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Research Article

Convergence of Variational Iteration Method for Second-Order Delay Differential Equations

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This paper employs the variational iteration method to obtain analytical solutions of second-order delay differential equations. The corresponding convergence results are obtained, and an effective technique for choosing a reasonable Lagrange multiplier is designed in the solving process. Moreover, some illustrative examples are given to show the efficiency of this method.

1. Introduction

The second-order delay differential equations often appear in the dynamical system, celestial mechanics, kinematics, and so forth. Some numerical methods for solving second-order delay differential equations have been discussed, which include θ-method [1], trapezoidal method [2], and Runge-Kutta-Nyström method [3]. The variational iteration method (VIM) was first proposed by He [4, 5] and has been extensively applied due to its flexibility, convenience, and efficiency. So far, the VIM is applied to autonomous ordinary differential systems [6], pantograph equations [7], integral equations [8], delay differential equations [9], fractional differential equations [10], the singular perturbation problems [11], and delay differential-algebraic equations [12]. Rafei et al. [13] and Marinca et al. [14] applied the VIM to oscillations. Tatari and Dehghan [15] consider the VIM for solving second-order initial value problems. For a more comprehensive survey on this method and its applications, the readers refer to the review articles [16–19] and the references therein. But the VIM for second-order delay differential equations has not been considered.

The article apply the VIM to second-order delay differential equations to obtain the analytical or approximate analytical solutions. The corresponding convergence results are obtained. Some illustrative examples confirm the theoretical results.

2. Convergence

2.1. The First Kind of Second-Order Delay Differential Equations. Consider the initial value problems of second-order delay differential equations

$$\begin{aligned} y''(t) &= f(t, y(t), y(\alpha(t))), \quad t \in [0, T], \\ y'(t) &= \varphi'(t), \quad t \in [-\tau, 0], \\ y(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{1}$$

where $\varphi(t)$ is a differentiable function, $\alpha(t) \in C^1[0,T]$ is a strictly monotone increasing function and satisfies that $-\tau \le \alpha(t) \le t$ and $\alpha(0) = -\tau$, there exists $t_1 \in [0,T]$ such that $\alpha(t_1) = 0$, and $f : D = [0,T] \times R \times R \to R$ is a given continuous mapping and satisfies the Lipschitz condition

$$\begin{aligned} \| f(t, u_1, v) - f(t, u_2, v) \| &\le \beta_0 \| u_1 - u_2 \|, \\ \| f(t, u, v_1) - f(t, u, v_2) \| &\le \beta_1 \| v_1 - v_2 \|, \end{aligned} \tag{2}$$

where $\beta_0,\ \beta_1$ are Lipschitz constants; $\|\cdot\|$ denotes the standard Euclidean norm.

Now the VIM for (1) can read

$$y_{m+1}(t) = y_m(t) + \int_0^t \lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi, \quad 0 < t < t_1; \tag{3}$$

$$y_{m+1}(t) = y_m(t) + \int_0^{t_1} \lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi + \int_{t_1}^t \lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), y_m(\alpha(\xi)))\right] d\xi, \quad t > t_1, \tag{4}$$

where $y_m(t) = \varphi(t)$ for $t \in [-\tau, 0]$; $\tilde{f}$ denotes the restrictive variation, that is, $\delta\tilde{f} = 0$. Thus, we have

$$\begin{aligned}\delta y_{m+1}(t) &= \delta y_m(t) + \int_0^t \delta\lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi \\ &= \delta y_m(t) + \int_0^t \delta\lambda(t,\xi)\, y_m''(\xi)\, d\xi, \quad 0 < t < t_1.\end{aligned} \tag{5}$$

Using integration by parts to (4), we have

$$\begin{aligned}\delta y_{m+1}(t) &= \delta y_m(t) + \int_0^{t_1} \delta\lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi + \int_{t_1}^t \delta\lambda(t,\xi)\left[y_m''(\xi)-\tilde{f}(\xi, y_m(\xi), y_m(\alpha(\xi)))\right] d\xi \\ &= \delta y_m(t) + \delta\lambda(t,\xi)\, y_m'(\xi)\Big|_{\xi=t} - \frac{\partial\lambda(t,\xi)}{\partial\xi}\delta y_m(\xi)\Big|_{\xi=t} + \int_0^t \frac{\partial^2\lambda(t,\xi)}{\partial\xi^2}\delta y_m(\xi)\, d\xi.\end{aligned} \tag{6}$$

From the above formula, the stationary conditions are obtained as

$$\frac{\partial^2\lambda(t,\xi)}{\partial\xi^2} = 0, \quad 1 - \frac{\partial\lambda(t,\xi)}{\partial\xi}\Big|_{\xi=t} = 0, \quad \lambda(t,\xi)\big|_{\xi=t} = 0. \tag{7}$$

Moreover, the general Lagrange multiplier

$$\lambda(t,\xi) = \xi - t \tag{8}$$

can be readily identified by (7). Thus, the variational iteration formula can be written as

$$y_{m+1}(t) = y_m(t) + \int_0^t (\xi - t)\left[y_m''(\xi) - f(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi, \quad 0 < t < t_1; \tag{9}$$

$$y_{m+1}(t) = y_m(t) + \int_0^{t_1} (\xi - t)\left[y_m''(\xi) - f(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi + \int_{t_1}^t (\xi - t)\left[y_m''(\xi) - f(\xi, y_m(\xi), y_m(\alpha(\xi)))\right] d\xi, \quad t > t_1. \tag{10}$$

Theorem 1. *Suppose that the initial value problems* (1) *satisfy the condition* (2), *and* $y(t), y_i(t) \in C^2[0,T]$, $i = 1, 2, \ldots$. *Then the sequence* $\{y_m(t)\}_{m=1}^{\infty}$ *defined by* (9) *and* (10) *with* $y_0(t)$ *converges to the solution of* (1).

Proof. From (1), we have

$$y(t) = y(t) + \int_0^t (\xi - t)\left[y''(\xi) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right] d\xi, \quad 0 < t < t_1; \tag{11}$$

$$y(t) = y(t) + \int_0^{t_1} (\xi - t)\left[y''(\xi) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right] d\xi + \int_{t_1}^t (\xi - t)\left[y''(\xi) - f(\xi, y(\xi), y(\alpha(\xi)))\right] d\xi, \quad t > t_1. \tag{12}$$

Let $E_i(t) = y_i(t) - y(t)$, $i = 0, 1, \ldots$. If $t \leqslant 0$, then $E_i(t) = 0$, $i = 0, 1, \ldots$. From (9) and (11), we have

$$E_{m+1}(t) = E_m(t) + \int_0^t (\xi - t)\left[E_m''(\xi) - \left(f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right)\right] d\xi, \quad 0 < t < t_1. \tag{13}$$

From (10) and (12), we have

$$\begin{aligned}E_{m+1}(t) &= E_m(t) + \int_0^{t_1}(\xi - t)\left[E_m''(\xi) - \left(f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right)\right]d\xi \\ &\quad + \int_{t_1}^{t}(\xi - t)\left[E_m''(\xi) - \left(f(\xi, y_m(\xi), y_m(\alpha(\xi))) - f(\xi, y(\xi), y(\alpha(\xi)))\right)\right]d\xi, \quad t > t_1.\end{aligned} \tag{14}$$

Using integration by parts, we have

$$\begin{aligned}E_{m+1}(t) &= E_m(t) + \int_0^t(\xi - t)E_m''(\xi)\,d\xi - \int_0^t(\xi - t)\left[f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &= -\int_0^t(\xi - t)\left[f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right]d\xi, \quad 0 < t < t_1; \\ E_{m+1}(t) &= E_m(t) + \int_0^t(\xi - t)E_m''(\xi)\,d\xi - \int_0^{t_1}(\xi - t)\left[f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &\quad - \int_{t_1}^t(\xi - t)\left[f(\xi, y_m(\xi), y_m(\alpha(\xi))) - f(\xi, y(\xi), y(\alpha(\xi)))\right]d\xi \\ &= -\int_0^{t_1}(\xi - t)\left[f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &\quad - \int_{t_1}^t(\xi - t)\left[f(\xi, y_m(\xi), y_m(\alpha(\xi))) - f(\xi, y(\xi), y(\alpha(\xi)))\right]d\xi, \quad t > t_1.\end{aligned} \tag{15}$$

Since $[\alpha^{-1}(t)]'$ is bounded, $M = \max_{-\tau\le\xi\le\alpha(T)}(\alpha^{-1}(\xi))'$ is bounded. Moreover, it follows from (2) and the inequality $|\xi - t| \le T$ that

$$\begin{aligned}\|E_{m+1}(t)\| &\le \int_0^t|t-\xi|\,\|f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\|\,d\xi \\ &\le \int_0^t T\beta_0\|y_m(\xi) - y(\xi)\|\,d\xi \\ &= \int_0^t T\beta_0\|E_m(\xi)\|\,d\xi, \quad 0 < t < t_1;\end{aligned} \tag{16}$$

$$\begin{aligned}\|E_{m+1}(t)\| &\le \int_0^{t_1}|t-\xi|\,\|f(\xi, y_m(\xi), \varphi(\alpha(\xi))) - f(\xi, y(\xi), \varphi(\alpha(\xi)))\|\,d\xi \\ &\quad + \int_{t_1}^t|t-\xi|\,\|f(\xi, y_m(\xi), y_m(\alpha(\xi))) - f(\xi, y(\xi), y(\alpha(\xi)))\|\,d\xi \\ &\le \int_0^{t_1}T\beta_0\|y_m(\xi) - y(\xi)\|\,d\xi + \int_{t_1}^t T\left(\beta_0\|E_m(\xi)\| + \beta_1\|E_m(\alpha(\xi))\|\right)d\xi \\ &= \int_0^t T\beta_0\|E_m(\xi)\|\,d\xi + \int_{t_1}^t T\beta_1\|E_m(\alpha(\xi))\|\,d\xi \\ &= \int_0^t T\beta_0\|E_m(\xi)\|\,d\xi + \int_{\alpha(t_1)}^{\alpha(t)} T\beta_1\|E_m(\xi)\|\left(\alpha^{-1}(\xi)\right)'d\xi \\ &\le TM\beta\int_0^t\|E_m(\xi)\|\,d\xi, \quad t > t_1,\end{aligned} \tag{17}$$

where $\beta = \max\beta_i$, $i = 1, 2$. Moreover,

$$\begin{aligned}\|E_{m+1}(t)\| &\le (TM\beta)^2\int_0^t\int_0^{s_1}\|E_{m-1}(s_2)\|\,ds_2ds_1 \\ &\le (TM\beta)^3\int_0^t\int_0^{s_1}\int_0^{s_2}\|E_{m-2}(s_3)\|\,ds_3ds_2ds_1 \\ &\le (TM\beta)^4\int_0^t\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\|E_{m-3}(s_4)\|\,ds_4ds_3ds_2ds_1 \\ &\cdots \\ &\le (TM\beta)^{m+1}\int_0^t\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_m}\|E_0(s_{m+1})\|\,ds_{m+1}\cdots ds_3ds_2ds_1,\end{aligned} \tag{18}$$

where $\|E_0(t)\|$ is constant. Therefore, we have

$$\|E_{m+1}(t)\| \le \|E_0(t)\| \frac{(TM\beta)^{m+1}}{(m+1)!} \longrightarrow 0, \quad (m \longrightarrow \infty). \tag{19}$$

□

2.2. The Second Kind of Second-Order Delay Differential Equations. Consider the initial value problems of second-order delay oscillation differential equations

$$\begin{gathered} y''(t) = -\omega^2 y(t) - f(t, y(t), y(\alpha(t))), \quad t \in [0, T], \\ y'(t) = \varphi'(t), \quad t \in [-\tau, 0], \\ y(t) = \varphi(t), \quad t \in [-\tau, 0], \end{gathered} \tag{20}$$

where $\varphi(t)$ is a differentiable function, $\alpha(t) \in C^1[0, T]$ is a strictly monotone increasing function and satisfies that $-\tau \le \alpha(t) \le t$ and $\alpha(0) = -\tau$, there exists $t_1 \in [0, T]$ such that $\alpha(t_1) = 0$, ω is a constant, and $f : D = [0, T] \times R \times R \to R$ is a given continuous mapping and satisfies the Lipschitz condition

$$\begin{gathered} \|f(t, u_1, v) - f(t, u_2, v)\| \le \kappa_0 \|u_1 - u_2\|, \\ \|f(t, u, v_1) - f(t, u, v_2)\| \le \kappa_1 \|v_1 - v_2\|, \end{gathered} \tag{21}$$

where κ_0, κ_1 are Lipschitz constants.

Now the VIM for (20) can read

$$\begin{aligned} y_{m+1}(t) = y_m(t) &+ \int_0^t \lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), \varphi_m(\alpha(\xi))) \right] d\xi, \\ &0 < t < t_1; \end{aligned} \tag{22}$$

$$\begin{aligned} y_{m+1}(t) = y_m(t) &+ \int_0^{t_1} \lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), \varphi_m(\alpha(\xi))) \right] d\xi \\ &+ \int_{t_1}^t \lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), y_m(\alpha(\xi))) \right] d\xi, \\ &t > t_1, \end{aligned} \tag{23}$$

where $y_m(t) = \varphi(t)$ for $t \in [-\tau, 0]$; $\tilde{f}$ denotes the restrictive variation, that is, $\delta \tilde{f} = 0$. Thus, we have

$$\begin{aligned} \delta y_{m+1}(t) &= \delta y_m(t) \\ &+ \int_0^t \delta\lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), \varphi_m(\alpha(\xi))) \right] d\xi \\ &= \delta y_m(t) + \int_0^t \delta\lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right] d\xi, \\ &0 < t < t_1. \end{aligned} \tag{24}$$

Using integration by parts to (23), we have

$$\begin{aligned} \delta y_{m+1}(t) &= \delta y_m(t) \\ &+ \int_0^{t_1} \delta\lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), \varphi_m(\alpha(\xi))) \right] d\xi \\ &+ \int_{t_1}^t \delta\lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right. \\ &\left. + \tilde{f}(\xi, y_m(\xi), y_m(\alpha(\xi))) \right] d\xi \\ &= \delta y_m(t) + \int_0^t \delta\lambda(t, \xi) \left[y_m''(\xi) + \omega^2 y_m(\xi) \right] d\xi \\ &= \delta y_m(t) - \frac{\partial\lambda(t, \xi)}{\partial\xi} \delta y_m(\xi) \Big|_{\xi=t} \\ &+ \lambda(t, \xi) \delta y_m'(\xi) \big|_{\xi=t} \\ &+ \int_0^t \left[\omega^2 \lambda(t, \xi) + \frac{\partial^2 \lambda(t, \xi)}{\partial\xi^2} \right] \delta y_m(\xi) \, d\xi. \end{aligned} \tag{25}$$

From the above formula, the stationary conditions are obtained as

$$\begin{gathered} \omega^2 \lambda(t, \xi) + \frac{\partial^2 \lambda(t, \xi)}{\partial\xi^2} = 0, \\ 1 - \frac{\partial\lambda(t, \xi)}{\partial\xi} \Big|_{\xi=t} = 0, \\ \lambda(t, \xi)\big|_{\xi=t} = 0. \end{gathered} \tag{26}$$

Moreover, the general Lagrange multiplier

$$\lambda(t, \xi) = \frac{1}{\omega} \sin \omega (\xi - t) \tag{27}$$

can be readily identified by (26). Thus, the variational iteration formula can be written as

$$\begin{aligned} y_{m+1}(t) &= y_m(t) + \int_0^t \frac{1}{\omega}\sin\omega(\xi - t)\left[y_m''(\xi) + \omega^2 y_m(\xi) + f(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi, \quad 0 < t < t_1; \\ y_{m+1}(t) &= y_m(t) + \int_0^{t_1} \frac{1}{\omega}\sin\omega(\xi - t)\left[y_m''(\xi) + \omega^2 y_m(\xi) + f(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &\quad + \int_{t_1}^t \frac{1}{\omega}\sin\omega(\xi - t)\left[y_m''(\xi) + \omega^2 y_m(\xi) + f(\xi, y_m(\xi), y_m(\alpha(\xi)))\right]d\xi, \quad t > t_1. \end{aligned} \tag{28}$$

Theorem 2. *Suppose that the initial value problems* (20) *satisfy the condition* (21), *and* $y(t), y_i(t) \in C^2[0,T]$, $i = 1, 2, \ldots$. *Then the sequence* $\{y_m(t)\}_{m=1}^{\infty}$ *defined by* (28) *with* $y_0(t)$ *converges to the solution of* (20).

Proof. The proof process is similar that in Theorem 1. □

2.3. The Third Kind of Second-Order Delay Differential Equations. In order to improve the iteration speed, we modify the above iterative formulas and reconstruct the Lagrange multiplier. Consider the initial value problems of second-order delay differential equations

$$\begin{aligned} y''(t) + a(t)y'(t) + b(t)y(t) + N(t, y(t), y(\alpha(t))) &= 0, \quad t \in [0, T], \\ y'(t) &= \varphi'(t), \quad t \in [-\tau, 0], \\ y(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{29}$$

where $\varphi(t)$ is a differentiable function, $\alpha(t) \in C^1[0,T]$ is a strictly monotone increasing function and satisfies that $-\tau \le \alpha(t) \le t$ and $\alpha(0) = -\tau$, there exists $t_1 \in [0,T]$ such that $\alpha(t_1) = 0$, $a(t), b(t)$ are bounded functions, and $N : D = [0,T] \times R \times R \to R$ is a given continuous mapping and satisfies the Lipschitz condition

$$\begin{aligned} \|N(t, u_1, v) - N(t, u_2, v)\| &\le \gamma_0 \|u_1 - u_2\|, \\ \|N(t, u, v_1) - N(t, u, v_2)\| &\le \gamma_1 \|v_1 - v_2\|, \end{aligned} \tag{30}$$

where γ_0, γ_1 are Lipschitz constants.

Now the VIM for (29) can read

$$\begin{aligned} y_{m+1}(t) = y_m(t) &+ \int_0^t \lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi, \quad 0 < t < t_1; \end{aligned} \tag{31}$$

$$\begin{aligned} y_{m+1}(t) = y_m(t) &+ \int_0^{t_1} \lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &+ \int_{t_1}^t \lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m, y_m(\alpha(\xi)))\right]d\xi, \quad t > t_1, \end{aligned} \tag{32}$$

where $y_m(t) = \varphi(t)$ for $t \in [-\tau, 0]$; $\widetilde{N}$ denotes the restrictive variation, that is, $\delta\widetilde{N} = 0$. Thus, we have

$$\begin{aligned} \delta y_{m+1}(t) &= \delta y_m(t) + \int_0^t \delta\lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &= \delta y_m(t) + \int_0^t \delta\lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi)\right]d\xi, \quad 0 < t < t_1. \end{aligned} \tag{33}$$

Using integration by parts to (32), we have

$$\begin{aligned} \delta y_{m+1}(t) &= \delta y_m(t) + \int_0^{t_1} \delta\lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right]d\xi \\ &\quad + \int_{t_1}^t \delta\lambda(t,\xi)\left[y_m''(\xi) + a(\xi)y_m'(\xi) + b(\xi)y_m(\xi) + \widetilde{N}(\xi, y_m(\xi), y_m(\alpha(\xi)))\right]d\xi \end{aligned}$$

$$\begin{aligned}
&= \delta y_m(t) \\
&\quad + \int_0^t \delta\lambda(t,\xi)\left[y_m''(\xi) + a(\xi)\, y_m'(\xi) + b(\xi)\, y_m(\xi)\right] d\xi \\
&= \delta y_m(t) + \int_0^t \lambda(t,\xi)\,\delta dy_m'(\xi) \\
&\quad + \int_0^t \lambda(t,\xi)\, a(\xi)\,\delta dy_m(\xi) \\
&\quad + \int_0^t \delta\lambda(t,\xi)\, b(\xi)\, y_m(\xi)\, d\xi \\
&= \left(1 - \frac{\partial\lambda(t,\xi)}{\partial\xi} + a(\xi)\,\lambda(\xi)\right)\delta y_m(\xi)\Big|_{\xi=t} \\
&\quad + \lambda(t,\xi)\,\delta\, y_m'(\xi)\big|_{\xi=t} \\
&\quad + \int_0^t \left[\frac{\partial^2\lambda(t,\xi)}{\partial^2(\xi)} - \frac{\partial(a(\xi)\,\lambda(t,\xi))}{\partial\xi} + b(\xi)\,\lambda(t,\xi)\right] \\
&\quad \times \delta y_m(\xi)\, d\xi.
\end{aligned} \tag{34}$$

From the above formula, the stationary conditions are obtained as

$$\begin{gathered}
\frac{\partial^2\lambda(t,\xi)}{\partial^2\xi} - \frac{\partial(a(\xi)\,\lambda(t,\xi))}{\partial\xi} + b(\xi)\,\lambda(t,\xi) = 0, \\
1 - \frac{\partial\lambda(t,\xi)}{\partial\xi} + a(\xi)\,\lambda(t,\xi)\Big|_{\xi=t} = 0, \\
\lambda(t,\xi)\big|_{\xi=t} = 0.
\end{gathered} \tag{35}$$

We suppose that $\lambda_1(t,\xi)$, $\lambda_2(t,\xi)$ are the fundamental solutions of (35); the corresponding general solution of (35) is

$$\lambda(t,\xi) = c_1\lambda_1(t,\xi) + c_2\lambda_2(t,\xi). \tag{36}$$

Using the initial conditions of (35), we have

$$\begin{gathered}
c_1\lambda_1(t,t) + c_2\lambda_2(t,t) = 0, \\
c_1\lambda_1'(t,t) + c_2\lambda_2'(t,t) = 1.
\end{gathered} \tag{37}$$

Note that $W(t,t) = \begin{vmatrix} \lambda_1(t,t) & \lambda_2(t,t) \\ \lambda_1'(t,t) & \lambda_2'(t,t) \end{vmatrix}$ is the Wronski determinant of $\lambda_1(t,\xi)$, $\lambda_2(t,\xi)$. We have

$$\lambda(t,\xi) = \frac{-\lambda_2(t,t)\,\lambda_1(t,\xi) + \lambda_1(t,t)\,\lambda_2(t,\xi)}{W(t,t)}. \tag{38}$$

Using the Liouville formula, we have

$$W(t,t) = W(t,0)\, e^{\int_0^t a(\xi)d\xi}. \tag{39}$$

So $\lambda(t,\xi)$ can be expressed as

$$\lambda(t,\xi) = \frac{-\lambda_2(t,t)\,\lambda_1(t,\xi) + \lambda_1(t,t)\,\lambda_2(t,\xi)}{\lambda_1(t,0)\,\lambda_2'(t,0) - \lambda_1'(t,0)\,\lambda_2(t,0)} e^{-\int_0^t a(\xi)d\xi}. \tag{40}$$

Note that

$$u_1(\xi) = \frac{\lambda_1(t,\xi)}{\sqrt{W(t,0)}}, \qquad u_2(\xi) = \frac{\lambda_2(t,\xi)}{\sqrt{W(t,0)}}. \tag{41}$$

Equation (40) can be expressed as

$$\lambda(t,\xi) = -e^{-\int_0^t a(\xi)d\xi}\left[u_1(\xi)\, u_2(t) - u_2(\xi)\, u_1(t)\right]. \tag{42}$$

Substituting (42) into (31) and (32), we obtain

$$\begin{aligned}
y_{m+1}(t) = {} & y_m(t) \\
& + \int_0^t e^{-\int_0^t a(\xi)d\xi}\left[u_1(\xi)\, u_2(t) - u_2(\xi)\, u_1(t)\right] \\
& \times \left[y_m''(\xi) + a(\xi)\, y_m'(\xi) + b(\xi)\, y_m(\xi)\right. \\
& \left. + N(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi, \\
& 0 < t < t_1;
\end{aligned} \tag{43}$$

$$\begin{aligned}
y_{m+1}(t) = {} & y_m(t) \\
& + \int_0^{t_1} e^{-\int_0^t a(\xi)d\xi}\left[u_1(\xi)\, u_2(t) - u_2(\xi)\, u_1(t)\right] \\
& \times \left[y_m''(\xi) + a(\xi)\, y_m'(\xi) + b(\xi)\, y_m(\xi)\right. \\
& \left. + N(\xi, y_m(\xi), \varphi(\alpha(\xi)))\right] d\xi \\
& + \int_{t_1}^t e^{-\int_0^t a(\xi)d\xi}\left[u_1(\xi)\, u_2(t) - u_2(\xi)\, u_1(t)\right] \\
& \times \left[y_m''(\xi) + a(\xi)\, y_m'(\xi) + b(\xi)\, y_m(\xi)\right. \\
& \left. + N(\xi, y_m, y_m(\alpha(\xi)))\right] d\xi, \quad t > t_1.
\end{aligned} \tag{44}$$

Theorem 3. *Suppose that the initial value problems* (29) *satisfy the condition* (30), *and* $y(t), y_i(t) \in C^2[0,T]$, $i = 1, 2, \ldots$. *Then the sequence* $\{y_m(t)\}_{m=1}^{\infty}$ *defined by* (43) *and* (44) *with* $y_0(t)$ *converges to the solution of* (29).

Proof. The proof process is similar to that in Theorem 1. □

3. Illustrative Examples

In this section, some illustrative examples are given to show the efficiency of the VIM for solving second-order delay differential equations.

Example 4. Consider the initial value problem of second-order differential equation with pantograph delay

$$\begin{gathered}
y''(t) = -y\left(\frac{t}{2}\right) - y^2(t) + \sin^4(t) + \sin^2\left(\frac{t}{2}\right) + 8, \quad t > 0, \\
\varphi'(0) = 0, \\
\varphi(0) = 2,
\end{gathered} \tag{45}$$

with the exact solution $y(t) = (5 - \cos 2t)/2$. Using the VIM given in formulas (9) and (10), we construct the correction functional

$$y_{m+1}(t) = y_m(t) + \int_0^t (\xi - t)\left(y_m''(\xi) + y_m\left(\frac{\xi}{2}\right) + y_m^2(\xi) - \sin^4(\xi) - \sin^2(\xi) - 8\right) d\xi, \quad m = 1, 2, \ldots. \tag{46}$$

We take $y_0(t) = 2$ as the initial approximation and obtain that

$$\begin{aligned}
y_1(t) &= \frac{5}{4} + \frac{23}{16}t^2 + \frac{1}{2}\cos t + \frac{5}{16}\cos^2 t - \frac{1}{16}\cos^4 t,\\
y_2(t) &= -\frac{989}{512} + \frac{815}{512}t^2 + \frac{23}{1536}t^4 + \frac{1}{2}t\sin\left(\frac{1}{2}t\right) + \frac{1}{16}t\sin t\\
&\quad - \frac{1}{256}t\sin t\cos t + 3\cos\left(\frac{1}{2}t\right) + \frac{11}{16}\cos t\\
&\quad + \frac{157}{512}\cos^2 t - \frac{1}{16}\cos^4 t,\\
y_3(t) &= -\frac{252541}{2048} + \frac{9781}{4096}t^2 + \frac{815}{49125}t^4 + \frac{23}{491520}t^6\\
&\quad + 14t\sin\left(\frac{1}{4}t\right) + \frac{15}{16}t\sin\left(\frac{1}{2}t\right) + \frac{121}{2048}t\sin t\\
&\quad + \frac{1}{256}t\sin t\cos t + 120\cos\left(\frac{1}{4}t\right) + \frac{39}{8}\cos\left(\frac{1}{2}t\right)\\
&\quad + \frac{1393}{2048}\cos t + \frac{157}{512}\cos^2 t\frac{1}{16}\cos^4 t + \frac{1}{2048}t^2\cos t\\
&\quad - \frac{1}{32}t^2\cos\left(\frac{1}{2}t\right) - \frac{1}{2}t^2\cos\left(\frac{1}{4}t\right),\\
&\ \ \vdots
\end{aligned} \tag{47}$$

The exact and approximate solutions are plotted in Figure 1, which shows that the method gives a very good approximation to the exact solution.

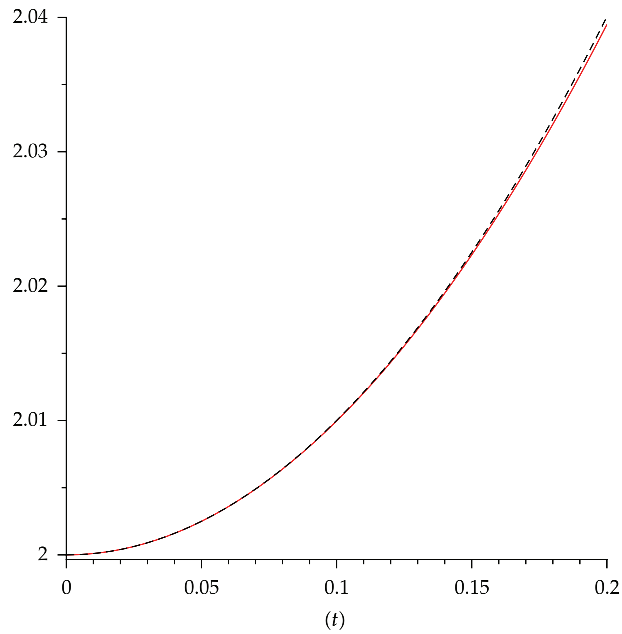

Figure 1: Results for Example 4.

Example 5. Consider the second-order delay differential equation

$$\begin{aligned}
y''(t) &= -16y(t) + y^2\left(\frac{t}{4}\right) - \sin^2 t, \quad t > 0,\\
\varphi'(0) &= 4,\\
\varphi(0) &= 0.
\end{aligned} \tag{48}$$

Using the VIM given in formulas (28), we construct the correction functional

$$y_{m+1}(t) = y_m(t) + \int_0^t \frac{1}{4}\sin(4\xi - 4t)\left(y_m''(\xi) + 16y_m(\xi) - y_m^2\left(\frac{\xi}{4}\right) + \sin^2\xi\right) d\xi, \quad m = 1, 2, \ldots. \tag{49}$$

We take $y_0(t) = 4t$ as the initial approximation, and obtain that

$$\begin{aligned}
y_1(t) &= -0.0390625 + 0.0625t^2 + \sin(4t)\\
&\quad + 0.04166666667\cos(2t)\\
&\quad - 0.002604166667\cos(4t),\\
y_2(t) &= 0.00613912861 - 3.998216869t + 0.1805413564t^2\\
&\quad - 0.4340277778t^3 + o\left(t^4\right)\\
&\quad + \left(0.006666666667 - 0.5208333333t + o\left(t^2\right)\right)\sin t\\
&\quad + \left(-0.1946373457 - 0.5555555556t + o\left(t^2\right)\right)\cos t\\
&\quad - 0.1085069444\sin(2t) - 2\cos 4t,\\
&\ \ \vdots
\end{aligned} \tag{50}$$

Table 1: The errors of the iteration solutions.

	$t = 0.01$	$t = 0.05$	$t = 0.1$	$t = 0.15$
Iterative formula (43)	$1.9048E-09$	$1.1905E-06$	$1.9048E-05$	$906428E-05$
Iterative formula (9)	$1.89278E-05$	$1.0972E-03$	$9.2763E-03$	$3.8062E-02$

Example 6. Consider the second-order delay differential equation

$$y''(t) = -\frac{2}{t}y'(t) + 16y^2\left(\frac{t}{2}\right) + 6 - t^4, \quad t > 0,$$
$$\varphi'(0) = 0, \tag{51}$$
$$\varphi(0) = 0,$$

with the exact solution $y(t) = t^2$. From (35), we can solve that $\lambda(t,\xi) = -\xi + \xi^2/t$. Using the VIM given in formulas (43) and (44), we construct the correction functional

$$y_{m+1}(t) = y_m(t) + \int_0^t \left(-\xi + \frac{\xi^2}{t}\right)\left(y_m''(\xi) + \frac{2}{\xi}y_m'(\xi) - 16y_m^2\left(\frac{\xi}{2}\right) - 6 + \xi^4\right)d\xi, \quad m = 1, 2, \ldots. \tag{52}$$

We take $y_0(t) = 2t$ as the initial approximation and obtain that

$$y_1(t) = -\frac{1}{42}t^6 + t^2,$$
$$y_2(t) = t^2 + \frac{4}{21}t^6 - \frac{1}{1760}t^{10} + \frac{1}{1935360}t^{14}, \tag{53}$$
$$\vdots$$

We use the iterative formulas (9) and (43) for Example 6, respectively. When the iteration number $n = 2$, the corresponding relative errors are showed in Table 1.

Table 1 shows that the iteration speed of the iterative formula (43) for Example 6 is much faster than that of iterative formula (9). This demonstrates that it is important to choose a reasonable Lagrange multiplier.

4. Conclusion

In this paper, we apply the VIM to obtain the analytical or approximate analytical solutions of second-order delay differential equations. Some illustrative examples show that this method gives a very good approximation to the exact solution. The VIM is a promising method for second-order delay differential equations.

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References

[1] Y. Xu, J. J. Zhao, and M. Z. Liu, "TH-stability of θ-method for a second-order delay differential equation," *Mathematica Numerica Sinica*, vol. 26, no. 2, pp. 189–192, 2004.

[2] C. M. Huang and W. H. Li, "Delay-dependent stability analysis of the trapezium rule for a class of second-order delay differential equations," *Mathematica Numerica Sinica*, vol. 29, no. 2, pp. 155–162, 2007.

[3] X. H. Ding, J. Y. Zou, and M. Z. Liu, "*P*-stability of continuous Runge-Kutta-Nyström method for a class of delayed second order differential equations," *Numerical Mathematics*, vol. 27, no. 2, pp. 123–132, 2005.

[4] J.-H. He, "Variational iteration method: a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[5] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[6] J.-H. He, "Variational iteration method for autonomous ordinary differential systems," *Applied Mathematics and Computation*, vol. 114, no. 2-3, pp. 115–123, 2000.

[7] Z.-H. Yu, "Variational iteration method for solving the multi-pantograph delay equation," *Physics Letters A*, vol. 372, no. 43, pp. 6475–6479, 2008.

[8] L. Xu, "Variational iteration method for solving integral equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 1071–1078, 2007.

[9] S.-P. Yang and A.-G. Xiao, "Convergence of the variational iteration method for solving multi-delay differential equations," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 2148–2151, 2011.

[10] S. Yang, A. Xiao, and H. Su, "Convergence of the variational iteration method for solving multi-order fractional differential equations," *Computers & Mathematics with Applications*, vol. 60, no. 10, pp. 2871–2879, 2010.

[11] Y. Zhao and A. Xiao, "Variational iteration method for singular perturbation initial value problems," *Computer Physics Communications*, vol. 181, no. 5, pp. 947–956, 2010.

[12] H. Liu, A. Xiao, and Y. Zhao, "Variational iteration method for delay differential-algebraic equations," *Mathematical & Computational Applications*, vol. 15, no. 5, pp. 834–839, 2010.

[13] M. Rafei, D. D. Ganji, H. Daniali, and H. Pashaei, "The variational iteration method for nonlinear oscillators with discontinuities," *Journal of Sound and Vibration*, vol. 305, no. 4-5, pp. 614–620, 2007.

[14] V. Marinca, N. Herişanu, and C. Bota, "Application of the variational iteration method to some nonlinear one-dimensional oscillations," *Meccanica*, vol. 43, no. 1, pp. 75–79, 2008.

[15] M. Tatari and M. Dehghan, "On the convergence of He's variational iteration method," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 121–128, 2007.

[16] J.-H. He and X.-H. Wu, "Variational iteration method: new development and applications," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 881–894, 2007.

[17] J.-H. He, "Variational iteration method—some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 3–17, 2007.

[18] J.-H. He, "A short remark on fractional variational iteration method," *Physics Letters A*, vol. 375, no. 38, pp. 3362–3364, 2011.

[19] J.-H. He, "Asymptotic methods for solitary solutions and compactons," *Abstract and Applied Analysis*, vol. 2012, Article ID 916793, 130 pages, 2012.