times (n+m+3)^2. \tag{11}$$

Proof. Applying Lemma 1, we have

$$\tau(K_3 \times K_{m,n}) = \frac{1}{9(m+n)^2} \det\left(3(m+n)I - \overline{D} + \overline{A}\right) = \frac{1}{9(m+n)^2}$$
$$\times \det \left(\begin{array}{cccccccccccc}
n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & n+3 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & m+3 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+3 & 1 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & n+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & n+3 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1
\end{array} \right.$$

$$\left.\begin{matrix}
1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 \\
m+3 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & m+3 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & \cdots & 1 & n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & n+3 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & m+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+3
\end{matrix}\right)$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)]$$

$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix}
n+3 & 0 & \cdots & 0 & -1 & \cdots & \cdots & -1 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & n+3 & -1 & \cdots & \cdots & -1 \\
-1 & \cdots & \cdots & -1 & m+3 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
-1 & \cdots & \cdots & -1 & 0 & \cdots & 0 & m+3
\end{pmatrix} \right)^2$$

$$\times \det \begin{pmatrix}
n+3 & 3 & \cdots & 3 & 2 & \cdots & \cdots & 2 \\
3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 3 & \vdots & \ddots & \ddots & \vdots \\
3 & \cdots & 3 & n+3 & 2 & \cdots & \cdots & 2 \\
2 & \cdots & \cdots & 2 & m+3 & 3 & \cdots & 3 \\
\vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 3 \\
2 & \cdots & \cdots & 2 & 3 & \cdots & 3 & m+3
\end{pmatrix}$$

$$= \frac{1}{9(m+n)^2}\left(\det\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}\right)^2 \times \det\begin{pmatrix} D & E \\ E^T & F \end{pmatrix}$$
$$= \frac{1}{9(m+n)^2} \times (\det A)^2\left(\det\left(C - B^T A^{-1} B\right)\right)^2 \times \det D \det\left(F - E^T D^{-1} E\right). \tag{12}$$

Thus,

$$\tau\left(K_3 \times K_{m,n}\right) = \frac{1}{9(m+n)^2}\left(\det\begin{pmatrix} n+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n+3 \end{pmatrix}_{m\times m}\right)^2$$
$$\times \left(\det\begin{pmatrix} \frac{nm+3n+2m+9}{n+3} & \frac{-m}{n+3} & \cdots & \frac{-m}{n+3} \\ \frac{-m}{n+3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{n+3} \\ \frac{-m}{n+3} & \cdots & \frac{-m}{n+3} & \frac{nm+3n+2m+9}{n+3} \end{pmatrix}_{n\times n}\right)^2$$
$$\times \det\begin{pmatrix} n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & n+3 \end{pmatrix}_{m\times m}$$
$$\times \det\begin{pmatrix} \frac{nm+3n+3m^2+5m}{n+3m} & \frac{3n+5m}{n+3m} & \cdots & \frac{3n+5m}{n+3m} \\ \frac{3n+5m}{n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{3n+5m}{n+3m} \\ \frac{3n+5m}{n+3m} & \cdots & \frac{3n+5m}{n+3m} & \frac{nm+3n+3m^2+5m}{n+3m} \end{pmatrix}_{n\times n}$$
$$= \frac{1}{9(m+n)^2}(m+3)^{2m} \times \left(\frac{-m}{n+3}\right)^{2n}$$
$$\times \left(\det\begin{pmatrix} \frac{mn+3n+2m+9}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{mn+3n+2m+9}{-m} \end{pmatrix}\right)^2$$
$$\times 3^m \times \det\begin{pmatrix} \frac{n+3}{3} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n+3}{3} \end{pmatrix}_{m\times m} \times \left(\frac{3n+5m}{n+3m}\right)^n$$

$$\times \det \begin{pmatrix} \dfrac{mn+3n+2m^2+5m}{3n+5m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \dfrac{mn+3n+2m^2+5m}{3n+5m} \end{pmatrix}. \quad (13)$$

Using Lemma 2, we have

$$\begin{aligned} &\tau(K_3 \times K_{m,n}) \\ &= \frac{1}{9(m+n)^2} \times (n+3)^{2m} \times \left(\frac{-m}{n+3}\right)^{2n} \\ &\quad \times \left[-\frac{nm+3n+2m+9}{m} + n - 1\right]^2 \\ &\quad \times \left[-\frac{nm+3n+2m+9}{m} - 1\right]^{2n-2} \times 3^m \left(\frac{n+3}{3} + m - 1\right) \\ &\quad \times \left(\frac{n+3}{3} - 1\right)^{m-1} \times \left(\frac{3n+5m}{n+3m}\right)^n \\ &\quad \times \left[\frac{nm+3n+3m^2+5m}{3n+5m} + n - 1\right] \\ &\quad \times \left[\frac{nm+3n+3m^2+5m}{3n+5m} - 1\right]^{n-1} \\ &= \frac{1}{9(m+n)^2}(n+3)^{2m} \\ &\quad \times \left[\frac{1}{(n+3)^{2n}} \times (3n+3m+9)^2 \right. \\ &\quad \left. \times (nm+3n+3m+9)^{2n-2}\right] \\ &\quad \times \left[(n+3m) \times n^{m-1} \times \frac{1}{(n+3m)^n} \right. \\ &\quad \times \left(6nm+3n^2+3m^2\right) \\ &\quad \left. \times \left(nm+3m^2\right)^{n-1}\right] \\ &= 3n^{m-1}m^{n-1}(m+3)^{2n-2}(n+3)^{2m-2}(n+m+3)^2. \end{aligned} \quad (14)$$

In particular,

$$\tau(K_3 \times K_{n,n}) = 3n^{2n-2}(2n+3)^2(n+3)^{4n-4}; \quad n \geq 1. \quad (15)$$

□

3. Number of Spanning Trees of Normal Product of Graphs

The normal product, or the strong product, $G_1 \circ G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2, u_1 is adjacent to v_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2 [13].

Theorem 7. *For $n, m \geq 1$, we have*

$$\begin{aligned} \tau(K_2 \circ K_{m,n}) = 2^{2m+2n-2} \times n^{m-1} \\ \times m^{n-1} \times (n+1)^m \times (m+1)^n. \end{aligned} \quad (16)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned} &\tau(K_2 \circ K_{m,n}) \\ &= \frac{1}{4(m+n)^2} \det\left(2(m+n)I - \overline{D} + \overline{A}\right) \\ &= \frac{1}{4(m+n)^2} \end{aligned}$$

$$
\times \det \begin{pmatrix}
2n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 2n+2 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 2m+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2m+2 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 2n+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+2 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 2m+2 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2m+2
\end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix}
2n+2 & 2 & \cdots & 2 & 0 & \cdots & \cdots & 0 \\
2 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \vdots \\
2 & \cdots & 2 & 2n+2 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 2m+2 & 2 & \cdots & 2 \\
\vdots & \ddots & \ddots & \vdots & 2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 2 \\
0 & \cdots & \cdots & 0 & 2 & \cdots & 2 & 2m+2
\end{pmatrix}
$$

$$
\times \det \begin{pmatrix}
2n+2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 2n+2 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 2m+2 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 2m+2
\end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix}
2n+2 & 2 & \cdots & 2 \\
2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 \\
2 & \cdots & 2 & 2n+2
\end{pmatrix}_{m\times m}
\times \det \begin{pmatrix}
2m+2 & 2 & \cdots & 2 \\
2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 \\
2 & \cdots & 2 & 2m+2
\end{pmatrix}_{n\times n}
$$

$$
\times \det \begin{pmatrix}
2n+2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 2n+2
\end{pmatrix}_{m\times m}
\times \det \begin{pmatrix}
2m+2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 2m+2
\end{pmatrix}_{n\times n}
$$

$$= \frac{1}{4(m+n)^2} \times 2^m (n+m) n^{m-1} \times 2^n (n+m) m^{n-1} \times 2^m (n+1)^m \times 2^n (m+1)^n$$
$$= 2^{2m+2n-2} \times n^{m-1} \times m^{n-1} \times (n+1)^m \times (m+1)^n. \tag{17}$$

In particular,

$$\tau\left(K_2 \circ K_{n,n}\right) = 2^{4n-2} \times n^{2n-2} \times (n+1)^{2n}; \quad n \geq 1. \tag{18}$$

□

Theorem 8. *For $m, n \geq 1$, we have*

$$\tau\left(K_3 \circ K_{m,n}\right) = 3^{3m+3n-2} \times n^{m-1} \times m^{n-1} \times (n+1)^{2m} \times (m+1)^{2n}. \tag{19}$$

Proof. Applying Lemma 1, we have

$$\tau\left(K_3 \circ K_{m,n}\right) = \frac{1}{9(m+n)^2} \det\left(3(m+n) I - \overline{D} + \overline{A}\right) = \frac{1}{9(m+n)^2}$$
$$\times \det \left(\begin{array}{cccccccccccc}
3n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 3n+3 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 3m+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+3 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 3n+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+3 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0
\end{array} \right.$$

$$\left.\begin{matrix}
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
3m+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 3m+3 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 3n+3 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+3 & 0 & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 3m+3 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+3
\end{matrix}\right)$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)]$$

$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix}
3n+3 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 3n+3 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3m+3 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 3m+3
\end{pmatrix} \right)^2$$

$$\times \det \begin{pmatrix}
3n+3 & 3 & \cdots & 3 & 0 & \cdots & \cdots & 0 \\
3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 3 & \vdots & \ddots & \ddots & \vdots \\
3 & \cdots & 3 & 3n+3 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3m+3 & 3 & \cdots & 3 \\
\vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 3 \\
0 & \cdots & \cdots & 0 & 3 & \cdots & 3 & 3m+3
\end{pmatrix}$$

$$= \frac{1}{9(m+n)^2}\left(\det\begin{pmatrix} 3n+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 3n+3 \end{pmatrix}_{m\times m}\right)^2$$

$$\times\left(\det\begin{pmatrix} 3m+3 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 3m+3 \end{pmatrix}_{n\times n}\right)^2$$

$$\times \det\begin{pmatrix} 3n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 3n+3 \end{pmatrix}_{m\times m} \times \det\begin{pmatrix} 3m+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 3m+3 \end{pmatrix}_{n\times n}. \quad (20)$$

Using Lemma 2, we have

$$\begin{aligned}\tau\left(K_3 \circ K_{m,n}\right) &= \frac{1}{9(m+n)^2} \times (3n+3)^{2m} \times (3m+3)^{2n} \\ &\quad \times \left(3^m \times (n+m) \times n^{m-1}\right) \\ &\quad \times \left(3^n \times (n+m) \times m^{n-1}\right) \\ &= 3^{3m+3n-2} \times n^{m-1} \times m^{n-1} \\ &\quad \times (n+1)^{2m} \times (m+1)^{2n}.\end{aligned} \quad (21)$$

In paricular,

$$\tau\left(K_3 \circ K_{n,n}\right) = 3^{6n-2} \times n^{2n-2} \times (n+1)^{4n}; \quad n \geq 1. \quad (22)$$

□

4. Number of Spanning Trees of Composition Product of Graphs

The composition, or lexicographic product, $G_1[G_2]$, is the simple graph with $V_1 \times V_2$ as the vertex set in which the vertices (u_1, u_2) and (v_1, v_2) are adjacent if either u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 [13].

Theorem 9. *For $n, m \geq 1$, we have*

$$\begin{aligned}\tau\left(K_2\left[K_{m,n}\right]\right) &= 4(m+n)^2 \\ &\quad \times (m+2n)^{2m-2}(n+2m)^{2n-2}.\end{aligned} \quad (23)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}&\tau\left(K_2\left[K_{m,n}\right]\right) \\ &= \frac{1}{4(m+n)^2}\det\left(2(m+n)I - \overline{D} + \overline{A}\right) \\ &= \frac{1}{4(m+n)^2}\end{aligned}$$

$$
\times \det \begin{pmatrix}
m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & m+2n+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & n+2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+2m+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+2n+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & n+2m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+2m+1
\end{pmatrix}
$$

$$
= \frac{1}{4(m+n)^2} \left(\det \begin{pmatrix}
m+2n+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & m+2n+1
\end{pmatrix}_{m\times m} \right)^2
$$

$$
\times \left(\det \begin{pmatrix}
n+2m+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & n+2m+1
\end{pmatrix}_{n\times n} \right)^2
$$

$$
\begin{aligned}
&= \frac{1}{4(m+n)^2}(2n+2m)^2(m+2n)^{2m-2} \times (2n+2m)^2(n+2m)^{2n-2} \\
&= 4(m+n)^2(m+2n)^{2m-2}(n+2m)^{2n-2}.
\end{aligned} \tag{24}
$$

In particular,

$$\tau\left(K_2\left[K_{n,n}\right]\right) = 16 \times 3^{4n-4} \times n^{4n-4}; \quad n \ge 1. \tag{25}$$

□

Theorem 10. *For $m, n \ge 1$, we have*

$$\tau\left(K_3\left[K_{m,n}\right]\right) = 3^4(m+n)^4(3m+2n)^{3n-3}(3n+2m)^{3m-3}. \tag{26}$$

Proof. Applying Lemma 1, we have

$$
\begin{aligned}
&\tau\left(K_3\left[K_{m,n}\right]\right) \\
&= \frac{1}{9(m+n)^2}\det\left(3(m+n)I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{9(m+n)^2} \\
&\times \det \left(\begin{array}{cccccccccccc}
3n+2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 3n+2m+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+2n+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3n+2m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+2m+1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{array}\right.
\end{aligned}
$$

$$
\left.\begin{matrix}
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
3m+2n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 3m+2n+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3n+2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3n+2m+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3m+2n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+2n+1
\end{matrix}\right)
$$

$$
= \frac{1}{9(m+n)^2}\left(\det\begin{pmatrix}
3n+2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 3n+2m+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3m+2n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 3m+2n+1
\end{pmatrix}\right)^3
$$

$$
= \frac{1}{9(m+n)^2}\left(\det\begin{pmatrix}
3n+2m+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 3n+2m+1
\end{pmatrix}_{m\times m}\right)^3
$$

$$
\times\left(\det\begin{pmatrix}
3m+2n+1 & 1 & \cdots & 1 \\
1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 3m+2n+1
\end{pmatrix}_{n\times n}\right)^3 .
\tag{27}
$$

Using Lemma 2, we have

$$\tau\left(K_3\left[K_{m,n}\right]\right) = 3^4(m+n)^4(3m+2n)^{\mathbf{3n-3}}(3n+2m)^{\mathbf{3m-3}}. \quad (28)$$

In particular,

$$\tau\left(K_3\left[K_{n,n}\right]\right) = 6^4 \times 5^{6n-6} \times n^{6n-2}; \quad n \geq 1. \quad (29)$$

□

5. Complexity of Tensor Product of Graphs

The tensor product, or Kronecker product, $G_1 \otimes G_2$, is the simple graph with $V(G_1 \otimes G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ if and only if u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [13].

Lemma 11. *For $m, n \geq 1$, we have*

$$\tau\left(K_2 \otimes K_{m,n}\right) = 0. \quad (30)$$

Theorem 12. *For $m, n \geq 1$, we have*

$$\tau\left(K_3 \otimes K_{m,n}\right) = 3 \times 2^{3m+3n-5} \times n^{3m-1} \times m^{3n-1}. \quad (31)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}
&\tau\left(K_3 \otimes K_{m,n}\right) \\
&= \frac{1}{9(m+n)^2} \det\left(3(m+n)I - \overline{D} + \overline{A}\right) \\
&= \frac{1}{9(m+n)^2} \\
&\times \det \left(\begin{array}{cccccccccccc}
2n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 2n+1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 2m+1 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 2n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0
\end{array}\right.
\end{aligned}$$

$$\left.\begin{matrix}
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
2m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 2m+1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & 2n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 2m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 2m+1
\end{matrix}\right)$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det (A-B)]^2 [\det (A+2B)]$$

$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix}
2n & 0 & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 2n & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 2m & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & 0 & 2m
\end{pmatrix} \right)^2$$

$$\times \det \begin{pmatrix}
2n+3 & 3 & \cdots & 3 & 1 & \cdots & \cdots & 1 \\
3 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 3 & \vdots & \ddots & \ddots & \vdots \\
3 & \cdots & 3 & 2n+3 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 2m+3 & 3 & \cdots & 3 \\
\vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 3 \\
1 & \cdots & \cdots & 1 & 3 & \cdots & 3 & 2m+3
\end{pmatrix}$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} = \frac{1}{9(m+n)^2} \times (\det A)^2 \left(\det \left(C - B^T A^{-1} B\right)\right)^2 \times \det D \det \left(F - E^T D^{-1} E\right). \tag{32}$$

Thus,

$$\tau\left(K_3 \otimes K_{m,n}\right) = \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 2n & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2n \end{pmatrix}_{m\times m} \right)^2$$
$$\times \left(\det \begin{pmatrix} \frac{m(4n-1)}{2n} & \frac{-m}{2n} & \cdots & \frac{-m}{2n} \\ \frac{-m}{2n} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{-m}{2n} \\ \frac{-m}{2n} & \cdots & \frac{-m}{2n} & \frac{m(4n-1)}{2n} \end{pmatrix}_{n\times n} \right)^2$$
$$\times \det \begin{pmatrix} 2n+3 & 3 & \cdots & 3 \\ 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 3 \\ 3 & \cdots & 3 & 2n+3 \end{pmatrix}_{m\times m}$$
$$\times \det \begin{pmatrix} \frac{n(4m+6)+6m^2+8m}{2n+3m} & \frac{6n+8m}{2n+3m} & \cdots & \frac{6n+8m}{2n+3m} \\ \frac{6n+8m}{2n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{6n+8m}{2n+3m} \\ \frac{6n+8m}{2n+3m} & \cdots & \frac{6n+8m}{2n+3m} & \frac{n(4m+6)+6m^2+8m}{2n+3m} \end{pmatrix}_{n\times n}$$
$$= \frac{1}{9(m+n)^2} (2n)^{2m} \times \left(\frac{-m}{2n}\right)^{2n} \times \left(\det \begin{pmatrix} \frac{m(4n-1)}{-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{m(4n-1)}{-m} \end{pmatrix}_{n\times n} \right)^2$$
$$\times 3^m \times \det \begin{pmatrix} \frac{2n+3}{3} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n+3}{3} \end{pmatrix} \times \left(\frac{6n+8m}{2n+3m}\right)^n$$

$$\times \det \begin{pmatrix} \frac{n(4m+6)+6m^2+8m}{6n+8m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n(4m+6)+6m^2+8m}{6n+8m} \end{pmatrix}_{n\times n}. \quad (33)$$

Using Lemma 2, we have

$$\begin{aligned}\tau(K_3 \otimes K_{m,n}) &= \frac{1}{9(m+n)^2}(2n)^{2m} \\ &\times \left[\left(\frac{m}{2n}\right)^{2n} \times (-3n)^2 \times (-4n)^{2n-2}\right] \\ &\times \left[3^m\left(\frac{2n+3m}{3}\right) \times \left(\frac{2n}{3}\right)^{m-1}\right] \\ &\times \left(\frac{6n+8m}{2n+3m}\right)^n \\ &\times \left[\left(\frac{4nm+6n+6m^2+8m}{6n+8m} + n - 1\right)\right. \\ &\quad \left.\times\left(\frac{4nm+6n+6m^2+8m}{6n+8m} - 1\right)^{n-1}\right] \\ &= 3 \times (2n)^{2m-2n} \times m^{3n-1} \\ &\quad \times n^{2n+m-1} \times 2^{5n+m-5} \\ &= 3 \times 2^{3m+3n-5} \times n^{3m-1} \times m^{3n-1}.\end{aligned} \quad (34)$$

In particular,

$$\tau(K_3 \otimes K_{n,n}) = 3 \times 2^{6n-5} \times n^{6n-2}; \quad n \geq 1. \quad (35)$$

□

6. Number of Spanning Trees of Symmetric Product of Graphs

The symmetric product, $G_1 \oplus G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$, where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \oplus G_2$ if and only if either u_1 is adjacent to v_1 in G_1 and u_2 is not adjacent to v_2 in G_2, or u_1 is not adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2 [13].

Theorem 13. *For $n, m \geq 1$, we have*

$$\tau(K_2 \oplus K_{m,n}) = (m+n)^{2(m+n-1)}. \quad (36)$$

Proof. Applying Lemma 1, we have

$$\begin{aligned}\tau&(K_2 \oplus K_{m,n}) \\ &= \frac{1}{4(m+n)^2}\det\left(2(m+n)I - \overline{D} + \overline{A}\right) \\ &= \frac{1}{4(m+n)^2}\end{aligned}$$

$$
\times \det \left(\begin{array}{cccccccc}
m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & m+n+1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & m+n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+n+1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0
\end{array} \right.
$$

$$
\left. \begin{array}{cccccccc}
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & m+n+1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & 0 & m+n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & m+n+1
\end{array} \right)
$$

$$
= \frac{1}{4(m+n)^2} \det \begin{pmatrix}
m+n+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & m+n+1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & m+n+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & m+n+1
\end{pmatrix}
$$

$$\times \det \begin{pmatrix} m+n+1 & 1 & \cdots & 1 & -1 & \cdots & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & m+n+1 & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & m+n+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & \cdots & -1 & 1 & \cdots & 1 & m+n+1 \end{pmatrix}$$

$$= \frac{1}{4(m+n)^2}(m+n+1+m+n-1)(m+n+1-1)^{m+n-1} \times \det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

$$= \frac{1}{2}(m+n)^{m+n-2} \times \det A \det \left(C - B^T A^{-1} B \right)$$

$$= \frac{1}{2}(m+n)^{m+n-2} \times (2m+n)(m+n)^{m-1}$$

$$\times \det \begin{pmatrix} \frac{n^2+(3m+1)n+2m^2+m}{(n+2m)} & \frac{n+m}{n+2m} & \cdots & \frac{n+m}{n+2m} \\ \frac{n+m}{n+2m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n+m}{n+2m} \\ \frac{n+m}{n+2m} & \cdots & \frac{n+m}{n+2m} & \frac{n^2+(3m+1)n+2m^2+m}{(n+2m)} \end{pmatrix}_{n\times n}$$

$$= \frac{1}{2}(m+n)^{2m+n-3} \times (2m+n) \times \left(\frac{n+m}{n+2m}\right)^n$$

$$\times \det \begin{pmatrix} \frac{n^2+(3m+1)n+2m^2+m}{n+m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{n^2+(3m+1)n+2m^2+m}{n+m} \end{pmatrix}_{n\times n}. \quad (37)$$

Thus,

$$\tau(K_2 \oplus K_{m,n}) = \frac{1}{2}(m+n)^{2m+n-3} \times (2m+n) \times \left(\frac{n+m}{n+2m}\right)^n \times (2n+2m)(n+2m)^{n-1} = (m+n)^{2(m+n-1)}. \quad (38)$$

In particular,

$$\tau(K_2 \oplus K_{n,n}) = (2n)^{2(2n-1)}; \quad n \geq 1. \quad (39)$$

□

Theorem 14. *For $m, n \geq 1$, we have*

$$\tau(K_3 \oplus K_{m,n}) = 3(2m+n)^{3m-3} \times (2n+m)^{3n-3}\left(m^2+n^2+3mn\right)^2. \quad (40)$$

Proof. Applying Lemma 1, we have

$$\tau(K_3 \oplus K_{m,n}) = \frac{1}{9(m+n)^2} \det\left(3(m+n)\ I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{9(m+n)^2}$$

$$
\times \det \left(\begin{array}{cccccccccccccccccccccccc}
2m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & 1 & 2m+n+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
0 & \cdots & \cdots & 0 & 2n+m+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+m+1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 2m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 2m+n+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 2n+m+1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+m+1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 2m+n+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 & \cdots & 1 & 2m+n+1 & 0 & \cdots & \cdots & 0 \\
1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 2n+m+1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\
1 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 2n+m+1
\end{array}\right)
$$

$$= \frac{1}{9(m+n)^2} \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = \frac{1}{9(m+n)^2} [\det(A-B)]^2 [\det(A+2B)]$$

$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} 2m+n+1 & 1 & \cdots & 1 & -1 & \cdots & \cdots & -1 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 2m+n+1 & -1 & \cdots & \cdots & -1 \\ -1 & \cdots & \cdots & -1 & 2n+m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & \cdots & \cdots & -1 & 1 & \cdots & 1 & 2n+m+1 \end{pmatrix} \right)^2$$

$$\times \det \begin{pmatrix} 2m+n+1 & 1 & \cdots & 1 & 2 & \cdots & \cdots & 2 \\ 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 2m+n+1 & 2 & \cdots & \cdots & 2 \\ 2 & \cdots & \cdots & 2 & 2n+m+1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots & 3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 1 \\ 2 & \cdots & \cdots & 2 & 1 & \cdots & 1 & 2n+m+1 \end{pmatrix}$$

$$= \frac{1}{9(m+n)^2} \left(\det \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \right)^2 \times \det \begin{pmatrix} D & E \\ E^T & F \end{pmatrix} = \frac{1}{9(m+n)^2} \times (\det A)^2 \left(\det \left(C - B^T A^{-1} B \right) \right)^2$$

$$\times \det D \det \left(F - E^T D^{-1} E \right)$$

$$= \frac{(3m+n)^2 (2m+n)^{2m-2}}{9(m+n)^2}$$

$$\times \left(\det \begin{pmatrix} \frac{2n^2 + n(7m+1) + 3m^2 + 2m}{n+3m} & \frac{n+2m}{n+3m} & \cdots & \frac{n+2m}{n+3m} \\ \frac{n+2m}{n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n+2m}{n+3m} \\ \frac{n+2m}{n+3m} & \cdots & \frac{n+2m}{n+3m} & \frac{2n^2 + n(7m+1) + 3m^2 + 2m}{n+3m} \end{pmatrix}_{n \times n} \right)^2$$

$$\times (3m+n)(2m+n)^{m-1}$$

$$\times \det \begin{pmatrix} \frac{2n^2 + n(7m+1) + 3m^2 - m}{n+3m} & \frac{n-m}{n+3m} & \cdots & \frac{n-m}{n+3m} \\ \frac{n-m}{n+3m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{n-m}{n+3m} \\ \frac{n-m}{n+3m} & \cdots & \frac{n-m}{n+3m} & \frac{2n^2 + n(7m+1) + 3m^2 - m}{n+3m} \end{pmatrix}_{n \times n}$$

$$= \frac{(3m+n)^2(2m+n)^{2m-2}}{9(m+n)^2}$$
$$\times \left(\frac{n+2m}{n+3m}\right)^{2n} \left(\det \begin{pmatrix} \frac{2n^2+n(7m+1)+3m^2+2m}{n+2m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n^2+n(7m+1)+3m^2+2m}{n+2m} \end{pmatrix}_{n\times n} \right)^2$$
$$\times (3m+n)(2m+n)^{m-1}$$
$$\times \left(\frac{n-m}{n+3m}\right)^{n} \det \begin{pmatrix} \frac{2n^2+n(7m+1)+3m^2-m}{n-m} & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & \frac{2n^2+n(7m+1)+3m^2-m}{n-m} \end{pmatrix}_{n\times n} . \quad (41)$$

Using Lemma 2, we have

$$\begin{aligned} \tau(K_3 \oplus K_{m,n}) &= \frac{1}{9(m+n)^2}(3m+n)^2(2m+n)^{2m-2} \\ &\quad \times \left(\frac{n+2m}{n+3m}\right)^{2n} \frac{1}{(n+3m)^{2n}} \\ &\quad \times \left(3n^2+3m^2+9nm\right)^2 \\ &\quad \times \left(2n^2+3m^2+7nm\right)^{2n-2} \\ &\quad \times (3m+n)(2m+n)^{m-1} \\ &\quad \times \left(\frac{n-m}{n+3m}\right)^{n} \times \frac{1}{(n-m)^n} \\ &\quad \times \left(3n^2+3m^2+6nm\right) \\ &\quad \times \left(2n^2+3m^2+7nm\right)^{n-1} \\ &= 3(2m+n)^{3m-3}(2n+m)^{3n-3} \\ &\quad \times \left(m^2+n^2+3mn\right)^2. \end{aligned} \quad (42)$$

In particular,

$$\tau(K_3 \oplus K_{n,n}) = 25 \times 3^{6n-5} \times n^{6n-2}; \quad n \geq 1. \quad (43)$$

□

7. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective but is also an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory, we introduced the above important theorems and lemmas and their proofs.

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