Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 654537, 7 pages
http://dx.doi.org/10.1155/2013/654537

Research Article

A System of Generalized Variational Inclusions Involving a New Monotone Mapping in Banach Spaces

Jinlin Guan and Changsong Hu

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

Correspondence should be addressed to Changsong Hu; huchang1004@yahoo.com.cn

Received 17 April 2013; Accepted 13 July 2013

Academic Editor: Micah Osilike

We introduce a new monotone mapping in Banach spaces, which is an extension of the C_n-monotone mapping studied by Nazemi (2012), and we generalize the variational inclusion involving the C_n-monotone mapping. Based on the new monotone mapping, we propose a new proximal mapping which combines the proximal mapping studied by Nazemi (2012) with the η mapping studied by Lan et al. (2011) and show its Lipschitz continuity. Based on the new proximal mapping, we give an iterative algorithm. Furthermore, we prove the convergence of iterative sequences generated by the algorithm under some appropriate conditions. Our results improve and extend corresponding ones announced by many others.

1. Introduction

Variational inequality theory has emerged as a powerful tool for a wide class of unrelated problems arising in various branches of physical, engineering, pure, and applied sciences in a unified and general framework. As the generalization of variational inequalities, variational inclusions have been widely studied in recent years. One of the most important problems in the theory of variational inclusions is the development of an efficient and implementable iterative algorithm. Therefore, many iterative algorithms and existence results for various variational inclusions have been studied see, for example, [1–3].

Several years ago, Xia and Huang [4] proposed the concept of general H-monotone operators in Banach spaces and studied a class of variational inclusions involving the general H-monotone operator in Banach spaces. In 2010, Luo and Huang [5] introduced a new notion of B-monotone operators in Banach spaces and gave a new proximal mapping related to these operators. Then, they used it to study a new class of variational inclusions in Banach spaces. Very recently, Nazemi [6] introduced the notion of a new class of C_n-monotone mappings which is an extension of B-monotone operators introduced in [5].

Motivated and inspired by the work going on in this direction, in this paper, we propose a new monotone mapping in Banach spaces named C_n-η-monotone mapping which generalizes the C_n-monotone mapping introduced in [6] from the same n-dimensional product space to different n-dimensional product space and reduces the C_n mapping from strictly monotone mapping to monotone mapping. Further, we consider a new proximal mapping which associates a η mapping introduced in [7] and generalizes the proximal mapping introduced in [6]. Furthermore, in the process of proving the convergence of iterative sequences generated by the algorithm, we change the condition of a uniformly smooth Banach space with $\rho_E(t) \leq Ct^2$ to a q-uniformly smooth Banach space, which extends the proof of the convergence of iterative sequences in [6]. The results presented in this paper generalize many known and important results in the recent literature and the references therein.

2. Preliminaries

Let E be a real Banach space, let E^* be the topological dual space of E, and let $\langle u, v\rangle$ be the dual pair between $u \in E^*$ and $v \in E$. Let $CB(E)$ denote the family of all nonempty, closed, and bounded subsets of E. Set $\prod_{i=1}^{n} E_i = E_1 \times E_2 \times \cdots \times E_n$. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$H(A,D) := \max\left\{\sup_{x\in A}\inf_{y\in D}\|x-y\|, \sup_{y\in D}\inf_{x\in A}\|x-y\|\right\}, \quad A, D \in CB(E). \tag{1}$$

We recall the following definitions and results which are needed in the sequel.

Definition 1 (see [7]). A single-valued mapping $\eta : E\times E \to E$ is said to be k-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\|\eta(x,y)\| \le k\|x-y\|, \quad \forall x, y \in E. \tag{2}$$

Definition 2 (see [8]). A Banach space E is called smooth if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^*$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$, defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = t\right\}. \tag{3}$$

Definition 3 (see [8]). The Banach space E is said to be

(i) uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0; \tag{4}$$

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that

$$\rho_E(t) \le ct^q, \quad t \in [0,\infty). \tag{5}$$

It is well known (see, e.g., [9]) that

$$L_q\left(\text{or } l_q\right) \text{ is } \begin{cases} q\text{-uniformly smooth}, & \text{if } 1 < q \le 2, \\ 2\text{-uniformly smooth}, & \text{if } q \ge 2. \end{cases} \tag{6}$$

Note that if E is uniformly smooth, j_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach space, Xu [8] established the following lemma.

Lemma 4 (see [8]). *Let $q > 1$ be a real number and let E be a smooth Banach space and $J_q : E \to 2^{E^*}$ the normalized duality mapping. Then, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for every $x, y \in E$,*

$$\|x+y\|^q \le \|x\|^q + q\left\langle y, j_q(x)\right\rangle + c_q\|y\|^q, \quad \forall j_q(x) \in J_q(x). \tag{7}$$

Definition 5. A single-valued mapping $g : E \to E$ is said to be (γ,μ)-relaxed cocoercive if there exist $j_q(x-y) \in J_q(x-y)$ and $\gamma, \mu > 0$ such that

$$\left\langle j_q(x-y), g(x) - g(y)\right\rangle \ge -\gamma\|g(x) - g(y)\|^q + \mu\|x-y\|^q, \quad \forall x, y \in E. \tag{8}$$

Definition 6. Let $n \ge 3$ and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ be a multivalued mapping, $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, and $\eta : E\times E \to E$ single-valued mappings.

(i) For each $1 \le i \le n$, $M(\ldots, f_i, \ldots)$ is said to be α_i-strongly η-monotone with respect to f_i (in the ith argument) if there exists a constant $\alpha_i > 0$ such that

$$\begin{gathered}\left\langle \omega_i - \omega_i', \eta(x,y)\right\rangle \ge \alpha_i\|x-y\|^2, \\ \omega_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), \\ \omega_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n), \\ \forall x, y \in E, u_1 \in E_1, u_2 \in E_2, \ldots, \\ u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n.\end{gathered} \tag{9}$$

(ii) For each $1 \le i \le n$, $M(\ldots, f_i, \ldots)$ is said to be β_i-relaxed η-monotone with respect to f_i (in the ith argument) if there exists a constant $\beta_i > 0$ such that

$$\begin{gathered}\langle \omega_i - \omega_i', \eta(x,y)\rangle \ge -\beta_i\|x-y\|^2, \\ \omega_i \in M(u_1, \ldots, u_{i-1}, f_i(x), u_{i+1}, \ldots, u_n), \\ \omega_i' \in M(u_1, \ldots, u_{i-1}, f_i(y), u_{i+1}, \ldots, u_n), \\ \forall x, y \in E, u_1 \in E_1, u_2 \in E_2, \ldots, \\ u_{i-1} \in E_{i-1}, u_{i+1} \in E_{i+1}, \ldots, u_n \in E_n.\end{gathered} \tag{10}$$

(iii) By assumption that n is an even number, M is said to be $\alpha_1\beta_2\alpha_3\beta_4\cdots\alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n-1\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly η-monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed η-monotone with respect to f_j (in the jth argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} > \beta_2 + \beta_4 + \cdots + \beta_n. \tag{11}$$

(iv) By assumption that n is an odd number, M is said to be $\alpha_1\beta_2\alpha_3\beta_4\cdots\alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ if, for each $i \in \{1, 3, \ldots, n\}$, $M(\ldots, f_i, \ldots)$ is α_i-strongly η-monotone with respect to f_i (in the ith argument) and for each $j \in \{2, 4, \ldots, n-1\}$, $M(\ldots, f_j, \ldots)$ is β_j-relaxed η-monotone with respect to f_j (in the jth argument) with

$$\alpha_1 + \alpha_3 + \cdots + \alpha_n > \beta_2 + \beta_4 + \cdots + \beta_{n-1}. \tag{12}$$

Definition 7 (see [10]). Let E be a Banach space. A multivalued mapping $A : E \rightrightarrows CB(E_i)$ is said to be H-Lipschitz continuous if there exists a constant $t > 0$ such that

$$H(A(x), A(y)) \leq t\|x - y\|, \quad \forall x, y \in E, \tag{13}$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E_i)$.

Definition 8. Let, for each $i = 1, 2, \ldots, n$, $T_i : E \rightrightarrows CB(E_i)$ be a multivalued mapping. A single-valued mapping $F : \prod_{i=1}^{n} E_i \to E^*$ is said to be λ_{F_i}-Lipschitz continuous in the ith argument with respect to T_i $(i = 1, 2, \ldots, n)$ if there exists a constant $\lambda_{F_i} > 0$ such that

$$\begin{aligned} &\|F(x_1, \ldots, x_{i-1}, u_{i1}, x_{i+1}, \ldots, x_n) \\ &\quad - F(x_1, \ldots, x_{i-1}, u_{i2}, x_{i+1}, \ldots, x_n)\| \\ &\qquad \leq \lambda_{F_i}\|u_{i1} - u_{i2}\|, \\ &\forall y_1, y_2 \in E, x_1 \in E_1, \ldots, x_{i-1} \in E_{i-1}, \\ &x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n, u_{i1} \in T_i(y_1), u_{i2} \in T_i(y_2). \end{aligned} \tag{14}$$

Definition 9. Let E be a Banach space with the dual space $E^*$and $\eta : E \times E \to E$ single-valued mappings; $C_n : E \to E^*$ is said to be η-monotone mapping if

$$\langle C_n(x) - C_n(y), \eta(x, y)\rangle \geq 0, \quad \forall x, y \in E. \tag{15}$$

3. C_n-η-Monotone Mapping

First, we define the notion of C_n-η-monotone mapping.

Definition 10. Let E be a Banach space with the dual space E^*. Let $n \geq 3$ and $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, $C_n : E \to E^*$ be single-valued mappings and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a multivalued mapping.

(i) In case that n is an even number, M is said to be a C_n-η-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\cdots\alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ and $(C_n + \lambda M(f_1, f_2, \ldots, f_n))(E) = E^*$, for every $\lambda > 0$.

(ii) In case that n is an odd number, M is said to be a C_n-η-monotone mapping if M is $\alpha_1\beta_2\alpha_3\beta_4\cdots\beta_{n-1}\alpha_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$ and $(C_n + \lambda M(f_1, f_2, \ldots, f_n))(E) = E^*$, for every $\lambda > 0$.

Remark 11. (i) If $M(f_1, f_2, \ldots, f_n) = M$, $\eta(y, x) = y - x$, and M is monotone, then the C_n-η-monotone mapping reduces to the general H-monotone mapping considered in [4].

(ii) If $M(f_1, f_2, \ldots, f_n) = M(f_1, f_2)$, $\eta(y, x) = y - x$, then the C_n-η-monotone mapping reduces to the B-monotone mapping considered in [5].

(iii) If $M(f_1, f_2, \ldots, f_n) = M$, $\eta(y, x) = y - x$, and M are m-relaxed monotone, then the C_n-η-monotone mapping reduces to the A-monotone mapping considered in [11].

(iv) If M reduces to $\underbrace{E \times E \times \cdots \times E}_{n} \to E^*$, $\eta(y, x) = y - x$, and $f_1, f_2, \ldots, f_n$ reduce to $E \to E$, then the C_n-η-monotone mapping reduces to the C_n-monotone mapping considered in [6].

Example 12. Let $E = l^2$ and then $E^* = l^2$; and assume n is an even number; let $E_i = (E, \|\cdot\|_i)$, $i = 1, 2, \ldots, n$, where $\|\cdot\|_i$ is the equivalent norm on l^2 space, $e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots) \in E_i$, for $\forall x \in E$; let $f_1(x) = \alpha_1 x + e_1 \in E_1$, $f_2(x) = -\beta_2 x + e_2 \in E_2$, $f_3(x) = \alpha_3 x + e_3 \in E_3$, $f_4(x) = -\beta_4 x + e_4 \in E_4, \ldots$, $f_{n-1}(x) = \alpha_{n-1}x + e_{n-1} \in E_{n-1}$, $f_n(x) = -\beta_n x + e_n \in E_n$, where $\alpha_1, \beta_2, \alpha_3, \beta_4, \ldots, \alpha_{n-1}, \beta_n > 0$ are constants such that

$$\alpha_1 - \beta_2 + \alpha_3 - \beta_4 + \cdots + \alpha_{n-1} - \beta_n = \gamma > 0. \tag{16}$$

Let $M(u_1, u_2, \ldots, u_n) = (u_1 - e_1) + (u_2 - e_2) + \cdots + (u_n - e_n)$, where $u_i \in E_i$, $i = 1, 2, \ldots, n$; let $C_n(x) = x + e_{n+1}, x \in E$, $\eta(x, y) = x - y$, $\forall x, y \in E$. Then M is a C_n-η-monotone mapping.

With no loss of generality, we may assume that n is an even number in the next text.

Lemma 13. *Let $\eta : E \times E \to E$, $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, be single-valued mappings; $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a $\alpha_1\beta_2\alpha_3\beta_4\cdots\alpha_{n-1}\beta_n$-symmetric η-monotone with respect to $f_1, f_2, \ldots, f_n$. Then for $\forall x, y \in E$ one has*

$$\begin{gathered} \langle u - v, \eta(y, x)\rangle \geq K_n\|x - y\|^2, \\ u \in M(f_1 y, f_2 y, \ldots, f_n y), \quad v \in M(f_1 x, f_2 x, \ldots, f_n x), \end{gathered} \tag{17}$$

where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.

Proof. Setting $\omega_1 \in M(f_1 x, f_2 y, \ldots, f_n y)$, $\omega_2 \in M(f_1 x, f_2 x, \ldots, f_n y), \ldots$, $\omega_{n-1} \in M(f_1 x, \ldots, f_{n-1} x, f_n y)$. From Definition 10, we have

$$\begin{aligned} \langle u - v, \eta(y, x)\rangle &\geq \langle u - \omega_1, \eta(y, x)\rangle + \langle \omega_1 - \omega_2, \eta(y, x)\rangle \\ &\quad + \cdots + \langle \omega_{n-1} - v, \eta(y, x)\rangle \\ &\geq \alpha_1\|x - y\|^2 - \beta_2\|x - y\|^2 + \cdots \\ &\quad + \alpha_{n-1}\|x - y\|^2 - \beta_n\|x - y\|^2 \\ &= K_n\|x - y\|^2, \end{aligned} \tag{18}$$

where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.

This completes the proof. □

Theorem 14. *Let E be a Banach space with the dual space E^*. Let $n \geq 3$ and $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, $\eta : E \times E \to E$ single-valued mappings, $C_n : E \to E^*$ a η-monotone mapping, and $M : \prod_{i=1}^{n} E_i \rightrightarrows E^*$ a C_n-η-monotone mapping. Then, $(C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}$ is a single-valued mapping.*

Proof. Suppose, on the contrary, that there exists $x_1, x_2 \in E$, $y^* \in E^*$, such that

$$x_1 = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(y^*), \quad x_2 = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(y^*); \tag{19}$$

then

$$\lambda^{-1}(y^* - C_n(x_1)) \in M(f_1, f_2, \ldots, f_n)(x_1), \quad \lambda^{-1}(y^* - C_n(x_2)) \in M(f_1, f_2, \ldots, f_n)(x_2). \tag{20}$$

Now, by using Lemma 13 and since C_n is a η-monotone mapping, we have

$$0 = \lambda \left\langle \lambda^{-1}(y^* - C_n(x_1)) - \lambda^{-1}(y^* - C_n(x_2)), \eta(x_1, x_2) \right\rangle + \langle C_n(x_1) - C_n(x_2), \eta(x_1, x_2) \rangle \geq \lambda K_n \|x_1 - x_2\|^2. \tag{21}$$

Thus, we have $x_1 = x_2$, which implies that $(C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}$ is single valued. This completes the proof. □

By Theorem 14, we can define the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}$ as follows.

Definition 15. Let E be a Banach space with the dual space E^*. Let $n \geq 3$ and $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, be single-valued mappings, $C_n : E \to E^*$ a η-monotone mapping, and $M : \prod_{i=1}^n E_i \rightrightarrows E^*$ a C_n-η-monotone mapping. A proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)} : E^* \to E$ is defined by

$$R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(x^*), \quad \forall x^* \in E^*. \tag{22}$$

Theorem 16. *Let E be a Banach space with the dual space E^*. Let $\eta : E \times E \to E$ be a k-Lipschitz continuous mapping. Let $n \geq 3$ and $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, be single-valued mappings, $C_n : E \to E^*$ a η-monotone mapping, and $M : \prod_{i=1}^n E_i \rightrightarrows E^*$ a C_n-η-monotone mapping. Then, the proximal mapping $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)} : E^* \to E$ is $k/\lambda K_n$-Lipschitz continuous, where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.*

Proof. Let $x^*, y^* \in E^*$ be any given points. It follows from Definition 15 that

$$R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(x^*), \quad R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(y^*) = (C_n + \lambda M(f_1, f_2, \ldots, f_n))^{-1}(y^*). \tag{23}$$

Setting

$$x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*), \quad y = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(y^*) \quad x, y \in E. \tag{24}$$

This implies that

$$\frac{1}{\lambda}(x^* - C_n(x)) \in M(f_1, f_2, \ldots, f_n)(x); \quad \frac{1}{\lambda}(y^* - C_n(y)) \in M(f_1, f_2, \ldots, f_n)(y). \tag{25}$$

By using Lemma 13, we have

$$\left\langle \frac{1}{\lambda}(x^* - C_n(x)) - \frac{1}{\lambda}(y^* - C_n(y)), \eta(x, y) \right\rangle \geq K_n \|x - y\|^2; \tag{26}$$

since η is K-Lipschitz continuous, we have

$$\lambda K_n \|x - y\|^2 \leq \langle x^* - y^*, \eta(x, y) \rangle \leq k \|x^* - y^*\| \|x - y\|; \tag{27}$$

thus

$$\|x - y\| \leq \frac{k}{\lambda K_n} \|x^* - y^*\|; \tag{28}$$

that is,

$$\left\| R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*) - R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*) \right\| \leq \frac{k}{\lambda K_n} \|x^* - y^*\|, \quad \forall x^*, y^* \in E^*, \tag{29}$$

where $K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n)$.

This completes the proof. □

4. System of Variational Inclusions: Iterative Algorithm

Let $n \geq 3$ and $A : E \to E^*$, $p : E \to E$, $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, $F : \prod_{i=1}^n E_i \to E^*$ be single-valued mappings and $T_i : E \rightrightarrows CB(E_i)$, $i = 1, 2, \ldots, n$, $M : \prod_{i=1}^n E_i \rightrightarrows E^*$ be multivalued mappings. We will study the following variational inclusion problem: for any given $a \in E^*$, find $x \in E$, $t_1 \in T_1(x)$, $t_2 \in T_2(x), \ldots, t_n \in T_n(x)$, such that

$$a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n). \tag{30}$$

We remark that problem (30) includes as special cases many kinds of variational inclusion and variational inequality of [4, 5, 10, 12, 13].

Theorem 17. *Let $n \geq 3$ and $A : E \to E^*$, $p : E \to E$, $f_n : E \to E_i$, $i = 1, 2, \ldots, n$, $F : \prod_{i=1}^n E_i \to E^*$ be single-valued mappings and let $T_i : E \rightrightarrows CB(E_i)$, $i = 1, 2, \ldots, n$, be multivalued mappings. Let $C_n : E \to E^*$ be a η-monotone mapping and $M : \prod_{i=1}^n E_i \rightrightarrows E^*$ a C_n-η-monotone mapping with respect to $f_1, f_2, \ldots, f_n$. Then, $(x, t_1, t_2, \ldots, t_n)$ is a solution of problem* (30) *if and only if*

$$x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}[C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)], \tag{31}$$

where $t_1 \in T_1(x)$, $t_2 \in T_2(x), \ldots, t_n \in T_n(x)$, *and* $\lambda > 0$ *is a constant.*

Proof. Let $(x, t_1, t_2, \ldots, t_n)$ be a solution of problem (30); then we have

$$a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n); \tag{32}$$

then

$$\lambda a + \lambda F(t_1, t_2, \ldots, t_n) - \lambda A(x - p(x)) \in \lambda M(f_1, f_2, \ldots, f_n)(x), \quad \lambda > 0; \tag{33}$$

thus

$$C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \in C_n(x) + \lambda M(f_1, f_2, \ldots, f_n)(x). \tag{34}$$

Setting $x^* = C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)$, from the definition of $R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}$, we have

$$x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}(x^*) = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)} \times [C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)]. \tag{35}$$

Conversely, let $x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}[C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n)]$; then

$$C_n(x) - \lambda A(x - p(x)) + \lambda a + \lambda F(t_1, t_2, \ldots, t_n) \in (C_n + \lambda M(f_1, f_2, \ldots, f_n))(x); \tag{36}$$

thus we have

$$a \in A(x - p(x)) + M(f_1(x), f_2(x), \ldots, f_n(x)) - F(t_1, t_2, \ldots, t_n). \tag{37}$$

This completes the proof. □

Based on Theorem 17, we construct the following iterative algorithm for solving problem (30).

Iterative Algorithm 1. For any given $x_0 \in E$, we choose $t_{1,0} \in T_1(x_0)$, $t_{2,0} \in T_2(x_0), \ldots, t_{n,0} \in T_n(x_0)$ and compute $\{x_m\}, \{t_{1,m}\}, \{t_{2,m}\}, \ldots, \{t_{n,m}\}$ by iterative schemes

$$\begin{aligned}
x_{m+1} &= R^{C_n,\lambda,\eta}_{M(f_1,f_2,\ldots,f_n)}[C_n(x_m) - \lambda A(x_m - p(x_m)) + \lambda a + \lambda F(t_{1,m}, t_{2,m}, \ldots, t_{n,m})]; \\
t_{1,m} &\in T_1(x_m), \quad \|t_{1,m+1} - t_{1,m}\| \le \left(1 + \frac{1}{m+1}\right) \times H(T_1(x_{m+1}), T_1(x_m)); \\
t_{2,m} &\in T_2(x_m), \quad \|t_{2,m+1} - t_{2,m}\| \le \left(1 + \frac{1}{m+1}\right) \times H(T_2(x_{m+1}), T_2(x_m)); \\
&\vdots \\
t_{n,m} &\in T_n(x_m), \quad \|t_{n,m+1} - t_{n,m}\| \le \left(1 + \frac{1}{m+1}\right) \times H(T_n(x_{m+1}), T_n(x_m)),
\end{aligned} \tag{38}$$

for all $m = 0, 1, 2, \ldots$.

Now, we give some sufficient conditions which guarantee the convergence of iterative sequences generated by Algorithm 1.

Theorem 18. *Let* E *be a* q*-uniformly smooth Banach space with* $q > 1$ *and* E^* *the dual space of* E. *Let* $\eta : E \times E \to E$ k*-Lipschitz continuous. Let* $n \ge 3$ *and* $f_i : E \to E_i$, $i = 1, 2, \ldots, n$, *be single-valued mappings,* $C_n : E \to E^*$ *a* η*-monotone and* δ*-Lipschitz continuous mapping,* $p : E \to E$ *a* (γ, μ)*-relaxed cocoercive and* λ_p*-Lipschitz continuous mapping, and* $M : \prod_{i=1}^n E_i \rightrightarrows E^*$ *a* C_n*-*η*-monotone mapping. Let* $A : E \to E^*$ *be a* τ*-Lipschitz continuous mapping and, for each* $i = 1, 2, \ldots, n$, *let* $T_i : E \rightrightarrows CB(E_i)$ *be* H*-Lipschitz continuous with constant* λ_{t_i}. *Suppose that* $F : \prod_{i=1}^n E_i \to E^*$ *is* λ_{F_i}*-Lipschitz continuous in the ith argument with respect to* T_i $(i = 1, 2, \ldots, n)$ *and the following condition is satisfied:*

$$0 < \frac{k}{\lambda K_n} \times \left[\delta + \lambda\tau\left(1 + q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q\right)^{1/q} + \lambda\sum_{i=1}^n \lambda_{F_i}\lambda_{t_i}\right] < 1, \tag{39}$$

where

$$K_n = (\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) - (\beta_2 + \beta_4 + \cdots + \beta_n). \tag{40}$$

Then, the iterative sequences $\{x_m\}, \{t_{1,m}\}, \{t_{2,m}\}, \ldots, \{t_{n,m}\}$ generated by Algorithm 1 converge strongly to $x, t_1, t_2, \ldots, t_n$, respectively, and $(x, t_1, t_2, \ldots, t_n)$ is a solution of problem (30).

Proof. By using Algorithm 1 and Theorem 16, we have

$$
\begin{aligned}
\|x_{m+1}-x_m\| &= \Big\| R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}\left[C_n(x_m)-\lambda A(x_m-p(x_m)) + \lambda a + \lambda F(t_{1,m},t_{2,m},\dots,t_{n,m})\right] \\
&\quad - R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}\left[C_n(x_{m-1}) - \lambda A(x_{m-1}-p(x_{m-1})) + \lambda a + \lambda F(t_{1,m-1},t_{2,m-1},\dots,t_{n,m-1})\right]\Big\| \\
&\le \frac{k}{\lambda K_n}\times \| C_n(x_m)-\lambda A(x_m-p(x_m)) + \lambda a + \lambda F(t_{1,m},t_{2,m},\dots,t_{n,m}) \\
&\quad - C_n(x_{m-1}) + \lambda A(x_{m-1}-p(x_{m-1})) - \lambda a - \lambda F(t_{1,m-1},t_{2,m-1},\dots,t_{n,m-1})\| \\
&\le \frac{k}{\lambda K_n}\times \big(\|C_n(x_m)-C_n(x_{m-1})\| + \lambda\|A(x_m-p(x_m)) - A(x_{m-1}-p(x_{m-1}))\| \\
&\quad + \lambda \|F(t_{1,m},t_{2,m},\dots,t_{n,m}) - F(t_{1,m-1},t_{2,m-1},\dots,t_{n,m-1})\|\big).
\end{aligned} \tag{41}
$$

From the Lipschitz continuity of C_n, p, A, and (γ,μ)-relaxed cocoercivity of p and Lemma 4, we have

$$\|C_n(x_m)-C_n(x_{m-1})\| \le \delta\|x_m-x_{m-1}\|, \tag{42}$$

$$
\begin{aligned}
&\|A(x_m-p(x_m)) - A(x_{m-1}-p(x_{m-1}))\| \\
&\quad \le \tau\|x_m-x_{m-1}-(p(x_m)-p(x_{m-1}))\|,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
&\|x_m-x_{m-1}-(p(x_m)-p(x_{m-1}))\|^q \\
&\quad \le \|x_m-x_{m-1}\|^q - q\left\langle p(x_m)-p(x_{m-1}), j_q(x_m-x_{m-1})\right\rangle + c_q\|p(x_m)-p(x_{m-1})\|^q \\
&\quad \le \|x_m-x_{m-1}\|^q + q\gamma\|p(x_m)-p(x_{m-1})\|^q - q\mu\|x_m-x_{m-1}\|^q + c_q\|p(x_m)-p(x_{m-1})\|^q \\
&\quad \le \|x_m-x_{m-1}\|^q + q\gamma\lambda_p^q\|x_m-x_{m-1}\|^q - q\mu\|x_m-x_{m-1}\|^q + c_q\lambda_p^q\|x_m-x_{m-1}\|^q \\
&\quad = \left(1+q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q\right)\|x_m-x_{m-1}\|^q,
\end{aligned} \tag{44}
$$

where $j_q : E \to 2^{E^*}$ is the normalized duality mapping.

Since $F, T_1, T_2, \dots, T_n$ are Lipschitz continuous, we have

$$
\begin{aligned}
&\|F(t_{1,m},t_{2,m},\dots,t_{n,m}) - F(t_{1,m-1},t_{2,m-1},\dots,t_{n,m-1})\| \\
&\quad \le \|F(t_{1,m},t_{2,m},\dots,t_{n,m}) - F(t_{1,m-1},t_{2,m},\dots,t_{n,m})\| \\
&\qquad + \|F(t_{1,m-1},t_{2,m},t_{3,m},\dots,t_{n,m}) - F(t_{1,m-1},t_{2,m-1},t_{3,m},\dots,t_{n,m})\| + \cdots \\
&\qquad + \|F(t_{1,m-1},t_{2,m-1},\dots,t_{n-1,m-1},t_{n,m}) - F(t_{1,m-1},t_{2,m-1},\dots,t_{n-1,m-1},t_{n,m-1})\| \\
&\quad \le \lambda_{F_1}\|t_{1,m}-t_{1,m-1}\| + \lambda_{F_2}\|t_{2,m}-t_{2,m-1}\| + \cdots + \lambda_{F_n}\|t_{n,m}-t_{n,m-1}\| \\
&\quad \le \lambda_{F_1}\left(1+\frac{1}{m}\right)H(T_1(x_m),T_1(x_{m-1})) + \lambda_{F_2}\left(1+\frac{1}{m}\right)H(T_2(x_m),T_2(x_{m-1})) + \cdots \\
&\qquad + \lambda_{F_n}\left(1+\frac{1}{m}\right)H(T_n(x_m),T_n(x_{m-1})) \\
&\quad \le \lambda_{F_1}\lambda_{t_1}\left(1+\frac{1}{m}\right)\|x_m-x_{m-1}\| + \lambda_{F_2}\lambda_{t_2}\left(1+\frac{1}{m}\right)\|x_m-x_{m-1}\| + \cdots \\
&\qquad + \lambda_{F_n}\lambda_{t_n}\left(1+\frac{1}{m}\right)\|x_m-x_{m-1}\| \\
&\quad = \sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}\left(1+\frac{1}{m}\right)\|x_m-x_{m-1}\|.
\end{aligned} \tag{45}
$$

It follows from (41)–(45) that

$$\|x_{m+1}-x_m\| \le \theta_m\|x_m-x_{m-1}\|, \tag{46}$$

where

$$
\begin{aligned}
\theta_m &= \frac{k}{\lambda K_n} \times \left[\delta + \lambda\tau\left(1+q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q\right)^{1/q} + \lambda\left(1+\frac{1}{m}\right)\sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}\right].
\end{aligned} \tag{47}
$$

Letting $m \to \infty$, we obtain $\theta_m \to \theta$, where

$$
\theta = \frac{k}{\lambda K_n} \times \left[\delta + \lambda\tau\left(1+q\gamma\lambda_p^q - q\mu + c_q\lambda_p^q\right)^{1/q} + \lambda\sum_{i=1}^{n}\lambda_{F_i}\lambda_{t_i}\right]. \tag{48}
$$

From condition (39), we know that $0 < \theta < 1$, and hence $\{x_m\}$ is a Cauchy sequence in E. Thus, there exists $x \in E$ such that $x_m \to x$, as $m \to \infty$. Now, we prove that $t_{1,m} \to t_1 \in T_1(x)$. In fact, it follows from the Lipschitz continuity of T_1 and Algorithm 1 that

$$\begin{aligned} \|t_{1,m} - t_{1,m-1}\| &\leq \left(1 + \frac{1}{m}\right) H\left(T_1\left(x_m\right), T_1\left(x_{m-1}\right)\right) \\ &\leq \left(1 + \frac{1}{m}\right) \lambda_{t_1} \|x_m - x_{m-1}\|. \end{aligned} \tag{49}$$

From (49), we know that $\{t_{1,m}\}$ is also a Cauchy sequence. In a similar way, $\{t_{2,m}\}, \{t_{3,m}\}, \dots, \{t_{n,m}\}$ are Cauchy sequences. Thus, there exist $t_1 \in E_1, t_2 \in E_2, \dots, t_n \in E_n$ such that $t_{1,m} \to t_1, t_{2,m} \to t_2, \dots, t_{n,m} \to t_n$, as $m \to \infty$. Furthermore,

$$\begin{aligned} d\left(t_1, T_1\left(x\right)\right) &\leq \|t_1 - t_{1,m}\| + d\left(t_{1,m}, T_1\left(x\right)\right) \\ &\leq \|t_1 - t_{1,m}\| + H\left(T_1\left(x_m\right), T_1\left(x\right)\right) \\ &\leq \|t_1 - t_{1,m}\| + \lambda_{t_1} \|x_m - x\| \longrightarrow 0, \\ &\qquad \text{as } m \longrightarrow \infty. \end{aligned} \tag{50}$$

Since $T_1(x)$ is closed, we have $t_1 \in T_1(x)$. In a similar way, we can show that $t_2 \in T_2(x)$, $t_3 \in T_3(x), \dots, t_n \in T_n(x)$. By continuity of $A, p, C_n, F, T_1, T_2, \dots, T_n, R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)}$ and Algorithm 1, we have

$$\begin{aligned} x = R^{C_n,\lambda,\eta}_{M(f_1,f_2,\dots,f_n)} &\left[C_n\left(x\right) - \lambda A\left(x - p\left(x\right)\right)\right. \\ &\left. + \lambda a + \lambda F\left(t_1, t_2, \dots, t_n\right)\right]. \end{aligned} \tag{51}$$

By Theorem 17, $(x, t_1, t_2, \dots, t_n)$ is a solution of problem (30). This completes the proof. □

5. Conclusions

The purpose of this paper is to study a new monotone mapping in Banach spaces, which generalizes the C_n-monotone mapping in [6], and generalizes the concepts of many monotone mappings. Moreover, the result of Theorem 18 improves and generalizes the corresponding results of [4–6, 10, 12, 13].

References

[1] R. P. Agarwal, Y. J. Cho, and N.-J. Huang, "Sensitivity analysis for strongly nonlinear quasi-variational inclusions," *Applied Mathematics Letters*, vol. 13, no. 6, pp. 19–24, 2000.

[2] S. S. Chang, "Set-valued variational inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 438–454, 2000.

[3] R. Ahmad and Q. H. Ansari, "An iterative algorithm for generalized nonlinear variational inclusions," *Applied Mathematics Letters*, vol. 13, no. 5, pp. 23–26, 2000.

[4] F.-Q. Xia and N.-J. Huang, "Variational inclusions with a general H-monotone operator in Banach spaces," *Computers & Mathematics with Applications*, vol. 54, no. 1, pp. 24–30, 2007.

[5] X.-P. Luo and N.-J. Huang, "A new class of variational inclusions with B-monotone operators in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 233, no. 8, pp. 1888–1896, 2010.

[6] S. Z. Nazemi, "A new class of monotone mappings and a new class of variational inclusions in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 155, no. 3, pp. 785–795, 2012.

[7] H.-Y. Lan, X. Wang, T. J. X. Xiong, and Y. M. Xiang, "An over-relaxed (A, η, m)-proximal point algorithm for system of nonlinear fuzzy-set valued operator equation frameworks and fixed point problems," in *Integrated Uncertainty in Knowledge Modelling and Decision Making*, Y. Tang, V.-N. Huynh, and J. Lawry, Eds., vol. 7027 of *Lecture Notes in Computer Science*, pp. 133–142, Springer, Berlin, Germany, 2011.

[8] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[9] L.-C. Zeng, S.-M. Guu, and J.-C. Yao, "Characterization of H-monotone operators with applications to variational inclusions," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 329–337, 2005.

[10] X. P. Ding and H. R. Feng, "Algorithm for solving a new class of generalized nonlinear implicit quasi-variational inclusions in Banach spaces," *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 547–555, 2009.

[11] H. R. Feng and X. P. Ding, "A new system of generalized nonlinear quasi-variational-like inclusions with A-monotone operators in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 365–373, 2009.

[12] Y.-P. Fang and N.-J. Huang, "H-monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.

[13] F.-Q. Xia and N.-J. Huang, "Algorithm for solving a new class of general mixed variational inequalities in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 632–642, 2008.