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Research Article

Implicit and Explicit Iterative Methods for Systems of Variational Inequalities and Zeros of Accretive Operators

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Based on Korpelevich's extragradient method, hybrid steepest-descent method, and viscosity approximation method, we propose implicit and explicit iterative schemes for computing a common element of the solution set of a system of variational inequalities and the set of zeros of an accretive operator, which is also a unique solution of a variational inequality. Under suitable assumptions, we study the strong convergence of the sequences generated by the proposed algorithms. The results of this paper improve and extend several known results in the literature.

1. Introduction and Preliminaries

Throughout the paper, we use the notation $\rightharpoonup$ to indicate the weak convergence and $\to$ to indicate the strong convergence.

Let X be a real Banach space whose dual space is denoted by X^*. Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. A Banach space X is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \Rightarrow \frac{\|x + y\|}{2} \leq 1 - \delta. \tag{1}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

A Banach space X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists for all $x, y \in U$; in this case, X is also said to have a *Gâteaux differentiable norm*. Moreover, it is said to be *uniformly smooth* if this limit is attained uniformly for all $x, y \in U$; in this case, X is also said to have a *uniformly Fréchet differentiable norm*. The norm of X is said to be *Fréchet differentiable* if for each $x \in U$, the limit is attained uniformly for all $y \in U$. A function $\rho : [0, \infty) \to [0, \infty)$ defined by

$$\rho(\tau) = \sup \left\{ \frac{1}{2} \left(\|x + y\| + \|x - y\| \right) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\} \tag{3}$$

is called the *modulus of smoothness* of X. It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. If $q \in (1, 2]$ is a fixed number, then a Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [1], no Banach space is q-uniformly smooth for $q > 2$.

The *normalized duality* mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}, \quad \forall x \in X, \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for all $x \in X$. It is also known that J is single-valued if and only if X is smooth, whereas if X

is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X. For further details on the geometry of Banach spaces, we refer to [2, 3] and the references therein.

Lemma 1 (see [4]). *Let X be a 2-uniformly smooth Banach space. Then*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|\kappa y\|^2, \quad \forall x, y \in X, \tag{5}$$

where κ is the 2-uniformly smooth constant of X and J is the normalized duality mapping from X into X^*.

The following lemma is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 2. *Let X be a smooth Banach space. Then,*

$$\|x\|^2 + 2\langle y, J(x)\rangle \le \|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall x, y \in X, \tag{6}$$

where J is the normalized duality mapping of X.

Proposition 3 (see [5]). *Let X be a real smooth and uniform convex Banach space and $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g(\|x-y\|) \le \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall x, y \in B_r, \tag{7}$$

where $B_r = \{x \in X : \|x\| \le r\}$.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $T : C \to C$ is said to be *L-Lipschitzian* (or *Lipschtiz continuous*) if there exists a constant $L > 0$ such that $\|Tx - Ty\| \le L\|x-y\|$ for all $x, y \in C$. If $L \in (0,1)$, then T is called *L-contraction*. If $L = 1$, then T is called *nonexpansive*.

The set of fixed points of T is denoted by $\mathrm{Fix}(T)$.

Lemma 4 (see [6]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X, and let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty}\lambda_n = 1$. Then, a mapping T on C defined by $Tx = \sum_{n=0}^{\infty}\lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive, and $\mathrm{Fix}(T) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$.*

Lemma 5 (see [7]). *Let C be a nonempty closed and convex subset of a Banach space X and $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

Let C be a nonempty subset of a Banach space X. A mapping $A : C \to X$ is said to be

(a) *accretive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge 0, \tag{8}$$

where J is the normalized duality mapping of X;

(b) *α-strongly accretive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge \alpha\|x-y\|^2, \quad \text{for some } \alpha \in (0,1); \tag{9}$$

(c) *pseudocontractive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \le \|x-y\|^2; \tag{10}$$

(d) *β-strongly pseudocontractive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \le \beta\|x-y\|^2, \quad \text{for some } \beta \in (0,1); \tag{11}$$

(e) *λ-strictly pseudocontractive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|x-y-(Ax-Ay)\|^2, \quad \text{for some } \lambda \in (0,1). \tag{12}$$

It is worth to mention that the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping, which was studied by so many authors; see, for example, [8, 9].

Lemma 6. *Let C be a nonempty closed convex subset of a real smooth Banach space X, and let $F : C \to X$ be a mapping.*

(a) *If $F : C \to X$ is α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda \ge 1$, then $I - F$ is nonexpansive and F is Lipschitz continuous with constant $1 + 1/\lambda$.*

(b) *If $F : C \to X$ is α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, then for any fixed $\tau \in (0,1)$, $I - \tau F$ is contractive with coefficient $1 - \tau(1 - \sqrt{(1-\alpha)/\lambda})$.*

Proof. (a) From the λ-strictly pseudocontractivity of F, we have for all $x, y \in C$,

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \langle (I-F)x-(I-F)y, J(x-y)\rangle \\ &\le \|(I-F)x-(I-F)y\|\,\|x-y\|,\end{aligned} \tag{13}$$

which implies that

$$\|(I-F)x-(I-F)y\| \le \frac{1}{\lambda}\|x-y\|. \tag{14}$$

Hence,

$$\begin{aligned}\|F(x)-F(y)\| &\le \|(I-F)x-(I-F)y\| + \|x-y\| \\ &\le \left(1+\frac{1}{\lambda}\right)\|x-y\|.\end{aligned} \tag{15}$$

From the λ-strictly pseudocontractiveness and α-strongly accretiveness of F, we have for all $x, y \in C$,

$$\begin{aligned}\lambda\|(I-F)x-(I-F)y\|^2 &\le \|x-y\|^2-\langle F(x)-F(y), J(x-y)\rangle \\ &\le (1-\alpha)\|x-y\|^2,\end{aligned} \tag{16}$$

which implies that

$$\|(I-F)x-(I-F)y\| \le \sqrt{\frac{1-\alpha}{\lambda}}\|x-y\|. \tag{17}$$

Since $\alpha+\lambda \ge 1 \Leftrightarrow \sqrt{(1-\alpha)/\lambda} \le 1$, $I-F$ is nonexpansive.

(b) Take a fixed $\tau \in (0,1)$ arbitrarily. Observe that for all $x, y \in C$,

$$\begin{aligned}&\|(I-\tau F)x-(I-\tau F)y\| \\ &\quad = \|(1-\tau)(x-y)+\tau[(I-F)x-(I-F)y]\| \\ &\quad \le (1-\tau)\|x-y\|+\tau\|(I-F)x-(I-F)y\| \\ &\quad \le (1-\tau)\|x-y\|+\tau\left(\sqrt{\frac{1-\alpha}{\lambda}}\right)\|x-y\| \\ &\quad = \left(1-\tau\left(1-\sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|x-y\|.\end{aligned} \tag{18}$$

Since $\alpha+\lambda > 1 \Leftrightarrow \sqrt{(1-\alpha)/\lambda} < 1$, $I-\tau F$ is contractive with coefficient $1-\tau(1-\sqrt{(1-\alpha)/\lambda})$. □

Let X be a real smooth Banach space X. An operator $V: X \to X^*$ is said to be *strongly positive* [10] if there exists a constant $\overline{\gamma} > 0$ such that

$$\begin{gathered}\langle Vx, J(x)\rangle \ge \overline{\gamma}\|x\|^2, \\ \|aI-bV\| = \sup_{\|x\|\le 1}|\langle (aI-bV)x, J(x)\rangle|, \\ a\in[0,1],\ b\in[-1,1],\end{gathered} \tag{19}$$

where I is the identity mapping.

Lemma 7 (see [7]). *Let V be a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|V\|^{-1}$. Then, $\|I-\rho A\| \le 1-\rho\overline{\gamma}$.*

Recall that a possibly multivalued operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ in X is *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1,2)$, there exists $j(x_1-x_2) \in J(x_1-x_2)$ such that $\langle y_1-y_2, j(x_1-x_2)\rangle \ge 0$. An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I+rA)$ for all $r > 0$. An accretive operator A is *m-accretive* if $R(I+rA) = X$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we can define, for each $r > 0$, a mapping $J_r: R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$, which is called the *resolvent* of A. It is well known that J_r is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}\mathbf{0}$ for all $r > 0$. Hence,

$$\mathrm{Fix}(J_r) = A^{-1}\mathbf{0} = \{z \in D(A) : \mathbf{0} \in Az\}. \tag{20}$$

If $A^{-1}\mathbf{0} \ne \emptyset$, then the inclusion $\mathbf{0} \in Az$ is solvable.

The following resolvent identity is well-known.

Lemma 8 (resolvent identity). *For $\lambda > 0$, $\mu > 0$, and $x \in X$,*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x+\left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right). \tag{21}$$

Recently, Aoyama et al. [11] studied an iterative scheme in the setting of a uniformly convex Banach space X having a uniformly Gâteaux differentiable norm to compute the approximate zero of an accretive operator A. They proved that the sequence generated by their scheme converges strongly to a zero of A under some appropriate assumptions. Subsequently, Ceng et al. [12] introduced and analyzed a composite iterative scheme in the setting of a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping. Further, Jung [13] proposed and analyzed a composite iterative scheme by the viscosity approximation method for finding a zero of an accretive operator A and established the strong convergence of the sequence generated by the proposed scheme.

Let C be a closed and convex subset of a Banach space X, and let D be a nonempty subset of C. A mapping $\Pi: C \to D$ is said to be *sunny* if

$$\Pi[\Pi(x)+t(x-\Pi(x))] = \Pi(x), \tag{22}$$

whenever $\Pi(x)+t(x-\Pi(x)) \in C$ for all $x \in C$ and $t \ge 0$. A mapping $\Pi: C \to C$ is called a *retraction* if $\Pi^2 = \Pi$. If a mapping $\Pi: C \to C$ is a retraction, then $\Pi(z) = z$ for all $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D.

The following lemma concerns the sunny nonexpansive retraction.

Lemma 9 (see [14]). *Let C be a nonempty closed convex subset of a real smooth Banach space X, let D be a nonempty subset of C, and let Π be a retraction of C onto D. Then, the following statements are equivalent:*

(a) Π *is sunny and nonexpansive;*

(b) $\|\Pi(x)-\Pi(y)\|^2 \le \langle x-y, J(\Pi(x)-\Pi(y))\rangle, \forall x, y \in C$;

(c) $\langle x-\Pi(x), J(y-\Pi(x))\rangle \le 0, \forall x \in C, y \in D$.

It is well known that if $X = H$ is a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T: C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \ne \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Let X be a smooth Banach space, let $C \subseteq X$ be a nonempty closed convex set, and let $B_1, B_2: C \to X$ be two nonlinear mappings. We consider the following system of variational inequalities (in short, SVI) which is earlier studied by Cai and Bu [15]. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{23}$$

where μ_1 and μ_2 are two positive constants. When $X = H$ is a real Hilbert space, then SVI (23) is considered and studied by Ceng et al. [16]. In addition, if $B_1 = B_2 = A$ and $x^* = y^*$, then the SVI reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{24}$$

It is worth to mention that the system of variational inequalities is used as a tool to study the Nash equilibrium problem; see, for example, [17–19] and the references therein. We believe that the problem (23) could be used to study Nash equilibrium problem for two-player game. The theory of variational inequalities is well established because it has a wide range of applications in science, engineering, management, and social sciences. In 1976, Korpelevič [20] proposed an iterative algorithm for solving VIP (24) in Euclidean space, which is known as the extragradient method or Korpelevich's extragradient method. For different generalizations of extragradient method for variational inequalities and fixed point problems, we refer to [21–23] and the references therein. For further details on variational inequalities, we refer to [2, 19, 24, 25] and the references therein.

Whenever X is a real smooth Banach space, $B_1 = B_2 = A$, and $x^* = y^*$, then SVI (23) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \tag{25}$$

which was considered by Aoyama et al. [26]. Very recently, Cai and Bu [15] constructed an iterative algorithm for solving SVI (23) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They proved the strong convergence of the sequence generated by the proposed algorithm.

Lemma 10. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $B_1, B_2 : C \to X$ be nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of SVI (23) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof. We can rewrite SVI (23) as

$$\begin{aligned} &\langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\ &\langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*) \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{26}$$

which is equivalent to

$$\begin{aligned} x^* &= \Pi_C (y^* - \mu_1 B_1 y^*), \\ y^* &= \Pi_C (x^* - \mu_2 B_2 x^*). \end{aligned} \tag{27}$$

From Lemma 9, we get the conclusion. □

Remark 11. In view of Lemma 10, we observe that

$$x^* = \Pi_C [\Pi_C (x^* - \mu_2 B_2 x^*) - \mu_1 B_1 \Pi_C (x^* - \mu_2 B_2 x^*)], \tag{28}$$

which implies that x^* is a fixed point of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$.

Lemma 12 (see [15, Lemma 2.8]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X, and for each $i = 1, 2$, let $B_i : C \to X$ be an α_i-inverse-strongly accretive. Then,*

$$\begin{aligned} &\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\|^2 \\ &\quad \leq \|x - y\|^2 + 2\mu_i (\mu_i \kappa^2 - \alpha_i) \|B_i x - B_i y\|^2, \\ &\qquad \forall x, y \in C, \ i = 1, 2, \end{aligned} \tag{29}$$

where $\mu_i > 0$. In particular, if $0 < \mu_i \leq \alpha_i/\kappa^2$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 13 (see [15, Lemma 2.9]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X, and let Π_C be a sunny nonexpansive retraction from X onto C. For each $i = 1, 2$, let $B_i : C \to X$ be a α_i-inverse-strongly accretive mapping and let $G : C \to C$ be the mapping defined by*

$$Gx = \Pi_C [\Pi_C (x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C (x - \mu_2 B_2 x)], \quad \forall x \in C. \tag{30}$$

If $0 < \mu_i \leq \alpha_i/\kappa^2$ for $i = 1, 2$, then $G : C \to C$ is nonexpansive.

Throughout this paper, the set of fixed points of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ is denoted by Ω.

Let X be a real uniformly convex and 2-uniformly smooth Banach space, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let V be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma} \in (1, 2)$. The purpose of this paper is to introduce and analyze implicit and explicit iterative schemes for computing a common element of the solution set of system of variational inequalities and set of zeros of an accretive operator A with domain $D(A)$ and range $R(A)$ in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Our methods are based on Korpelevich's extragradient method, hybrid steepest-descent method, and viscosity approximation method. In particular, we propose the iterative methods for computing a point $p \in \Omega \cap A^{-1}\mathbf{0}$, which is also a unique solution of the following variational inequality:

$$\langle (V - I) p, J(p - u) \rangle \leq 0, \quad \forall u \in \Omega \cap A^{-1}\mathbf{0}. \tag{31}$$

Under suitable assumptions, the strong convergence of the sequences generated by the proposed schemes is studied. Our results extend and improve several results presented in the recent past in the literature; see, for example, [10, 13, 15, 27–29] and the references therein.

Now we present some known results and definitions which will be used in the sequel.

Let μ be a mean if μ is a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbf{N}$ if and only if

$$\inf \{a_n : n \in \mathbf{N}\} \leq \mu(a) \leq \sup \{a_n : n \in \mathbf{N}\} \tag{32}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbf{N}$ is called a *Banach limit* if and only if

$$\mu_n(a_n) = \mu_n(a_{n+1}) \tag{33}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n \tag{34}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. So, if $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in l^\infty$ and $a_n \to c$ (resp., $a_n - b_n \to 0$), as $n \to \infty$, we have

$$\mu_n(a_n) = \mu(a) = c \quad (\text{resp.}, \mu_n(a_n) = \mu_n(b_n)). \tag{35}$$

Lemma 14 (see [30]). *Let C be a nonempty closed convex subset of a uniformly smooth Banach space X, and let $\{x_n\}$ be a bounded sequence in X. Let μ be a mean on $\mathbf{N}$ and $z \in C$. Then,*

$$\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2 \tag{36}$$

if and only if

$$\mu_n \langle y - z, J(x_n - z)\rangle \leq 0, \forall y \in C, \tag{37}$$

where J is the normalized duality mapping of X.

Recall that a Banach space X is said to satisfy *Opial's condition* if whenever $\{x_n\}$ is a sequence in X which converges weakly to x as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x. \tag{38}$$

Lemma 15 (demiclosedness principle, see [31, Theorem 10.3]). *Let X be a reflexive Banach space satisfying Opial's condition, let C be a nonempty closed convex subset of X, and let $T : C \to C$ be a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, if $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to y$, then $(I - T)x = y$.*

Remark 16. If the duality mapping $J : x \mapsto \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ from X into X^* is single-valued and weakly sequentially continuous, then X satisfies Opial's condition; see [32, Theorem 1]. Each Hilbert space and the sequence spaces l^p with $1 < p < \infty$ satisfy Opial's condition; see [32]. Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed, so that it satisfies Opial's condition; see [33].

We close this section by mentioning a result on the sequence which will be used in the proof of the main results of this paper.

Lemma 17 (see [34]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \geq 0, \tag{39}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty}\beta_n \leq 0$;

(iii) $\gamma_n \geq 0, \forall n \geq 0$, *and* $\sum_{n=0}^\infty \gamma_n < \infty$.

Then $\lim_{n\to\infty}s_n = 0$.

2. Implicit Iterative Scheme and Its Convergence Criteria

In this section, we propose an implicit hybrid steepest-descent viscosity scheme for finding the solution of SVI (23) such that such solution is also a zero of an accretive operator. We also study the strong convergence of the sequence generated by the proposed algorithm.

Theorem 18. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X such that $C \pm C \subset C$. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. For each $i = 1, 2$, let $B_i : C \to X$ be an α_i-inverse-strongly accretive mapping such that $\Delta = \Omega \cap A^{-1}\mathbf{0} \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha+\lambda > 1$, and let $f : C \to C$ be a fixed Lipschitzian strongly pseudocontractive mapping with pseudocontractive coefficient $\beta \in (0, \gamma_0)$, $\gamma_0 = 1 - \sqrt{(1 - \alpha)/\lambda}$, and Lipschitzian constant $L > 0$. Let $V : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma} \in (1, 2)$. For each integer $n \geq 0$, let $\{x_{t,n}\}$ be defined by*

$$x_{t,n} = (I - \theta_t V) J_{r_n} G x_{t,n} + \theta_t \left[J_{r_n} G x_{t,n} - t\left(F\left(J_{r_n} G x_{t,n}\right) - f(x_{t,n})\right)\right], \tag{40}$$

where $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$ and $0 < \theta_t \leq \|V\|^{-1}$ for all $t \in (0, (2 - \overline{\gamma})/(\gamma_0 - \beta))$ with $\lim_{t\to 0}\theta_t = 0$. Then, as $t \to 0$, $\{x_{t,n}\}$ converges strongly to $p \in \Delta$, which is a unique solution of the following variational inequality:

$$\langle (V - I) p, J(p - u)\rangle \leq 0, \quad \forall u \in \Delta. \tag{41}$$

Proof. We first show that $\{x_{t,n}\}$ is defined well. Since $0 < \theta_t \leq \|V\|^{-1}$, for all $t \in (0, (2-\overline{\gamma})/(\gamma_0-\beta))$ and $\|V\| \geq \overline{\gamma} > 1$, we have $0 < \theta_t \leq \|V\|^{-1} < 1$, for all $t \in (0, (2 - \overline{\gamma})/(\gamma_0 - \beta))$. Define a mapping $\Gamma_{t,n} : C \to C$ by

$$\Gamma_{t,n}x = (I - \theta_t V) J_{r_n} Gx + \theta_t \left[J_{r_n} Gx - t\left(F\left(J_{r_n} Gx\right) - f(x)\right)\right], \quad \forall x \in C. \tag{42}$$

Observe that

$$\begin{aligned}
&\langle \Gamma_{t,n}x - \Gamma_{t,n}y, J(x-y)\rangle \\
&= \left\langle (I-\theta_t V) J_{r_n} Gx - (I-\theta_t V) J_{r_n} Gy, J(x-y)\right\rangle \\
&\quad + \theta_t \left\langle (I-tF) J_{r_n} Gx - (I-tF) J_{r_n} Gy, J(x-y)\right\rangle \\
&\quad + \theta_t t \langle f(x) - f(y), J(x-y)\rangle \\
&\le \|I-\theta_t V\| \left\|J_{r_n} Gx - J_{r_n} Gy\right\| \|x-y\| \\
&\quad + \theta_t \left\|(I-tF) J_{r_n} Gx - (I-tF) J_{r_n} Gy\right\| \|x-y\| \\
&\quad + t\theta_t \beta \|x-y\|^2 \\
&\le (1-\theta_t \overline{\gamma}) \left\|J_{r_n} Gx - J_{r_n} Gy\right\| \|x-y\| \\
&\quad + \theta_t \left(1 - t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \left\|J_{r_n} Gx - J_{r_n} Gy\right\| \|x-y\| \\
&\quad + t\theta_t \beta \|x-y\|^2 \\
&\le (1-\theta_t \overline{\gamma}) \|Gx - Gy\| \|x-y\| \\
&\quad + \theta_t \left(1 - t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \|Gx - Gy\| \|x-y\| \\
&\quad + t\theta_t \beta \|x-y\|^2 \\
&\le (1-\theta_t \overline{\gamma}) \|x-y\|^2 \\
&\quad + \theta_t \left(1 - t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right) \|x-y\|^2 \\
&\quad + t\theta_t \beta \|x-y\|^2 \\
&= \left[1 - \theta_t \left(\overline{\gamma} - 1 + t(\gamma_0 - \beta)\right)\right] \|x-y\|^2.
\end{aligned} \tag{43}$$

Whenever $n \ge 0$ and $t \in (0, (2-\overline{\gamma})/(\gamma_0 - \beta))$, $\Gamma_{t,n} : C \to C$ is a continuous and strongly pseudocontractive mapping with pseudocontractive coefficient $1 - \theta_t(\overline{\gamma} - 1 + t(\gamma_0 - \beta)) \in (0,1)$. By Lemma 5, we deduce that there exists a unique fixed point in C, denoted by $x_{t,n}$, which uniquely solves the fixed point equation

$$\begin{aligned}
x_{t,n} &= (I-\theta_t V) J_{r_n} G x_{t,n} \\
&\quad + \theta_t \left[J_{r_n} G x_{t,n} - t\left(F\left(J_{r_n} G x_{t,n}\right) - f(x_{t,n})\right)\right].
\end{aligned} \tag{44}$$

Let us show the uniqueness of a solution of VIP (41). Suppose that both $p_1 \in \Delta$ and $p_2 \in \Delta$ are solutions to VIP (41). Then, we have

$$\begin{aligned}
&\left\langle (V-I) p_1, J(p_1 - p_2)\right\rangle \le 0, \\
&\left\langle (V-I) p_2, J(p_2 - p_1)\right\rangle \le 0.
\end{aligned} \tag{45}$$

Adding up the above two inequalities, we obtain

$$\left\langle (V-I) p_1 - (V-I) p_2, J(p_1 - p_2)\right\rangle \le 0. \tag{46}$$

Note that

$$\begin{aligned}
&\left\langle (V-I) p_1 - (V-I) p_2, J(p_1 - p_2)\right\rangle \\
&\quad = \langle V(p_1 - p_2), J(p_1 - p_2)\rangle - \langle p_1 - p_2, J(p_1 - p_2)\rangle \\
&\quad \ge \overline{\gamma}\|p_1 - p_2\|^2 - \|p_1 - p_2\|^2 \\
&\quad = (\overline{\gamma} - 1)\|p_1 - p_2\|^2 \ge 0.
\end{aligned} \tag{47}$$

Taking into account $0 < \overline{\gamma} - 1 < 1$, we have $p_1 = p_2$, and hence the uniqueness is proved.

Next, we show that $\{x_{t,n}\}$ is bounded. Indeed, we note that $0 < \theta_t \le \|V\|^{-1}$, for all $t \in (0, (2-\overline{\gamma})/(\gamma_0 - \beta))$. Take a fixed $p \in \Delta$ arbitrarily. Utilizing Lemmas 6 and 7, we obtain

$$\begin{aligned}
&\|x_{t,n} - p\|^2 \\
&= \Big\langle (I-\theta_t V) J_{r_n} G x_{t,n} \\
&\qquad + \theta_t \left[J_{r_n} G x_{t,n} - t\left(F\left(J_{r_n} G x_{t,n}\right) - f(x_{t,n})\right)\right] \\
&\qquad - p, J(x_{t,n} - p) \Big\rangle \\
&= \left\langle (I-\theta_t V) J_{r_n} G x_{t,n} - (I-\theta_t V) J_{r_n} G p, J(x_{t,n} - p)\right\rangle \\
&\quad + \theta_t \left\langle J_{r_n} G x_{t,n} - t\left(F\left(J_{r_n} G x_{t,n}\right) - f(x_{t,n})\right) - p, J(x_{t,n} - p)\right\rangle \\
&\quad + \theta_t \langle (I-V) p, J(x_{t,n} - p)\rangle \\
&= \left\langle (I-\theta_t V) J_{r_n} G x_{t,n} - (I-\theta_t V) J_{r_n} G p, J(x_{t,n} - p)\right\rangle \\
&\quad + \theta_t \Big[\left\langle (I-tF) J_{r_n} G x_{t,n} - (I-tF) J_{r_n} G p, J(x_{t,n} - p)\right\rangle \\
&\qquad + t\left(\langle f(x_{t,n}) - f(p), J(x_{t,n} - p)\rangle\right. \\
&\qquad\quad \left. + \langle f(p) - F(p), J(x_{t,n} - p)\rangle\right)\Big] \\
&\quad + \theta_t \langle (I-V) p, J(x_{t,n} - p)\rangle \\
&\le \left\|(I-\theta_t V) J_{r_n} G x_{t,n} - (I-\theta_t V) J_{r_n} G p\right\| \|x_{t,n} - p\| \\
&\quad + \theta_t \Big[\left\|(I-tF) J_{r_n} G x_{t,n} - (I-tF) J_{r_n} G p\right\| \|x_{t,n} - p\| \\
&\qquad + t\left(\langle f(x_{t,n}) - f(p), J(x_{t,n} - p)\rangle\right. \\
&\qquad\quad \left. + \|f(p) - F(p)\| \|x_{t,n} - p\|\right)\Big] \\
&\quad + \theta_t \|I-V\| \|p\| \|x_{t,n} - p\| \\
&\le (1-\theta_t \overline{\gamma}) \left\|J_{r_n} G x_{t,n} - J_{r_n} G p\right\| \|x_{t,n} - p\| \\
&\quad + \theta_t \Big[(1-t\gamma_0) \left\|J_{r_n} G x_{t,n} - J_{r_n} G p\right\| \|x_{t,n} - p\| \\
&\qquad + t\left(\beta\|x_{t,n} - p\|^2 + \|f(p) - F(p)\| \|x_{t,n} - p\|\right)\Big] \\
&\quad + \theta_t \|I-V\| \|p\| \|x_{t,n} - p\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\theta_t\overline{\gamma})\|Gx_{t,n}-Gp\|\|x_{t,n}-p\| \\
&\quad+\theta_t[(1-t\gamma_0)\|Gx_{t,n}-Gp\|\|x_{t,n}-p\| \\
&\qquad+t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|)] \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&\leq (1-\theta_t\overline{\gamma})\|x_{t,n}-p\|^2 \\
&\quad+\theta_t[(1-t\gamma_0)\|x_{t,n}-p\|^2 \\
&\qquad+t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|)] \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\|,
\end{aligned} \tag{48}$$

which immediately yields

$$\begin{aligned}
\|x_{t,n}-p\| \leq{}& (1-\theta_t\overline{\gamma})\|x_{t,n}-p\| \\
&+\theta_t[(1-t\gamma_0)\|x_{t,n}-p\| \\
&\quad+t(\beta\|x_{t,n}-p\|+\|f(p)-F(p)\|)] \\
&+\theta_t\|I-V\|\|p\|.
\end{aligned} \tag{49}$$

It follows that

$$\begin{aligned}
\|x_{t,n}-p\| &\leq \frac{\|I-V\|\|p\|+t\|f(p)-F(p)\|}{\overline{\gamma}-1+t(\gamma_0-\beta)} \\
&\leq \frac{\|I-V\|\|p\|+\|f(p)-F(p)\|}{\overline{\gamma}-1}.
\end{aligned} \tag{50}$$

Thus, $\{x_{t,n}\}$ is bounded and so are $\{f(x_{t,n})\}$, $\{Gx_{t,n}\}$, and $\{J_{r_n}Gx_{t,n}\}$.

Let us show that $\|x_{t,n}-Gx_{t,n}\| \to 0$ as $t \to 0$.

Indeed, for simplicity we put $q=\Pi_C(I-\mu_2B_2)p$, $u_{t,n}=\Pi_C(I-\mu_2B_2)x_{t,n}$ and $v_{t,n}=\Pi_C(I-\mu_1B_1)u_{t,n}$. Then, $p=\Pi_C(I-\mu_1B_1)q$ and $v_{t,n}=Gx_{t,n}$. Hence from (48), we get

$$\begin{aligned}
&\|x_{t,n}-p\|^2 \\
&\leq (1-\theta_t\overline{\gamma})\|v_{t,n}-p\|\|x_{t,n}-p\| \\
&\quad+\theta_t[(1-t\gamma_0)\|v_{t,n}-p\|\|x_{t,n}-p\| \\
&\qquad+t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|)] \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&=[1-\theta_t(\overline{\gamma}-1+t\gamma_0)]\|v_{t,n}-p\|\|x_{t,n}-p\| \\
&\quad+\theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&\leq[1-\theta_t(\overline{\gamma}-1+t\gamma_0)]\frac{1}{2}(\|v_{t,n}-p\|^2+\|x_{t,n}-p\|^2) \\
&\quad+\theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\|.
\end{aligned} \tag{51}$$

From Lemma 12, we have

$$\begin{aligned}
\|u_{t,n}-q\|^2 &= \|\Pi_C(x_{t,n}-\mu_2B_2x_{t,n})-\Pi_C(p-\mu_2B_2p)\|^2 \\
&\leq \|x_{t,n}-p-\mu_2(B_2x_{t,n}-B_2p)\|^2 \\
&\leq \|x_{t,n}-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_{t,n}-B_2p\|^2, \\
\|v_{t,n}-p\|^2 &= \|\Pi_C(u_{t,n}-\mu_1B_1u_{t,n})-\Pi_C(q-\mu_1B_1q)\|^2 \\
&\leq \|u_{t,n}-q-\mu_1(B_1u_{t,n}-B_1q)\|^2 \\
&\leq \|u_{t,n}-q\|^2-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_{t,n}-B_1q\|^2.
\end{aligned} \tag{52}$$

From the last two inequalities, we obtain

$$\begin{aligned}
\|v_{t,n}-p\|^2 \leq{}& \|x_{t,n}-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_{t,n}-B_2p\|^2 \\
&-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_{t,n}-B_1q\|^2,
\end{aligned} \tag{53}$$

which together with (51) implies that

$$\begin{aligned}
&\|x_{t,n}-p\|^2 \\
&\leq[1-\theta_t(\overline{\gamma}-1+t\gamma_0)]\frac{1}{2}(\|x_{t,n}-p\|^2+\|v_{t,n}-p\|^2) \\
&\quad+\theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&\leq[1-\theta_t(\overline{\gamma}-1+t\gamma_0)] \\
&\quad\times\frac{1}{2}\{\|x_{t,n}-p\|^2+\|x_{t,n}-p\|^2 \\
&\qquad-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_{t,n}-B_2p\|^2 \\
&\qquad-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_{t,n}-B_1q\|^2\} \\
&\quad+\theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&=[1-\theta_t(\overline{\gamma}-1+t(\gamma_0-\beta))]\|x_{t,n}-p\|^2 \\
&\quad+\theta_t t\|f(p)-F(p)\|\|x_{t,n}-p\| \\
&\quad+\theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&\quad-[1-\theta_t(\overline{\gamma}-1+t\gamma_0)]
\end{aligned}$$

$$\times \left\{\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 x_{t,n} - B_2 p\|^2 + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right\}$$
$$\leq \|x_{t,n} - p\|^2 + \theta_t t \|f(p) - F(p)\| \|x_{t,n} - p\| + \theta_t \|I - V\| \|p\| \|x_{t,n} - p\|$$
$$- [1 - \theta_t (\overline{\gamma} - 1 + t\gamma_0)] \times \left\{\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 x_{t,n} - B_2 p\|^2 + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right\}. \quad (54)$$

Therefore,

$$[1 - \theta_t (\overline{\gamma} - 1 + t\gamma_0)] \left\{\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 x_{t,n} - B_2 p\|^2 + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_{t,n} - B_1 q\|^2\right\}$$
$$\leq \theta_t t \|f(p) - F(p)\| \|x_{t,n} - p\| + \theta_t \|I - V\| \|p\| \|x_{t,n} - p\|. \quad (55)$$

Since $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$, it follows from $\lim_{t\to 0}\theta_t = 0$ that

$$\lim_{t\to 0} \|B_2 x_{t,n} - B_2 p\| = 0, \qquad \lim_{t\to 0} \|B_1 u_{t,n} - B_1 q\| = 0. \quad (56)$$

Utilizing Proposition 3 and Lemma 9, we have

$$\|u_{t,n} - q\|^2 = \|\Pi_C (x_{t,n} - \mu_2 B_2 x_{t,n}) - \Pi_C (p - \mu_2 B_2 p)\|^2$$
$$\leq \langle x_{t,n} - \mu_2 B_2 x_{t,n} - (p - \mu_2 B_2 p), J(u_{t,n} - q)\rangle$$
$$= \langle x_{t,n} - p, J(u_{t,n} - q)\rangle + \mu_2 \langle B_2 p - B_2 x_{t,n}, J(u_{t,n} - q)\rangle$$
$$\leq \frac{1}{2} \left[\|x_{t,n} - p\|^2 + \|u_{t,n} - q\|^2 - g_1 (\|x_{t,n} - u_{t,n} - (p - q)\|)\right] + \mu_2 \|B_2 p - B_2 x_{t,n}\| \|u_{t,n} - q\|, \quad (57)$$

which implies that

$$\|u_{t,n} - q\|^2 \leq \|x_{t,n} - p\|^2 - g_1 (\|x_{t,n} - u_{t,n} - (p - q)\|) + 2\mu_2 \|B_2 p - B_2 x_{t,n}\| \|u_{t,n} - q\|. \quad (58)$$

Similarly, we derive

$$\|v_{t,n} - p\|^2 = \|\Pi_C(u_{t,n} - \mu_1 B_1 u_{t,n}) - \Pi_C(q - \mu_1 B_1 q)\|^2$$
$$\leq \langle u_{t,n} - \mu_1 B_1 u_{t,n} - (q - \mu_1 B_1 q), J(v_{t,n} - p)\rangle$$
$$= \langle u_{t,n} - q, J(v_{t,n} - p)\rangle + \mu_1 \langle B_1 q - B_1 u_{t,n}, J(v_{t,n} - p)\rangle$$
$$\leq \frac{1}{2} \left[\|u_{t,n} - q\|^2 + \|v_{t,n} - p\|^2 - g_2 (\|u_{t,n} - v_{t,n} + (p - q)\|)\right] + \mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|, \quad (59)$$

which implies that

$$\|v_{t,n} - p\|^2 \leq \|u_{t,n} - q\|^2 - g_2 (\|u_{t,n} - v_{t,n} + (p - q)\|) + 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|. \quad (60)$$

Combining (58) and (60), we get

$$\|v_{t,n} - p\|^2 \leq \|x_{t,n} - p\|^2 - g_1 (\|x_{t,n} - u_{t,n} - (p - q)\|) - g_2 (\|u_{t,n} - v_{t,n} + (p - q)\|) + 2\mu_2 \|B_2 p - B_2 x_{t,n}\| \|u_{t,n} - q\| + 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|, \quad (61)$$

which together with (51) implies that

$$\|x_{t,n} - p\|^2$$
$$\leq [1 - \theta_t (\overline{\gamma} - 1 + t\gamma_0)] \frac{1}{2} \left(\|x_{t,n} - p\|^2 + \|v_{t,n} - p\|^2\right) + \theta_t t \left(\beta\|x_{t,n} - p\|^2 + \|f(p) - F(p)\| \|x_{t,n} - p\|\right) + \theta_t \|I - V\| \|p\| \|x_{t,n} - p\|$$
$$\leq [1 - \theta_t (\overline{\gamma} - 1 + t\gamma_0)] \times \frac{1}{2} \left\{\|x_{t,n} - p\|^2 + \|x_{t,n} - p\|^2 - g_1 (\|x_{t,n} - u_{t,n} - (p - q)\|) - g_2 (\|u_{t,n} - v_{t,n} + (p - q)\|) + 2\mu_2 \|B_2 p - B_2 x_{t,n}\| \|u_{t,n} - q\| + 2\mu_1 \|B_1 q - B_1 u_{t,n}\| \|v_{t,n} - p\|\right\}$$
$$+ \theta_t t \left(\beta\|x_{t,n} - p\|^2 + \|f(p) - F(p)\| \|x_{t,n} - p\|\right) + \theta_t \|I - V\| \|p\| \|x_{t,n} - p\|$$

$$\begin{aligned}
&= [1-\theta_t(\overline{\gamma}-1+t(\gamma_0-\beta))]\|x_{t,n}-p\|^2 \\
&\quad -\frac{1}{2}[1-\theta_t(\overline{\gamma}-1+t\gamma_0)] \\
&\quad \times \{g_1(\|x_{t,n}-u_{t,n}-(p-q)\|) \\
&\qquad + g_2(\|u_{t,n}-v_{t,n}+(p-q)\|)\} \\
&\quad + [1-\theta_t(\overline{\gamma}-1+t\gamma_0)] \\
&\quad \times (\mu_2\|B_2p-B_2x_{t,n}\|\|u_{t,n}-q\| \\
&\qquad + \mu_1\|B_1q-B_1u_{t,n}\|\|v_{t,n}-p\|) \\
&\quad + \theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad + \theta_t\|I-V\|\|p\|\|x_{t,n}-p\| \\
&\le \|x_{t,n}-p\|^2-\frac{1}{2}[1-\theta_t(\overline{\gamma}-1+t\gamma_0)] \\
&\quad \times \{g_1(\|x_{t,n}-u_{t,n}-(p-q)\|) \\
&\qquad + g_2(\|u_{t,n}-v_{t,n}+(p-q)\|)\} \\
&\quad + [1-\theta_t(\overline{\gamma}-1+t\gamma_0)] \\
&\quad \times (\mu_2\|B_2p-B_2x_{t,n}\|\|u_{t,n}-q\| \\
&\qquad + \mu_1\|B_1q-B_1u_{t,n}\|\|v_{t,n}-p\|) \\
&\quad + \theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad + \theta_t\|I-V\|\|p\|\|x_{t,n}-p\|.
\end{aligned} \tag{62}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}[1-\theta_t(\overline{\gamma}-1+t\gamma_0)]\{g_1(\|x_{t,n}-u_{t,n}-(p-q)\|) \\
&\qquad + g_2(\|u_{t,n}-v_{t,n}+(p-q)\|)\} \\
&\le [1-\theta_t(\overline{\gamma}-1+t\gamma_0)](\mu_2\|B_2p-B_2x_{t,n}\|\|u_{t,n}-q\| \\
&\qquad + \mu_1\|B_1q-B_1u_{t,n}\|\|v_{t,n}-p\|) \\
&\quad + \theta_t t(\beta\|x_{t,n}-p\|^2+\|f(p)-F(p)\|\|x_{t,n}-p\|) \\
&\quad + \theta_t\|I-V\|\|p\|\|x_{t,n}-p\|.
\end{aligned} \tag{63}$$

Hence, from (56) and $\lim_{t\to 0}\theta_t=0$, we conclude that

$$\begin{aligned}
&\lim_{t\to 0} g_1(\|x_{t,n}-u_{t,n}-(p-q)\|)=0, \\
&\lim_{t\to 0} g_2(\|u_{t,n}-v_{t,n}+(p-q)\|)=0.
\end{aligned} \tag{64}$$

Utilizing the properties of g_1 and g_2, we get

$$\begin{aligned}
&\lim_{t\to 0}\|x_{t,n}-u_{t,n}-(p-q)\|=0, \\
&\lim_{t\to 0}\|u_{t,n}-v_{t,n}+(p-q)\|=0,
\end{aligned} \tag{65}$$

which leads to

$$\begin{aligned}
\|x_{t,n}-v_{t,n}\| &\le \|x_{t,n}-u_{t,n}-(p-q)\| \\
&\quad + \|u_{t,n}-v_{t,n}+(p-q)\| \longrightarrow 0 \quad \text{as } t\longrightarrow 0;
\end{aligned} \tag{66}$$

that is,

$$\lim_{t\to 0}\|x_{t,n}-Gx_{t,n}\|=\lim_{t\to 0}\|x_{t,n}-v_{t,n}\|=0. \tag{67}$$

Note that $\mathrm{Fix}(J_{r_n})=A^{-1}0$ for all $n\ge 0$ and that $\{x_{t,n}: t\in(0,b), n\ge 0\}$ is bounded and so are $\{f(x_{t,n}): t\in(0,b), n\ge 0\}$, $\{J_{r_n}x_{t,n}: t\in(0,b), n\ge 0\}$ and $\{J_{r_n}Gx_{t,n}: t\in(0,b), n\ge 0\}$, where $b=(2-\overline{\gamma})/(\gamma_0-\beta)$. Hence, we have

$$\begin{aligned}
&\|x_{t,n}-J_{r_n}Gx_{t,n}\| \\
&\quad = \theta_t\|(A-I)J_{r_n}Gx_{t,n}-t(f(x_{t,n})-F(J_{r_n}Gx_{t,n}))\| \longrightarrow 0, \\
&\qquad \text{as } t\longrightarrow 0.
\end{aligned} \tag{68}$$

Also, observe that

$$\begin{aligned}
\|x_{t,n}-J_{r_n}x_{t,n}\| &\le \|x_{t,n}-J_{r_n}Gx_{t,n}\|+\|J_{r_n}Gx_{t,n}-J_{r_n}x_{t,n}\| \\
&\le \|x_{t,n}-J_{r_n}Gx_{t,n}\|+\|Gx_{t,n}-x_{t,n}\|.
\end{aligned} \tag{69}$$

This together with (67) and (68) implies that

$$\lim_{t\to 0}\|x_{t,n}-J_{r_n}x_{t,n}\|=0. \tag{70}$$

Since $r_n\ge\varepsilon$ for all $n\ge 0$, utilizing Lemma 8, we have

$$\begin{aligned}
&\|x_{t,n}-J_\varepsilon x_{t,n}\| \\
&\le \|x_{t,n}-J_{r_n}x_{t,n}\|+\|J_{r_n}x_{t,n}-J_\varepsilon x_{t,n}\| \\
&= \|x_{t,n}-J_{r_n}x_{t,n}\|+\left\|J_\varepsilon\left(\frac{\varepsilon}{r_n}x_{t,n}+\left(1-\frac{\varepsilon}{r_n}\right)J_{r_n}x_{t,n}\right)-J_\varepsilon x_{t,n}\right\| \\
&\le 2\|x_{t,n}-J_{r_n}x_{t,n}\| \longrightarrow 0, \quad \text{as } t\longrightarrow 0.
\end{aligned} \tag{71}$$

For any integer $n\ge 0$, for simplicity we put $w_t=x_{t,n}$ for all $t\in(0,b)$. Now let $\{t_k\}$ be a sequence in $(0,b)$ that converges to 0 as $k\to\infty$, and define a function $g: C\to\mathbb{R}$ by

$$g(w)=\mu_k\frac{1}{2}\|w_{t_k}-w\|^2, \quad \forall w\in C, \tag{72}$$

where μ is a Banach limit. Define a set

$$K:=\{w\in C: g(w)=\min\{g(y): y\in C\}\} \tag{73}$$

and a mapping

$$Wx=(1-\theta)J_\varepsilon x+\theta Gx, \quad \forall x\in C, \tag{74}$$

where θ is a constant in $(0,1)$. Then by Lemma 4, we have $\mathrm{Fix}(W)=\mathrm{Fix}(J_\varepsilon)\cap\mathrm{Fix}(G)=\Delta$. We observe that

$$\begin{aligned}\|w_t-Ww_t\|&=\|(1-\theta)(w_t-J_\varepsilon w_t)+\theta(w_t-Gw_t)\|\\&\le(1-\theta)\|w_t-J_\varepsilon w_t\|+\theta\|w_t-Gw_t\|.\end{aligned}\tag{75}$$

So, from (67) and (71), we obtain

$$\lim_{n\to\infty}\|w_t-Ww_t\|=0.\tag{76}$$

Since X is a uniformly smooth Banach space, K is a nonempty bounded closed convex subset of C (see [30]). We claim that K is also invariant under the nonexpansive mapping W. Indeed, noticing (76), we have for all $w\in K$,

$$\begin{aligned}g(Ww)&=\mu_k\frac{1}{2}\|w_{t_k}-Ww\|^2=\mu_k\frac{1}{2}\|Ww_{t_k}-Ww\|^2\\&\le\mu_k\frac{1}{2}\|w_{t_k}-w\|^2=g(w).\end{aligned}\tag{77}$$

Since every nonempty closed bounded convex subset of uniformly smooth Banach space X has the fixed point property for nonexpansive mappings and W is a nonexpansive mapping of K, W has a fixed point in K, say, p. Utilizing Lemma 15, we get

$$\mu_k\left\langle x-p,J\left(w_{t_k}-p\right)\right\rangle\le0,\quad\forall x\in C.\tag{78}$$

Putting $x=(2I-V)p$, we have

$$\mu_k\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\le0.\tag{79}$$

Noticing $w_{t_k}=x_{t_k,n}$, we conclude from (48) that

$$\begin{aligned}&\|w_{t_k}-p\|^2\\&=\left\langle\left(I-\theta_{t_k}V\right)J_{r_n}Gw_{t_k}-\left(I-\theta_{t_k}V\right)J_{r_n}Gp,J\left(w_{t_k}-p\right)\right\rangle\\&\quad+\theta_{t_k}\left[\left\langle(I-t_kF)J_{r_n}Gw_{t_k}-(I-t_kF)J_{r_n}Gp,J\left(w_{t_k}-p\right)\right\rangle\right.\\&\qquad+t_k\left(\left\langle f\left(w_{t_k}\right)-f(p),J\left(w_{t_k}-p\right)\right\rangle\right.\\&\qquad\quad\left.\left.+\left\langle f(p)-F(p),J\left(w_{t_k}-p\right)\right\rangle\right)\right]\\&\quad+\theta_{t_k}\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\\&\le\left(1-\theta_{t_k}\overline{\gamma}\right)\|J_{r_n}Gw_{t_k}-J_{r_n}Gp\|\|w_{t_k}-p\|\\&\quad+\theta_{t_k}\left[(1-t_k\gamma_0)\|J_{r_n}Gw_{t_k}-J_{r_n}Gp\|\|w_{t_k}-p\|\right.\\&\qquad\left.+t_k\left(\beta\|w_{t_k}-p\|^2+\|f(p)-F(p)\|\|w_{t_k}-p\|\right)\right]\\&\quad+\theta_{t_k}\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\\&\le\left(1-\theta_{t_k}\overline{\gamma}\right)\|w_{t_k}-p\|^2\\&\quad+\theta_{t_k}\left[(1-t_k\gamma_0)\|w_{t_k}-p\|^2\right.\\&\qquad\left.+t_k\left(\beta\|w_{t_k}-p\|^2+\|f(p)-F(p)\|\|w_{t_k}-p\|\right)\right]\\&\quad+\theta_{t_k}\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\\&=\left[1-\theta_{t_k}\left(\overline{\gamma}-1+t_k(\gamma_0-\beta)\right)\right]\|w_{t_k}-p\|^2\\&\quad+\theta_{t_k}t_k\|f(p)-F(p)\|\|w_{t_k}-p\|\\&\quad+\theta_{t_k}\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle.\end{aligned}\tag{80}$$

It follows that

$$\begin{aligned}&\|w_{t_k}-p\|^2\\&\le\frac{1}{\overline{\gamma}-1+t_k(\gamma_0-\beta)}\left[\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\right.\\&\qquad\left.+t_k\|f(p)-F(p)\|\|w_{t_k}-p\|\right]\\&\le\frac{1}{\overline{\gamma}-1}\left[\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\right.\\&\qquad\left.+t_k\|f(p)-F(p)\|\|w_{t_k}-p\|\right].\end{aligned}\tag{81}$$

Since $\lim_{k\to\infty}t_k=0$, from (79) and the boundedness of $\{w_{t_k}\}$, it follows that

$$\begin{aligned}&\mu_k\|w_{t_k}-p\|^2\\&\quad\le\frac{1}{\overline{\gamma}-1}\mu_k\left[(I-V)p,J\left(w_{t_k}-p\right)\right.\\&\qquad\left.+t_k\|f(p)-F(p)\|\|w_{t_k}-p\|\right]\\&\quad=\frac{1}{\overline{\gamma}-1}\left[\mu_k\left\langle(I-V)p,J\left(w_{t_k}-p\right)\right\rangle\right.\\&\qquad\left.+\mu_kt_k\|f(p)-F(p)\|\|w_{t_k}-p\|\right]\le0.\end{aligned}\tag{82}$$

Therefore, it is known that for any sequence $\{w_{t_k}\}$ in $\{w_t:t\in(0,b)\}$, there exists a subsequence which is still denoted by $\{w_{t_k}\}$ that converges strongly to some fixed point p of W.

Now we claim that such a $p\in\Delta$ is a unique solution of variational inequality (41).

Indeed, from (48), it follows that for all $u \in \Delta = \Omega \cap A^{-1}\mathbf{0}$

$$\begin{aligned}
&\|w_t - u\|^2 \\
&= \langle (I-\theta_t V) J_{r_n} G w_t - (I-\theta_t V) J_{r_n} G u, J(w_t - u)\rangle \\
&\quad + \theta_t [\langle (I - tF) J_{r_n} G w_t - (I - tF) J_{r_n} G u, J(w_t - u)\rangle \\
&\qquad + t(\langle f(w_t) - f(u), J(w_t - u)\rangle \\
&\qquad + \langle f(u) - F(u), J(w_t - u)\rangle)] \\
&\quad + \theta_t \langle (I - V) u, J(w_t - u)\rangle \\
&\le (1 - \theta_t \overline{\gamma}) \|J_{r_n} G w_t - J_{r_n} G u\| \|w_t - u\| \\
&\quad + \theta_t [(1 - t\gamma_0) \|J_{r_n} G w_t - J_{r_n} G u\| \|w_t - u\| \\
&\qquad + t(\beta \|w_t - u\|^2 + \|f(u) - F(u)\| \|w_t - u\|)] \\
&\quad + \theta_t \langle (I - V) u, J(w_t - u)\rangle \\
&\le (1 - \theta_t \overline{\gamma}) \|w_t - u\|^2 \\
&\quad + \theta_t [(1 - t\gamma_0) \|w_t - u\|^2 \\
&\qquad + t(\beta \|w_t - u\|^2 + \|f(u) - F(u)\| \|w_t - u\|)] \\
&\quad + \theta_t \langle (I - V) u, J(w_t - u)\rangle \\
&= [1 - \theta_t (\overline{\gamma} - 1 + t(\gamma_0 - \beta))] \|w_t - u\|^2 \\
&\quad + \theta_t t \|f(u) - F(u)\| \|w_t - u\| \\
&\quad + \theta_t \langle (I - V) u, J(w_t - u)\rangle \\
&\le \|w_t - u\|^2 + \theta_t t \|f(u) - F(u)\| \|w_t - u\| \\
&\quad + \theta_t \langle (I - V) u, J(w_t - u)\rangle,
\end{aligned} \tag{83}$$

which implies that

$$\langle (V - I) u, J(w_t - u)\rangle \le t \|f(u) - F(u)\| \|w_t - u\|, \quad \forall u \in \Delta. \tag{84}$$

Since $w_{t_k} \to p$ as $t_k \to 0$, we obtain from the last inequality that

$$\langle (V - I) u, J(p - u)\rangle \le 0, \quad \forall u \in \Delta. \tag{85}$$

Utilizing the well-known Minty-type Lemma [25], we get

$$\langle (V - I) p, J(p - u)\rangle \le 0, \quad \forall u \in \Delta. \tag{86}$$

So, $p \in \Delta$ is a solution of variational inequality (41).

In order to prove that the net $\{x_t : t \in (0, b)\}$ converges strongly to p as $t \to 0$, suppose that there exists another subsequence $\{w_{s_k}\} \subset \{w_t : t \in (0, b)\}$ such that $w_{s_k} \to q$ as $s_k \to 0$; then we also have $q \in \mathrm{Fix}(W) = \Omega \cap A^{-1}\mathbf{0} =: \Delta$ due to (76). Repeating the same argument as above, we know that $q \in \Delta$ is another solution of variational inequality (41). In terms of the uniqueness of solutions in Δ to the variational inequality (41), we get $p = q$. □

Remark 19. In the assertion of Theorem 18, "as $t \to 0$, $\{x_{t,n}\}$ converges strongly to $p \in \Delta$", p does not depend on n. Indeed, it is known that $\{r_n\} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$. Moreover, in the proof of Theorem 18, it can be readily seen that p is a fixed point of the nonexpansive self-mapping W of K. This shows that p depends on neither n nor t.

Remark 20. Theorem 18 improves and extends [15, Theorem 3.1], [10, Lemma 2.5], and [13, Theorem 3.1] in the following aspects.

(a) The problem and iterative scheme in Theorem 18 are different from those considered in [15, Theorem 3.1], [10, Lemma 2.5], and [13, Theorem 3.1].

(b) Theorem 18 drops the assumption of the asymptotical regularity for the iterative scheme in [13, Theorem 3.1].

3. Explicit Iterative Scheme and Its Convergence Criteria

In this section, we propose an explicit iterative scheme to compute a common element of the solution set of SVI (23) and the solution set of zeros of an accretive operator. We study the strong convergence of the sequence generated by the proposed scheme.

Theorem 21. *Let X be an uniformly convex and 2-uniformly smooth Banach space which has the weakly sequentially continuous duality mapping J. Let C be a nonempty closed convex subset of X such that $C \pm C \subset C$, and let Π_C be a sunny nonexpansive retraction from X onto C. Let $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. For each $i = 1, 2$, let $B_i : C \to X$ be an α_i-inverse-strongly accretive mapping such that $\Delta = \Omega \cap A^{-1}\mathbf{0} \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$. Let $F : C \to C$ be α-strongly accretive and λ-strictly pseudocontractive with $\alpha + \lambda > 1$, and let $f : C \to C$ be a contractive mapping with coefficient $\beta \in (0, \gamma_0)$, where $\gamma_0 = 1 - \sqrt{(1-\alpha)/\lambda}$. Let $V : C \to C$ be a $\overline{\gamma}$-strongly positive linear bounded operator with $\overline{\gamma} \in (1, 2)$. For any given point $x_0 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{aligned}
y_n &= \alpha_n x_n + (1 - \alpha_n) J_{r_n} G x_n, \\
x_{n+1} &= (I - \beta_n V) J_{r_n} G y_n \\
&\quad + \beta_n [J_{r_n} G y_n - \lambda_n (F(J_{r_n} G y_n) - f(y_n))], \\
&\qquad \forall n \ge 0,
\end{aligned} \tag{87}$$

where $\{r_n\}_{n=0}^{\infty} \subset [\varepsilon, \infty)$ for some $\varepsilon > 0$ and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\lambda_n\}_{n=0}^{\infty}$ are the sequences in $(0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\{\alpha_n\} \subset [a, b]$ *for some* $a, b \in (0, 1)$;

(iv) $\sum_{n=0}^{\infty} (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n| + |r_{n+1} - r_n|) < \infty$.

Then, $\{x_n\}$ converges strongly to $p \in \Delta$, which is a unique solution of the following variational inequality:

$$\langle (V-I)\,p, J(p-u)\rangle \le 0, \quad \forall u \in \Delta. \tag{88}$$

Proof. We first show that $\{x_n\}$ is bounded. Indeed, since $\lim_{n\to\infty}\beta_n = 0$ and $\|V\| \ge \overline{\gamma} > 1$, without loss of generality, we may assume that $0 < \beta_n \le \|V\|^{-1} < 1$, for all $n \ge 0$. Take a fixed $p \in \Delta$ arbitrarily. Then, $p = Gp$, $p = J_{r_n}p$, for all $n \ge 0$, and

$$\begin{aligned}\|y_n - p\| &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\left\|J_{r_n}Gx_n - p\right\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|Gx_n - p\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned} \tag{89}$$

Note that

$$\begin{aligned}&x_{n+1} - p \\ &\quad= (I-\beta_n V) J_{r_n}Gx_n - (I-\beta_n V) J_{r_n}Gp \\ &\qquad+ \beta_n \left[J_{r_n}Gy_n - \lambda_n\left(F\left(J_{r_n}Gy_n\right) - f(y_n)\right) - p\right] \\ &\qquad+ \beta_n (I-V)\,p \\ &\quad= (I-\beta_n V)\left(J_{r_n}Gx_n - J_{r_n}Gp\right) \\ &\qquad+ \beta_n \left[\lambda_n (f(y_n) - F(p)) + (I-\lambda_n F) J_{r_n}Gy_n \right.\\ &\qquad\qquad \left. - (I-\lambda_n F) J_{r_n}Gp\right] + \beta_n (I-V)\,p.\end{aligned} \tag{90}$$

Therefore, we have

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad= \left\|(I-\beta_n V)\left(J_{r_n}Gy_n - J_{r_n}Gp\right)\right. \\ &\qquad\quad+ \beta_n \left[\lambda_n (f(y_n) - F(p))\right. \\ &\qquad\qquad\quad \left.+ (I-\lambda_n F) J_{r_n}Gy_n - (I-\alpha_n F) J_{r_n}Gp\right] \\ &\qquad\quad \left.+\beta_n (I-V)\,p\right\| \\ &\quad\le \|(I-\beta_n V)\| \left\|J_{r_n}Gy_n - J_{r_n}Gp\right\| \\ &\qquad+ \beta_n \left[\lambda_n \|f(y_n) - F(p)\|\right. \\ &\qquad\qquad \left.+ \left\|(I-\lambda_n F) J_{r_n}Gy_n - (I-\lambda_n F) J_{r_n}Gp\right\|\right] \\ &\qquad+ \beta_n \|(I-V)\,p\| \\ &\quad\le (1-\beta_n\overline{\gamma})\|y_n - p\| \\ &\qquad+ \beta_n \left[\lambda_n (\|f(y_n) - f(p)\| + \|f(p) - F(p)\|)\right. \\ &\qquad\qquad \left.+ (1-\lambda_n\gamma_0)\left\|J_{r_n}Gy_n - J_{r_n}Gp\right\|\right] \\ &\qquad+ \beta_n \|I-V\|\|p\| \\ &\quad\le (1-\beta_n\overline{\gamma})\|y_n - p\| \\ &\qquad+ \beta_n [\lambda_n (\beta\|y_n - p\| + \|f(p) - F(p)\|) \\ &\qquad\qquad + (1-\lambda_n\gamma_0)\|y_n - p\|] + \beta_n \|I-V\|\|p\| \\ &\quad\le (1-\beta_n\overline{\gamma})\|y_n - p\| \\ &\qquad+ \beta_n [(1-\lambda_n(\gamma_0 - \beta))\|y_n - p\| \\ &\qquad\qquad +\alpha_n \|f(p) - F(p)\|] + \beta_n \|I-V\|\|p\| \\ &\quad\le (1-\beta_n\overline{\gamma})\|y_n - p\| \\ &\qquad+ \beta_n [\|y_n - p\| + \|f(p) - F(p)\|] \\ &\qquad+ \beta_n \|I-V\|\|p\| \\ &\quad= (1-\beta_n(\overline{\gamma}-1))\|y_n - p\| \\ &\qquad+ \beta_n (\|f(p) - F(p)\| + \|I-V\|\|p\|) \\ &\quad= (1-\beta_n(\overline{\gamma}-1))\|y_n - p\| \\ &\qquad+ \beta_n (\overline{\gamma}-1)\frac{\|f(p) - F(p)\| + \|I-V\|\|p\|}{\overline{\gamma}-1} \\ &\quad\le \max\left\{\|y_n - p\|, \frac{\|f(p) - F(p)\| + \|I-V\|\|p\|}{\overline{\gamma}-1}\right\} \\ &\quad\le \max\left\{\|x_n - p\|, \frac{\|f(p) - F(p)\| + \|I-V\|\|p\|}{\overline{\gamma}-1}\right\}.\end{aligned} \tag{91}$$

By induction, we obtain

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - F(p)\| + \|I-V\|\|p\|}{\overline{\gamma}-1}\right\}, \quad \forall n \ge 0, \tag{92}$$

and hence $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{J_{r_n}Gx_n\}$, $\{J_{r_n}Gy_n\}$, $\{f(x_n)\}$, and $\{F(J_{r_n}Gy_n)\}$. From conditions (i), (ii), we have

$$\begin{aligned}\left\|x_{n+1} - J_{r_n}Gy_n\right\| = \beta_n &\left\|(I-V) J_{r_n}Gy_n\right. \\ &\left.- \lambda_n\left(F\left(J_{r_n}Gy_n\right) - f(y_n)\right)\right\| \longrightarrow 0, \\ &\text{as } n \longrightarrow \infty.\end{aligned} \tag{93}$$

Now we claim that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{94}$$

Indeed, (88) can be rewritten as

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n) J_{r_n} G x_n, \\ z_n &= J_{r_n} G y_n - \lambda_n \left(F\left(J_{r_n} G y_n\right) - f(y_n)\right), \\ x_{n+1} &= (I - \beta_n V) J_{r_n} G y_n + \beta_n z_n, \quad \forall n \geq 0. \end{aligned} \tag{95}$$

If $r_{n-1} \leq r_n$, using the resolvent identity in Lemma 8,

$$J_{r_n} G x_n = J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} G x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} G x_n\right). \tag{96}$$

We get

$$\begin{aligned} &\left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| \\ &= \left\|J_{r_{n-1}} \left(\frac{r_{n-1}}{r_n} G x_n + \left(1 - \frac{r_{n-1}}{r_n}\right) J_{r_n} G x_n\right) - J_{r_{n-1}} G x_{n-1}\right\| \\ &\leq \frac{r_{n-1}}{r_n} \left\|G x_n - G x_{n-1}\right\| + \left(1 - \frac{r_{n-1}}{r_n}\right) \left\|J_{r_n} G x_n - G x_{n-1}\right\| \\ &\leq \left\|x_n - x_{n-1}\right\| + \frac{r_n - r_{n-1}}{r_n} \left\|J_{r_n} G x_n - G x_{n-1}\right\| \\ &\leq \left\|x_n - x_{n-1}\right\| + \frac{1}{\varepsilon} \left|r_n - r_{n-1}\right| \left\|J_{r_n} G x_n - G x_{n-1}\right\|. \end{aligned} \tag{97}$$

If $r_n \leq r_{n-1}$, then it can be easily seen that

$$\begin{aligned} &\left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| \\ &\quad \leq \left\|x_{n-1} - x_n\right\| + \frac{1}{\varepsilon} \left|r_{n-1} - r_n\right| \left\|J_{r_{n-1}} G x_{n-1} - G x_n\right\|. \end{aligned} \tag{98}$$

By combining the above two cases, we obtain

$$\begin{aligned} &\left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| \\ &\quad \leq \left\|x_{n-1} - x_n\right\| + \frac{\left|r_{n-1} - r_n\right|}{\varepsilon} \\ &\quad\quad \times \sup_{n \geq 1} \left\{\left\|J_{r_n} G x_n - G x_{n-1}\right\| + \left\|J_{r_{n-1}} G x_{n-1} - G x_n\right\|\right\}, \\ &\quad\quad \forall n \geq 1. \end{aligned} \tag{99}$$

Similarly, we obtain

$$\begin{aligned} &\left\|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\right\| \\ &\quad \leq \left\|y_{n-1} - y_n\right\| + \frac{\left|r_{n-1} - r_n\right|}{\varepsilon} \\ &\quad\quad \times \sup_{n \geq 1} \left\{\left\|J_{r_n} G y_n - G y_{n-1}\right\| + \left\|J_{r_{n-1}} G y_{n-1} - G y_n\right\|\right\}, \\ &\quad\quad \forall n \geq 1. \end{aligned} \tag{100}$$

Thus,

$$\begin{aligned} &\left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| \leq \left\|x_{n-1} - x_n\right\| + \left|r_{n-1} - r_n\right| M_0, \\ &\left\|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\right\| \leq \left\|y_{n-1} - y_n\right\| + \left|r_{n-1} - r_n\right| M_0, \\ &\quad \forall n \geq 1, \end{aligned} \tag{101}$$

where $\sup_{n\geq 1}\{(1/\varepsilon)(\|J_{r_n} G x_n - G x_{n-1}\| + \|J_{r_{n-1}} G x_{n-1} - G x_n\|)\} \leq M_0$ and $\sup_{n\geq 1}\{(1/\varepsilon)(\|J_{r_n} G y_n - G y_{n-1}\| + \|J_{r_{n-1}} G y_{n-1} - G y_n\|)\} \leq M_0$ for some $M_0 > 0$. From (95) and (101), we have

$$\begin{aligned} &\left\|y_n - y_{n-1}\right\| \\ &= \left\|(1-\alpha_n)\left(J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right) + \alpha_n (x_n - x_{n-1}) \right. \\ &\quad \left. + \left(x_{n-1} - J_{r_{n-1}} G x_{n-1}\right)(\alpha_n - \alpha_{n-1})\right\| \\ &\leq (1-\alpha_n) \left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| + \alpha_n \left\|x_n - x_{n-1}\right\| \\ &\quad + \left\|x_{n-1} - J_{r_{n-1}} G x_{n-1}\right\| \left|\alpha_n - \alpha_{n-1}\right| \\ &\leq (1-\alpha_n) \left\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\right\| + \alpha_n \left\|x_n - x_{n-1}\right\| \\ &\quad + \left\|x_{n-1} - J_{r_{n-1}} G x_{n-1}\right\| \left|\alpha_n - \alpha_{n-1}\right| \\ &\leq (1-\alpha_n) \left[\left\|x_n - x_{n-1}\right\| + \left|r_n - r_{n-1}\right| M_0\right] \\ &\quad + \alpha_n \left\|x_n - x_{n-1}\right\| + \left\|x_{n-1} - J_{r_{n-1}} G x_{n-1}\right\| \left|\alpha_n - \alpha_{n-1}\right| \\ &\leq \left\|x_n - x_{n-1}\right\| + \left|r_n - r_{n-1}\right| M_0 \\ &\quad + \left\|x_{n-1} - J_{r_{n-1}} G x_{n-1}\right\| \left|\alpha_n - \alpha_{n-1}\right|, \\ &\left\|z_n - z_{n-1}\right\| \\ &= \left\|(I - \lambda_n F) J_{r_n} G y_n + \lambda_n f(y_n) \right. \\ &\quad \left. - (I - \lambda_{n-1} F) J_{r_{n-1}} G y_{n-1} - \lambda_{n-1} f(y_{n-1})\right\| \\ &= \left\|\lambda_n (f(y_n) - f(y_{n-1})) + (\lambda_n - \lambda_{n-1}) \right. \\ &\quad\quad \times \left(f(y_{n-1}) - F\left(J_{r_{n-1}} G y_{n-1}\right)\right) \\ &\quad\quad \left. + (I - \lambda_n F) J_{r_n} G y_n - (I - \lambda_n F) J_{r_{n-1}} G y_{n-1}\right\| \\ &\leq \lambda_n \left\|f(y_n) - f(y_{n-1})\right\| + \left|\lambda_n - \lambda_{n-1}\right| \\ &\quad \times \left\|f(y_{n-1}) - F\left(J_{r_{n-1}} G y_{n-1}\right)\right\| \\ &\quad + \left\|(I - \lambda_n F) J_{r_n} G y_n - (I - \lambda_n F) J_{r_{n-1}} G y_{n-1}\right\| \\ &\leq \lambda_n \beta \left\|y_n - y_{n-1}\right\| + \left|\lambda_n - \lambda_{n-1}\right| \\ &\quad \times \left\|f(y_{n-1}) - F\left(J_{r_{n-1}} G y_{n-1}\right)\right\| \\ &\quad + (1 - \lambda_n \gamma_0) \left\|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\right\| \\ &\leq \lambda_n \beta \left\|y_n - y_{n-1}\right\| + \left|\lambda_n - \lambda_{n-1}\right| \\ &\quad \times \left\|f(y_{n-1}) - F\left(J_{r_{n-1}} G y_{n-1}\right)\right\| \\ &\quad + (1 - \lambda_n \gamma_0) \left[\left\|y_n - y_{n-1}\right\| + \left|r_n - r_{n-1}\right| M_0\right] \end{aligned}$$

$$\begin{aligned} &= (1-\lambda_n(\gamma_0-\beta))\|y_n-y_{n-1}\| \\ &\quad + (1-\lambda_n\gamma_0)|r_n-r_{n-1}|M_0 \\ &\quad + |\lambda_n-\lambda_{n-1}|\left\|f(y_{n-1})-F\left(J_{r_{n-1}}Gy_{n-1}\right)\right\| \\ &\le \|y_n-y_{n-1}\| + |r_n-r_{n-1}|M_0 \\ &\quad + |\lambda_n-\lambda_{n-1}|\left\|f(y_{n-1})-F\left(J_{r_{n-1}}Gy_{n-1}\right)\right\|, \end{aligned} \tag{102}$$

and hence,

$$\begin{aligned} &\|x_{n+1}-x_n\| \\ &= \|(I-\beta_nV)J_{r_n}Gy_n+\beta_nz_n \\ &\quad -(I-\beta_{n-1}V)J_{r_{n-1}}Gy_{n-1}-\beta_{n-1}z_{n-1}\| \\ &= \left\|\beta_n(z_n-z_{n-1})+(\beta_n-\beta_{n-1})\left(z_{n-1}-VJ_{r_{n-1}}Gy_{n-1}\right)\right. \\ &\quad \left.+(I-\beta_nV)\left(J_{r_n}Gy_n-J_{r_{n-1}}Gy_{n-1}\right)\right\| \\ &\le \beta_n\|z_n-z_{n-1}\|+|\beta_n-\beta_{n-1}|\left\|z_{n-1}-VJ_{r_{n-1}}Gy_{n-1}\right\| \\ &\quad +(1-\beta_n\overline{\gamma})\left\|J_{r_n}Gy_n-J_{r_{n-1}}Gy_{n-1}\right\| \\ &\le \beta_n\left[\|y_n-y_{n-1}\|+|r_n-r_{n-1}|M_0\right. \\ &\quad \left.+|\lambda_n-\lambda_{n-1}|\left\|f(y_{n-1})-F\left(J_{r_{n-1}}Gy_{n-1}\right)\right\|\right] \\ &\quad +|\beta_n-\beta_{n-1}|\left\|z_{n-1}-VJ_{r_{n-1}}Gy_{n-1}\right\| \\ &\quad +(1-\beta_n\overline{\gamma})\left[\|y_n-y_{n-1}\|+|r_n-r_{n-1}|M_0\right] \\ &= (1-\beta_n(\overline{\gamma}-1))\left[\|y_n-y_{n-1}\|+|r_n-r_{n-1}|M_0\right] \\ &\quad +\beta_n|\lambda_n-\lambda_{n-1}|\left\|f(y_{n-1})-F\left(J_{r_{n-1}}Gy_{n-1}\right)\right\| \\ &\quad +|\beta_n-\beta_{n-1}|\left\|z_{n-1}-VJ_{r_{n-1}}Gy_{n-1}\right\| \\ &\le (1-\beta_n(\overline{\gamma}-1)) \\ &\quad \times\left[\|x_n-x_{n-1}\|+2|r_n-r_{n-1}|M_0\right. \\ &\quad \left.+\left\|x_{n-1}-J_{r_{n-1}}Gx_{n-1}\right\||\alpha_n-\alpha_{n-1}|\right] \\ &\quad +\beta_n|\lambda_n-\lambda_{n-1}|\left\|f(y_{n-1})-F\left(J_{r_{n-1}}Gy_{n-1}\right)\right\| \\ &\quad +|\beta_n-\beta_{n-1}|\left\|z_{n-1}-VJ_{r_{n-1}}Gy_{n-1}\right\| \\ &\le (1-\beta_n(\overline{\gamma}-1))\|x_n-x_{n-1}\|+\beta_n|\lambda_n-\lambda_{n-1}|M_1 \\ &\quad +(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+|r_n-r_{n-1}|)M_1, \end{aligned} \tag{103}$$

where $\sup_{n\ge0}\{\|x_n-J_{r_n}Gx_n\|+\|z_n-VJ_{r_n}Gy_n\|+\|f(y_n)-F(J_{r_n}Gy_n)\|+2M_0\}\le M_1$ for some $M_1>0$ (it is easy to see that $\{z_n\}$ is bounded due to the boundedness of $\{y_n\}$). Utilizing Lemma 17, we conclude from conditions (i), (ii), and (iv) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{104}$$

By condition (iii) and (95), we have

$$\begin{aligned} &\|y_n-x_n\| \\ &\quad = (1-\alpha_n)\left\|J_{r_n}Gx_n-x_n\right\| \\ &\quad \le (1-a)\left(\left\|J_{r_n}Gx_n-J_{r_n}Gy_n\right\|\right. \\ &\qquad \left.+\left\|J_{r_n}Gy_n-x_{n+1}\right\|+\|x_{n+1}-x_n\|\right) \\ &\quad \le (1-a)\left(\|x_n-y_n\|\right. \\ &\qquad \left.+\left\|J_{r_n}Gy_n-x_{n+1}\right\|+\|x_{n+1}-x_n\|\right), \end{aligned} \tag{105}$$

which implies that

$$\|y_n-x_n\|\le\frac{1-a}{a}\left(\left\|J_{r_n}Gy_n-x_{n+1}\right\|+\|x_{n+1}-x_n\|\right). \tag{106}$$

This together with (93)-(94) implies that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{107}$$

Therefore,

$$\begin{aligned} \left\|x_n-J_{r_n}Gx_n\right\| &\le \|x_n-y_n\|+\left\|y_n-J_{r_n}Gx_n\right\| \\ &\le \|x_n-y_n\|+\alpha_n\left\|x_n-J_{r_n}Gx_n\right\| \\ &\le \|x_n-y_n\|+b\left\|x_n-J_{r_n}Gx_n\right\|, \end{aligned} \tag{108}$$

which implies that

$$\left\|x_n-J_{r_n}Gx_n\right\|\le\frac{1}{1-b}\|x_n-y_n\|, \tag{109}$$

and hence,

$$\lim_{n\to\infty}\left\|x_n-J_{r_n}Gx_n\right\|=0. \tag{110}$$

Next, we show that $\lim_{n\to\infty}\|x_n-J_{r_n}x_n\|=0$ and $\lim_{n\to\infty}\|x_n-Gx_n\|=0$.

Indeed, for simplicity, put $q=\Pi_C(p-\mu_2B_2p)$, $u_n=\Pi_C(x_n-\mu_2B_2x_n)$, and $v_n=\Pi_C(u_n-\mu_1B_1u_n)$. Then, $p=\Pi_C(q-\mu_1B_1q)$ and $v_n=Gx_n$ for all $n\ge0$. From (95), we have

$$\begin{aligned} \|y_n-p\|^2 &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\left\|J_{r_n}Gx_n-p\right\|^2 \\ &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2. \end{aligned} \tag{111}$$

Utilizing Lemma 12, we have

$$\begin{aligned} \|u_n-q\|^2 &= \|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\|^2 \\ &\le \|x_n-p-\mu_2(B_2x_n-B_2p)\|^2 \\ &\le \|x_n-p\|^2-2\mu_2\left(\alpha_2-\kappa^2\mu_2\right)\|B_2x_n-B_2p\|^2, \end{aligned} \tag{112}$$

$$\begin{aligned} \|v_n-p\|^2 &= \|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\|^2 \\ &\le \|u_n-q-\mu_1(B_1u_n-B_1q)\|^2 \\ &\le \|u_n-q\|^2-2\mu_1\left(\alpha_1-\kappa^2\mu_1\right)\|B_1u_n-B_1q\|^2. \end{aligned} \tag{113}$$

Combining (112) and (113), we obtain

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - 2\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 x_n - B_2 p\|^2 - 2\mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_n - B_1 q\|^2, \tag{114}$$

which together with (111) implies that

$$\begin{aligned} \|y_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \|v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \\ &\quad \times \left[\|x_n - p\|^2 - 2\mu_2 \left(\alpha_2 - \kappa^2\mu_2\right) \|B_2 x_n - B_2 p\|^2 \right. \\ &\qquad \left. - 2\mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_n - B_1 q\|^2\right] \\ &= \|x_n - p\|^2 \\ &\quad - 2(1-\alpha_n) \left[\mu_2 \left(\alpha_2 - \kappa^2 \mu_2\right) \|B_2 x_n - B_2 p\|^2 \right. \\ &\qquad \left. + \mu_1 \left(\alpha_1 - \kappa^2\mu_1\right) \|B_1 u_n - B_1 q\|^2\right]. \end{aligned} \tag{115}$$

It immediately follows that

$$\begin{aligned} &2(1-\alpha_n) \left[\mu_2 \left(\alpha_2 - \kappa^2\mu_2\right) \|B_2 x_n - B_2 p\|^2 \right. \\ &\qquad \left. + \mu_1 \left(\alpha_1 - \kappa^2 \mu_1\right) \|B_1 u_n - B_1 q\|^2\right] \\ &\quad \le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad \le \left(\|x_n - p\| + \|y_n - p\|\right) \|x_n - y_n\|. \end{aligned} \tag{116}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$, we deduce from (107) and condition (iii) that

$$\lim_{n\to\infty} \|B_2 x_n - B_2 p\| = 0, \qquad \lim_{n\to\infty} \|B_1 u_n - B_1 q\| = 0. \tag{117}$$

Utilizing Proposition 3 and Lemma 9, we have

$$\begin{aligned} \|u_n - q\|^2 &= \|\Pi_C (x_n - \mu_2 B_2 x_n) - \Pi_C (p - \mu_2 B_2 p)\|^2 \\ &\le \langle x_n - \mu_2 B_2 x_n - (p - \mu_2 B_2 p), J(u_n - q)\rangle \\ &= \langle x_n - p, J(u_n - q)\rangle + \mu_2 \langle B_2 p - B_2 x_n, J(u_n - q)\rangle \\ &\le \frac{1}{2}\left[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right)\right] \\ &\quad + \mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\|, \end{aligned} \tag{118}$$

which implies that

$$\|u_n - q\|^2 \le \|x_n - p\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right) + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\|. \tag{119}$$

Similarly, we derive

$$\begin{aligned} \|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\|^2 \\ &\le \langle u_n - \mu_1 B_1 u_n - (q - \mu_1 B_1 q), J(v_n - p)\rangle \\ &= \langle u_n - q, J(v_n - p)\rangle + \mu_1 \langle B_1 q - B_1 u_n, J(v_n - p)\rangle \\ &\le \frac{1}{2}\left[\|u_n - q\|^2 + \|v_n - p\|^2 - g_2\left(\|u_n - v_n + (p - q)\|\right)\right] \\ &\quad + \mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|, \end{aligned} \tag{120}$$

which implies that

$$\|v_n - p\|^2 \le \|u_n - q\|^2 - g_2\left(\|u_n - v_n + (p - q)\|\right) + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|. \tag{121}$$

Combining (119) and (121), we get

$$\begin{aligned} \|v_n - p\|^2 \le \|x_n - p\|^2 &- g_1\left(\|x_n - u_n - (p - q)\|\right) \\ &- g_2\left(\|u_n - v_n + (p - q)\|\right) \\ &+ 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\ &+ 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|, \end{aligned} \tag{122}$$

which together with (111) implies that

$$\begin{aligned} \|y_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \|v_n - p\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \\ &\quad \times \left[\|x_n - p\|^2 - g_1\left(\|x_n - u_n - (p - q)\|\right) \right. \\ &\qquad - g_2\left(\|u_n - v_n + (p - q)\|\right) \\ &\qquad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\ &\qquad \left. + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|\right] \\ &= \|x_n - p\|^2 - (1-\alpha_n) \\ &\quad \times \left[g_1\left(\|x_n - u_n - (p - q)\|\right) + g_2\left(\|u_n - v_n + (p - q)\|\right)\right] \\ &\quad + 2(1-\alpha_n)\left(\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \right. \\ &\qquad \left. + \mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|\right). \end{aligned} \tag{123}$$

It immediately follows that

$$\begin{aligned}(1-\alpha_n)&\left[g_1\left(\|x_n-u_n-(p-q)\|\right)\right.\\&\left.+g_2\left(\|u_n-v_n+(p-q)\|\right)\right]\\&\leq\|x_n-p\|^2-\|y_n-p\|^2+2(1-\alpha_n)\\&\quad\times\left(\mu_2\|B_2p-B_2x_n\|\|u_n-q\|+\mu_1\|B_1q-B_1u_n\|\|v_n-p\|\right)\\&\leq\left(\|x_n-p\|+\|y_n-p\|\right)\|x_n-y_n\|\\&\quad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\&\quad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.\end{aligned}\tag{124}$$

Since $\{x_n\},\{y_n\},\{u_n\}$, and $\{v_n\}$ are bounded, we deduce from (107), (117) and condition (iii) that

$$\begin{aligned}\lim_{n\to\infty}g_1\left(\|x_n-u_n-(p-q)\|\right)&=0,\\\lim_{n\to\infty}g_2\left(\|u_n-v_n+(p-q)\|\right)&=0.\end{aligned}\tag{125}$$

Utilizing the properties of g_1 and g_2, we get

$$\begin{aligned}\lim_{n\to\infty}\|x_n-u_n-(p-q)\|&=0,\\\lim_{n\to\infty}\|u_n-v_n+(p-q)\|&=0,\end{aligned}\tag{126}$$

which yields

$$\begin{aligned}\|x_n-v_n\|&\leq\|x_n-u_n-(p-q)\|\\&\quad+\|u_n-v_n+(p-q)\|\longrightarrow0\quad\text{as }n\longrightarrow\infty;\end{aligned}\tag{127}$$

that is,

$$\lim_{n\to\infty}\|x_n-Gx_n\|=0.\tag{128}$$

Note that

$$\begin{aligned}\left\|x_n-J_{r_n}x_n\right\|&\leq\left\|x_n-J_{r_n}Gx_n\right\|+\left\|J_{r_n}Gx_n-J_{r_n}x_n\right\|\\&\leq\left\|x_n-J_{r_n}Gx_n\right\|+\|Gx_n-x_n\|.\end{aligned}\tag{129}$$

From (110) and (128), we have

$$\lim_{n\to\infty}\left\|x_n-J_{r_n}x_n\right\|=0.\tag{130}$$

By utilizing the resolvent identity in Lemma 8, we obtain from $\{r_n\}\subset[\varepsilon,\infty)$ that

$$\begin{aligned}\|x_n-J_\varepsilon x_n\|&\leq\left\|x_n-J_{r_n}x_n\right\|+\left\|J_{r_n}x_n-J_\varepsilon x_n\right\|\\&=\left\|x_n-J_{r_n}x_n\right\|\\&\quad+\left\|J_{r_n}x_n-J_{r_n}\left(\frac{r_n}{\varepsilon}x_n+\left(1-\frac{r_n}{\varepsilon}\right)J_{r_n}x_n\right)\right\|\\&\leq2\left\|x_n-J_{r_n}x_n\right\|,\end{aligned}\tag{131}$$

which together with (130) implies that

$$\lim_{n\to\infty}\|x_n-J_\varepsilon x_n\|=0.\tag{132}$$

For every $n\geq0$ and $t\in(0,(2-\overline{\gamma})/(\gamma_0-\beta))$, let

$$\begin{aligned}x_{t,n}&=(I-\theta_tV)J_{r_n}Gx_{t,n}\\&\quad+\theta_t\left[J_{r_n}Gx_{t,n}-t\left(F\left(J_{r_n}Gx_{t,n}\right)-f(x_{t,n})\right)\right],\end{aligned}\tag{133}$$

where $\{r_n\}\subset[\varepsilon,\infty)$ and $0<\theta_t\leq\|V\|^{-1}$, for all $t\in(0,(2-\overline{\gamma})/(\gamma_0-\beta))$ with $\lim_{t\to0}\theta_t=0$. Then, by Theorem 18, $\{x_{t,n}\}$ converges strongly to $p\in\Delta$ as $t\to0$, which is the unique solution of the variational inequality (88).

Define a mapping

$$Wx=(1-\theta)J_\varepsilon x+\theta Gx,\quad\forall x\in C,\tag{134}$$

where θ is a constant in $(0,1)$. Then by Lemma 4, we have $\mathrm{Fix}(W)=\mathrm{Fix}(J_\varepsilon)\cap\mathrm{Fix}(G)=\Delta$. Observe that

$$\begin{aligned}\|x_n-Wx_n\|&=\|(1-\theta)(x_n-J_\varepsilon x_n)+\theta(x_n-Gx_n)\|\\&\leq(1-\theta)\|x_n-J_\varepsilon x_n\|+\theta\|x_n-Gx_n\|.\end{aligned}\tag{135}$$

From (128) and (132), we obtain

$$\lim_{n\to\infty}\|x_n-Wx_n\|=0.\tag{136}$$

Further, we show that

$$\limsup_{n\to\infty}\langle(I-V)p,J(x_n-p)\rangle\leq0.\tag{137}$$

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(I-V)p,J(x_n-p)\rangle\\&\quad=\lim_{i\to\infty}\left\langle(I-V)p,J\left(x_{n_i}-p\right)\right\rangle.\end{aligned}\tag{138}$$

Without loss of generality, we may assume that $x_{n_i}\rightharpoonup\tilde{x}$. Utilizing Lemma 15, we obtain from (136) that $\tilde{x}\in\mathrm{Fix}(W)=\mathrm{Fix}(J_\varepsilon)\cap\mathrm{Fix}(G)=\Delta$. Hence from (88) and (138), we get

$$\limsup_{n\to\infty}\langle(I-V)p,J(x_n-p)\rangle=\langle(I-V)p,J(\tilde{x}-p)\rangle\leq0.\tag{139}$$

Finally, we show that $x_n\to p$ as $n\to\infty$.

Indeed,

$$\begin{aligned}
\|y_n - p\| &\le \alpha_n \|x_n - p\| + (1-\alpha_n) \|J_{r_n} G x_n - p\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n) \|x_n - p\| = \|x_n - p\|, \\
\|x_{n+1} - p\|^2 & \\
= \langle (I - \beta_n V) & J_{r_n} G y_n - (I - \beta_n V) J_{r_n} G p, J(x_{n+1} - p) \rangle \\
&+ \beta_n \Big[\langle (I - \lambda_n F) J_{r_n} G y_n - (I - \lambda_n F) J_{r_n} G p, J(x_{n+1} - p) \rangle \\
&\quad + \lambda_n (\langle f(y_n) - f(p), J(x_{n+1} - p) \rangle \\
&\quad\quad + \langle f(p) - F(p), J(x_{n+1} - p) \rangle) \Big] \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
\le (1 - \beta_n \overline{\gamma}) & \|J_{r_n} G y_n - J_{r_n} G p\| \|x_{n+1} - p\| \\
&+ \beta_n \Big[(1 - \lambda_n \gamma_0) \|J_{r_n} G y_n - J_{r_n} G p\| \|x_{n+1} - p\| \\
&\quad + \lambda_n (\beta \|y_n - p\| \|x_{n+1} - p\| \\
&\quad\quad + \|f(p) - F(p)\| \|x_{n+1} - p\|) \Big] \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
\le (1 - \beta_n \overline{\gamma}) & \|y_n - p\| \|x_{n+1} - p\| \\
&+ \beta_n [(1 - \lambda_n \gamma_0) \|y_n - p\| \|x_{n+1} - p\| \\
&\quad + \lambda_n (\beta \|y_n - p\| \|x_{n+1} - p\| \\
&\quad\quad + \|f(p) - F(p)\| \|x_{n+1} - p\|)] \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
= (1 - \beta_n \overline{\gamma}) & \|y_n - p\| \|x_{n+1} - p\| \\
&+ \beta_n [(1 - \lambda_n (\gamma_0 - \beta)) \|y_n - p\| \|x_{n+1} - p\| \\
&\quad + \lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\|] \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
\le (1 - \beta_n \overline{\gamma}) & \|y_n - p\| \|x_{n+1} - p\| \\
&+ \beta_n [\|y_n - p\| \|x_{n+1} - p\| \\
&\quad + \lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\|] \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
= (1 - \beta_n (\overline{\gamma} - 1)) & \|y_n - p\| \|x_{n+1} - p\| \\
&+ \beta_n \lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\| \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle \\
\le (1 - \beta_n (\overline{\gamma} - 1)) & \frac{1}{2} \left(\|y_n - p\|^2 + \|x_{n+1} - p\|^2 \right) \\
&+ \beta_n \lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\| \\
&+ \beta_n \langle (I - V) p, J(x_{n+1} - p) \rangle,
\end{aligned} \tag{140}$$

which yields

$$\begin{aligned}
\|x_{n+1} - p\|^2 & \\
&\le \frac{1 - \beta_n (\overline{\gamma} - 1)}{1 + \beta_n (\overline{\gamma} - 1)} \|y_n - p\|^2 + \beta_n \frac{1}{1 + \beta_n (\overline{\gamma} - 1)} \\
&\quad \times [2\lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\| \\
&\quad\quad + 2 \langle (I - V) p, J(x_{n+1} - p) \rangle] \\
&\le (1 - \beta_n (\overline{\gamma} - 1)) \|x_n - p\|^2 + \beta_n \frac{1}{1 + \beta_n (\overline{\gamma} - 1)} \\
&\quad \times [2\lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\| \\
&\quad\quad + 2 \langle (I - V) p, J(x_{n+1} - p) \rangle] \\
&= (1 - \gamma_n) \|x_n - p\|^2 + \gamma_n \delta_n,
\end{aligned} \tag{141}$$

where $\gamma_n = \beta_n(\overline{\gamma} - 1)$ and

$$\begin{aligned}
\delta_n = & \frac{1}{(1 + \beta_n (\overline{\gamma} - 1)) (\overline{\gamma} - 1)} \\
& \times [2\lambda_n \|f(p) - F(p)\| \|x_{n+1} - p\| \\
& \quad + 2 \langle (I - V) p, J(x_{n+1} - p) \rangle].
\end{aligned} \tag{142}$$

It can be easily seen from (137) and conditions (i), (ii) that

$$\sum_{n=0}^{\infty} \gamma_n = \infty, \qquad \limsup_{n \to \infty} \delta_n \le 0. \tag{143}$$

Applying Lemma 17 to (141), we infer that $x_n \to p$ as $n \to \infty$. This completes the proof. □

Remark 22. By a careful analysis of the proof of Theorem 21, we reached to the conclusion that the iterative method (87) in Theorem 21 can be replaced by the following iterative method:

$$\begin{aligned}
y_n &= \alpha_n x_n + (1 - \alpha_n) J_{r_n} G x_n, \\
x_{n+1} &= (I - \beta_n V) y_n + \beta_n [y_n - \lambda_n (F(y_n) - f(y_n))], \\
& \qquad \forall n \ge 0.
\end{aligned} \tag{144}$$

Remark 23. Theorem 21 improves and extends [15, Theorem 3.1], [10, Theorem 3.1], and [13, Theorem 3.1] in the following aspects.

(a) The problem in Theorem 21 is different from the one considered in [15, Theorem 3.1], [10, Theorem 3.1], and [13, Theorem 3.1]. Also, the iterative scheme in Theorem 21 is different from the one considered in [15, Theorem 3.1], [10, Theorem 3.1] and [13, Theorem 3.1].

(b) Theorem 21 is more advantageous and more flexible than the iterative scheme in [13, Theorem 3.1].

(c) Theorem 21 drops the assumption of the asymptotical regularity for the iterative scheme in [13, Theorem 3.1].

Remark 24. Theorem 21 extends and improves [28, Theorem 3.1] to a great extent in the following aspects.

(a) The u is replaced by a fixed contractive mapping.

(b) One continuous pseudocontractive mapping (including nonexpansive mapping) is replaced by the mapping $J_{r_n} \circ G$.

(c) We add a strongly positive linear bounded operator A and a strongly accretive and strictly pseudocontractive mapping F in our iterative algorithm.

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