

Research Article

On the Homomorphisms of the Lie Groups $SU(2)$ and S^3

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We first construct all the homomorphisms from the Heisenberg group to the 3-sphere. Also, defining a topology on these homomorphisms, we regard the set of these homomorphisms as a topological space. Next, using the kernels of homomorphisms, we define an equivalence relation on this topological space. We finally show that the quotient space is a topological group which is isomorphic to S^1 .

1. Introduction

Discrete and continuous forms of the Heisenberg group have been studied in mathematics and physics such as analysis [1–3], geometry [4–6], topology [3, 7], and quantum physics [8–14]. An introductory review can be also found in [15].

In [16–18], it was shown that the Heisenberg group \mathbb{H} is nilpotent, and any arbitrary nilpotent subgroup of $SU(2)$ is conjugate to a subgroup of $U(1)$, which is identified with the set of diagonal matrices in $SU(2)$. Since groups can be considered as metric spaces, it leads us to examine if there exists any geometry in these groups.

It is known that the matrices

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \quad (1)$$

form a linear group which is isomorphic to \mathbb{R}^1 . The Lie groups \mathbb{R}^1 and S^1 are related, for the mapping $\rho: \mathbb{R}^1 \rightarrow S^1$ defined by $\rho(x) = e^{2\pi i x}$ is a continuous homomorphism from \mathbb{R}^1 onto S^1 [10]. The Lie algebras \mathbb{R}^1 and S^1 are trivially isomorphic. In this work, we search if there is a similar relationship between the only other sphere Lie group S^3 and the linear group of matrices which is diffeomorphic to \mathbb{R}^3 [17].

The subgroup \mathbb{H} of $GL(3, \mathbb{R})$ formed by the matrices of type

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \quad (2)$$

is called three-dimensional Heisenberg group. It is convenient to denote the elements of this group by three-tuples of numbers. Using this convention, that is, $\mathbb{H} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, the multiplicative operation of elements can be expressed as

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'). \quad (3)$$

The identity element of this group is $(0, 0, 0)$, and the inverse of an element (x, y, x) is $(-x, -y, -z)$.

This Lie group is diffeomorphic to \mathbb{R}^3 [17].

Moreover, the Lie groups $SU(2)$ and S^3 are isomorphic, and they are diffeomorphic as manifolds. Since the respective Lie algebras \mathfrak{h} and $\mathfrak{su}(2)$ of \mathbb{H} and $SU(2)$ are three-dimensional real vector spaces, they are isomorphic as real vector spaces. But they are not isomorphic as Lie algebras, because there is no Lie algebra isomorphism between a compact and a noncompact Lie algebra. However, there may be a Lie

algebra homomorphism between them. We try to find the relationship between h and $su(2)$ as Lie algebras.

Consider the following matrices:

$$U_1 = -\frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad U_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad (4)$$

which span the Lie algebra $su(2)$ of the Lie group $SU(2)$, and also consider the following matrices:

$$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5)$$

which span the Lie algebra h of the Heisenberg group \mathbb{H} .

Using the exponential map from h to \mathbb{H} we may convert our equations in h to equations in \mathbb{H} . Letting $\bar{V}_i = \exp(V_i)$ for each i , we see that

$$\bar{V}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{V}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{V}_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

The standard generating set for the Heisenberg group is $\{\bar{V}_1, \bar{V}_2, \bar{V}_1^{-1}, \bar{V}_2^{-1}\}$, and the group has the relation

$$\begin{aligned} \bar{V}_1 \bar{V}_2 \bar{V}_1^{-1} \bar{V}_2^{-1} &= \bar{V}_2 \bar{V}_1^{-1} \bar{V}_2^{-1} \bar{V}_1 \\ &= \bar{V}_1^{-1} \bar{V}_2^{-1} \bar{V}_1 \bar{V}_2 = \bar{V}_2^{-1} \bar{V}_1 \bar{V}_2 \bar{V}_1^{-1} = \bar{V}_3. \end{aligned} \quad (7)$$

In the above, the notations \bar{V}_1^{-1} and \bar{V}_2^{-1} denote the inverses of the elements \bar{V}_1 and \bar{V}_2 in the Heisenberg group.

From (4) and (5), we find that

$$\begin{aligned} [U_1, U_2] &= U_3, & [U_2, U_3] &= U_1, & [U_3, U_1] &= U_2, \\ [V_1, V_2] &= V_3, & [V_2, V_3] &= [V_3, V_1] &= 0. \end{aligned} \quad (8)$$

It was shown in ([13, 16]) that there is no nontrivial Lie algebra homomorphism from $su(2)$ to h . Here, we study Lie algebra homomorphisms from h to $su(2)$. We observe that there exists a nontrivial Lie algebra homomorphism from h to $su(2)$. Moreover, we describe all Lie algebra homomorphisms from h to $su(2)$.

2. Homomorphisms from \mathbb{H} to $SU(2)$

Let $\varphi : h \rightarrow su(2)$ be a Lie algebra homomorphism. Suppose that

$$\varphi(V_i) = \sum_{j=1}^3 \alpha_{ij} U_j, \quad i = 1, 2, 3 \quad (9)$$

for some real numbers α_{ij} .

By using (8) and the following commutators:

$$\begin{aligned} \varphi[V_1, V_2] &= [\varphi V_1, \varphi V_2], & \varphi V_3 &= [\varphi V_1, \varphi V_2], \\ \varphi[V_1, V_3] &= [\varphi V_1, \varphi V_3], & \varphi[V_1, V_3] &= 0, \\ \varphi[V_2, V_3] &= [\varphi V_2, \varphi V_3], & \varphi[V_2, V_3] &= 0, \end{aligned} \quad (10)$$

we obtain the system of equations of coefficients in the real constants α_{ij} 's

$$\begin{aligned} \alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22} &= \alpha_{31}, & \alpha_{13}\alpha_{21} - \alpha_{11}\alpha_{23} &= \alpha_{32}, \\ \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} &= \alpha_{33}, & \alpha_{12}\alpha_{33} - \alpha_{13}\alpha_{32} &= 0, \\ \alpha_{13}\alpha_{31} - \alpha_{11}\alpha_{33} &= 0, & \alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} &= 0, \\ \alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} &= 0, & \alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33} &= 0, \\ \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} &= 0. \end{aligned} \quad (11)$$

By using the Mathematica program, we may solve the system (11). We write below only two of the non-trivial solutions, because a group homomorphism induced by any of the other solutions will be equal to a group homomorphism induced by one of the solutions given below. Here, a nontrivial solution means a solution in which at least one of the constants α_{ij} is nonzero.

- (1) $\alpha_{22} = \alpha_{12}\alpha_{21}/\alpha_{11}$, $\alpha_{23} = \alpha_{13}\alpha_{21}/\alpha_{11}$, $\alpha_{31} = \alpha_{32} = \alpha_{33} = 0$, $\alpha_{11} \neq 0$, $\alpha_{12}, \alpha_{13}, \alpha_{21}$ arbitrary.
- (2) $\alpha_{23} = \alpha_{13}\alpha_{22}/\alpha_{12}$, $\alpha_{11} = \alpha_{21} = \alpha_{31} = \alpha_{32} = \alpha_{33} = 0$, $\alpha_{12} \neq 0$, α_{13}, α_{22} arbitrary.

Also, in the system of (11), if $\alpha_{11} \neq 0$ and $\alpha_{12} = 0$, then the first set of solutions is obtained. If $\alpha_{12} \neq 0$ and $\alpha_{11} = 0$, then the second set of solutions is obtained. Moreover, if α_{11} and α_{12} are both 0, then (11) has trivial solution.

In the set of solutions, we consider α_{22} and α_{23} in terms of the other arbitrary constants α_{11} , α_{12} , α_{13} , and α_{21} and obtain that the group homomorphism is spanned by elements $\{U_1, U_2, U_3\}$ or $\{U_2, U_3\}$. If another configuration, say α_{11}, α_{13} , was chosen, then we see that the homomorphism would be spanned by the same elements.

Since φ is a Lie algebra homomorphism, we observe from solution sets 1 and 2 that the subalgebra of $su(2)$ is only generated by $a_1U_1 + a_2U_2 + a_3U_3$ or $aU_2 + bU_3$ and not by $a_1U_1 + a_2U_3$.

As we will use the constants α_{ij} frequently, to simplify the notations we put a_1, a_2, a_3 , and b for the coefficients $\alpha_{11}, \alpha_{12}, \alpha_{13}$, and α_{21} , respectively, and, a, b , and c for α_{12}, α_{13} , and α_{22} .

Hence, for the first set of solutions, φ has the following form:

$$\begin{aligned} \varphi(V_1) &= a_1U_1 + a_2U_2 + a_3U_3, \\ \varphi(V_2) &= \frac{b}{a_1}\varphi(V_1), & a_1 &\neq 0, \\ \varphi(V_3) &= 0. \end{aligned} \quad (12)$$

We note that the rank of φ is one, and thus, $\varphi(h)$ is a one-dimensional Lie subalgebra of $su(2)$ generated by $a_1U_1 + a_2U_2 + a_3U_3$.

For the second set of the solutions, φ is of the following form:

$$\begin{aligned} \varphi(V_1) &= aU_2 + bU_3, \\ \varphi(V_2) &= \frac{c}{a}\varphi(V_1), \quad a \neq 0, \\ \varphi(V_3) &= 0. \end{aligned} \tag{13}$$

Here, we note again that the rank of φ is one, and thus, $\varphi(h)$ is a one-dimensional Lie subalgebra of $su(2)$ generated by $aU_2 + bU_3$.

It is known from ([8, 18]) that if $\Phi : \mathbb{H} \rightarrow SU(2)$ is a homomorphism, then the following diagram commutes:

$$\begin{array}{ccc} h & \xrightarrow{d\Phi} & su(2) \\ \exp \downarrow & & \downarrow \exp \\ \mathbb{H} & \xrightarrow{\Phi} & SU(2), \end{array} \tag{14}$$

where $d\Phi$ is the differential of Φ , and it is a Lie algebra homomorphism. It is also known that for the matrix groups the exponential map is given by the exponentiation of matrices. In our notation, $\varphi = d\Phi$ for some Φ , and we will determine Φ .

By using (5), for any element of $V = c_1V_1 + c_2V_2 + c_3V_3 \in h$ we have

$$\begin{aligned} V &= \begin{pmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}, & V^2 &= \begin{pmatrix} 0 & 0 & c_1c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ V^3 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & c_1, c_2, c_3 &\in \mathbb{R}. \end{aligned} \tag{15}$$

By using (15), we obtain $\exp(V)$ as

$$\exp(V) = 1 + V + \frac{1}{2}V^2 + \frac{1}{3!}V^3 + \dots = \begin{pmatrix} 1 & c_1 & c_3 + \frac{c_1c_2}{2} \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

To guarantee that

$$\exp(V) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (x, y, z) \tag{17}$$

we must take $V = xV_1 + yV_2 + (z - xy/2)V_3 \in h$. Here, we put (x, y, z) for the matrix (2) for any real x, y , and z .

For the first set of solutions, we obtain

$$\begin{aligned} \varphi(V) &= \varphi\left(xV_1 + yV_2 + \left(z - \frac{xy}{2}\right)V_3\right) \\ &= \left(x + \frac{b}{a_1}y\right)(a_1U_1 + a_2U_2 + a_3U_3). \end{aligned} \tag{18}$$

Thus, the kernel of φ is the plane $x + (b/a_1)y = 0$ in \mathbb{R}^3 .

For $V \notin \text{Ker } \varphi$, we obtain

$$\begin{aligned} \exp \varphi(V) &= I \cos A \left(x + \frac{b}{a_1}y\right) \\ &\quad + \frac{1}{A(x + (b/a_1)y)} \sin \left(A \left(x + \frac{b}{a_1}y\right)\right) \varphi(V), \end{aligned} \tag{19}$$

where I is the 2×2 identity matrix, and $A = (1/2)\sqrt{a_1^2 + a_2^2 + a_3^2}$.

In this case, Φ is of the following form:

$$\begin{aligned} (x, y, z) &\longrightarrow I \cos A \left(x + \frac{b}{a_1}y\right) \\ &\quad + \frac{1}{A} \sin \left(A \left(x + \frac{b}{a_1}y\right)\right) (a_1U_1 + a_2U_2 + a_3U_3). \end{aligned} \tag{20}$$

For the second set of solutions, we obtain

$$\varphi(V) = \left(x + \frac{c}{a}y\right)(aU_2 + bU_3). \tag{21}$$

From (21), the kernel of φ is the plane $x + (c/a)y = 0$ in \mathbb{R}^3 .

For $V \notin \text{Ker } \varphi$, we have

$$\begin{aligned} \exp \varphi(V) &= I \cos A \left(x + \frac{c}{a}y\right) \\ &\quad + \frac{1}{A(x + (c/a)y)} \sin \left(A \left(x + \frac{c}{a}y\right)\right) \varphi(V), \end{aligned} \tag{22}$$

where I is the 2×2 identity matrix, and $A = (1/2)\sqrt{a^2 + b^2}$.

In this case, Φ is of the form

$$\begin{aligned} (x, y, z) &\longrightarrow I \cos A \left(x + \frac{c}{a}y\right) \\ &\quad + \frac{1}{A} \sin \left(A \left(x + \frac{c}{a}y\right)\right) (aU_2 + bU_3). \end{aligned} \tag{23}$$

Hence, we can state the following theorem.

Theorem 1. Any nontrivial homomorphism from the Heisenberg group to the 3-sphere is one of (20) and (23).

We now observe the following properties for the first type of homomorphisms. It can be shown that same observations are valid for the second set of solutions.

(a) By considering Φ as a map from \mathbb{R}^3 to \mathbb{R}^4 , we can write

$$\Phi : (x, y, z) \longrightarrow (u_1, u_2, u_3, u_4), \tag{24}$$

where

$$\begin{aligned} u_1 &= \cos At, & u_2 &= -\frac{a_3}{2A} \sin At, & u_3 &= -\frac{a_2}{2A} \sin At, \\ u_4 &= -\frac{a_1}{2A} \sin At, & t &= x + \frac{b}{a_1}y. \end{aligned} \tag{25}$$

Then, we find that the image of the planes P_t in \mathbb{H} , with the equation $x + b/a_1 = t$, is a periodic (closed) curve in \mathbb{S}^3 .

- (b) In the first case, every homomorphism Φ from \mathbb{H} to $SU(2)$ depends on four parameters, namely, a_1, a_2, a_3 , and b with $a_1 \neq 0$.

We now concentrate on the parameters a_1 and b , since they are involved in the kernel of Φ . To each kernel $x + (b/a_1)y = 0$, we associate a point $(b, -a_1)$ in the xy -plane. Furthermore, we normalize the vector $(b, -a_1)$ as $(b/\sqrt{b^2 + a_1^2}, -a_1/\sqrt{b^2 + a_1^2})$ which can also be considered as a point of \mathbb{S}^1 in the xy -plane.

By considering the principal value of $\arctan(-b/a_1)$, we define a topology on the set \mathbb{H}' of all homomorphisms from \mathbb{H} to $SU(2)$ as follows.

For any $\Phi_1(a_1, a_2, a_3, b) \in \mathbb{H}'$, $\Phi_2(a'_1, a'_2, a'_3, b') \in \mathbb{H}'$ is in the ϵ -neighborhood of Φ_1 , if and only if $|\arctan(-b/a_1) - \arctan(-b'/a'_1)| < \epsilon$, $|a_2 - a'_2| < \epsilon$, and $|a_3 - a'_3| < \epsilon$.

Now let us define an equivalence relation on \mathbb{H}' . For $\Phi_1, \Phi_2 \in \mathbb{H}'$, Φ_1 is equivalent to Φ_2 if and only if Φ_1 and Φ_2 have the same kernel. Denote the set of equivalence classes by $\widetilde{\mathbb{H}}$. We define a multiplication on $\widetilde{\mathbb{H}}$ such that, for any $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ in $\widetilde{\mathbb{H}}$, $\widetilde{\Phi}_1 \cdot \widetilde{\Phi}_2$ denotes the element of $\widetilde{\mathbb{H}}$ whose kernel is the plane obtained by the product of the elements of \mathbb{S}^1 corresponding to Φ_1 and Φ_2 . This multiplication makes $\widetilde{\mathbb{H}}$ into a group which is isomorphic to \mathbb{S}^1 .

- (c) The set of the kernels of all the homomorphisms from \mathbb{H} to \mathbb{S}^3 is a subset of the Grassmann manifold of 2 planes in \mathbb{R}^3 . It is known that the Grassmann manifold of 2 planes in \mathbb{R}^3 is diffeomorphic to \mathbb{S}^1 . Any point p of \mathbb{S}^1 corresponds to the plane which is orthogonal to the normal of \mathbb{S}^1 at p and which contains the z -axis. Hence, there exists a 1-1 correspondence between the set of the kernels of all the homomorphisms from \mathbb{H} to \mathbb{S}^3 and the equator \mathbb{S}^1 of \mathbb{S}^2 .

Thus, we state the following theorem.

Theorem 2. *The set of all homomorphisms from the Heisenberg group to the 3-sphere is isomorphic (up to a certain equivalence relation concerning kernels) with the topological group \mathbb{S}^1 .*

3. Conclusion

In this paper, our aim is to construct all homomorphisms between the Heisenberg group and the 3-sphere which is isomorphic to $SU(2)$. In the literature, it has been shown that no nontrivial Lie algebra homomorphism from $su(2)$ to h exists ([8, 11, 17]). So, a natural question arises: is there a homomorphism from h to $su(2)$? Here, we answer this question completely.

We here observe that there are nontrivial homomorphisms from the Heisenberg group to the 3-sphere, and they can be only in the form of (20) or (23).

Also, we use these maps to define a topology in order to construct an equivalence relation, and we show that quotient space is isomorphic to \mathbb{S}^1 .

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