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Research Article

The Hyperorder of Solutions of Second-Order Linear Differential Equations

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We prove that the hyperorder of every nontrivial solution of homogenous linear differential equations of type $f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0$ and nonhomogeneous equation of type $f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = H(z)$ is one, where $A_0, A_1, H(z)$ are entire functions of order less than one, improving the previous results of Chen, Wang, and Laine.

1. Introduction

We assume that the reader is familiar with the usual notations and the basic results of the Nevanlinna theory (see [1–4]). We also use basic notions and the results of the Wiman-Valiron theory; see [5]. Let $f(z)$ be a nonconstant meromorphic function in the complex plane. We remark that $\sigma(f)$, respectively, $\sigma_2(f)$ will be used to denote the order, respectively, the hyperorder, of f. In particular, the hyperorder $\sigma_2(f)$ is defined as

$$\sigma_2(f) = \limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}; \tag{1}$$

see [1, 2, 4]. For a set $E \subset R^+$, let $m(E)$, respectively, $\lambda(E)$, denote the linear measure, respectively, the logarithmic measure of E. Moreover, the upper logarithmic density and the lower logarithmic density of E are defined by

$$\begin{aligned}\overline{\text{logdens}}(E) &= \limsup_{r\to\infty}\frac{\lambda(E\bigcap[1,r])}{\log r},\\ \underline{\text{logdens}}(E) &= \liminf_{r\to\infty}\frac{\lambda(E\bigcap[1,r])}{\log r}.\end{aligned} \tag{2}$$

Observe that E may have a different meaning at different occurrences in what follows.

We now recall some previous results concerning linear differential equations of type

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0, \tag{3}$$

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = H(z), \tag{4}$$

where $A_0, A_1, H(z)$ are entire functions of order less than one, and a, b are complex constants.

Chen proved the following theorem; see [6].

Theorem A. *Let $A_0 \not\equiv 0$, $A_1 \not\equiv 0$ be entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. Then every nontrivial solution f of (3) is of infinite order.*

Wang and Laine investigated the nonhomogeneous equation (4) and got the following; see [7].

Theorem B. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $b \neq a$. Then every nontrivial solution f of (4) is of infinite order.*

Theorem C. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, D_0, D_1, H are entire functions of order less than one, and the complex* *constants a, b satisfy $ab \neq 0$ and $b/a < 0$. Then every nontrivial solution f of equation*

$$f'' + \left(A_1(z)e^{az} + D_1(z)\right)f' + \left(A_0(z)e^{bz} + D_0(z)\right)f = H(z) \tag{5}$$

is of infinite order.

In this paper, we investigate the hyperorder of the nontrivial solutions of (3), (4), and (5) and obtain the following theorems.

Theorem 1. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $b \neq a$. Then the hyperorder of every nontrivial solution f of (4) is one.*

Corollary 2. *Let $A_0 \not\equiv 0$, $A_1 \not\equiv 0$ be entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $a = cb$ $(c > 1)$. Then the hyperorder of every nontrivial solution f of (3) is one.*

Theorem 3. *Suppose that $A_0 \not\equiv 0$, $A_1 \not\equiv 0$, D_0, D_1, H are entire functions of order less than one, and the complex constants a, b satisfy $ab \neq 0$ and $b/a < 0$. Then the hyperorder of every nontrivial solution f of (5) is one.*

2. Lemmas

Lemma 4 (see [5]). *Let f be an entire function of infinite order and let $\nu_f(r)$ be the central index of $f(z)$, then the hyperorder*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log \nu_f(r)}{\log r}. \tag{6}$$

Lemma 5 (see [8]). *Let f be an entire function of infinite order with $\sigma_2(f) = \alpha$ $(0 \leq \alpha < \infty)$, and there exists a set $E_2 \subset [1,\infty)$ which have a finite logarithmic measure. Then there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi)$, and $r_n \notin E_2$, $r_n \to \infty$ and such that*

(1) *if $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then, for any given ε_1 $(0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_n^{\alpha-\varepsilon_1}\} < \nu(r_n) < \exp\{r_n^{\alpha+\varepsilon_1}\}; \tag{7}$$

(2) *if $\sigma_2(f) = 0$, then, for any given ε_2 $(0 < \varepsilon_2 < 1/2)$ and for any large M_1 (> 0),*

$$r_n^{M_1} < \nu(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{8}$$

Lemma 6 (see [7]). *Suppose that $P(z) = (\alpha + i\beta)z$, where α, β are real numbers, $|\alpha| + |\beta| \neq 0$, and that $A(z)$ $(\not\equiv 0)$ is a meromorphic function with $\sigma(A) < 1$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \alpha\cos\theta - \beta\sin\theta$. Then, for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1,\infty)$ of finite linear measure such that, for any $\theta \in [0, 2\pi) \setminus H$, there exists $R > 0$ such that, for $|z| = r > R$ and $r \notin E_3$, we have*

(1) *if $\delta(P,\theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r\} < \left|g\left(re^{i\theta}\right)\right| < \exp\{(1+\varepsilon)\delta(P,\theta)r\}; \tag{9}$$

(2) *if $\delta(P,\theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r\} < \left|g\left(re^{i\theta}\right)\right| < \exp\{(1-\varepsilon)\delta(P,\theta)r\}, \tag{10}$$

where $H = \{\theta \in [0, 2\pi) \mid \delta(P,\theta) = 0\}$.

Lemma 7 (see [6]). *Let A, B be entire functions of finite order and if f is a solution of equation*

$$f'' + Af' + Bf = 0, \tag{11}$$

then the hyperorder $\sigma_2(f) \leq \max\{\sigma(A), \sigma(B)\}$.

The proof of the lemma below follows the idea of Bergweiler et al.; see [9, Theorem 3.1].

Lemma 8. *Let $f(z)$ be an entire function, and $M(r, f) = f(re^{i\theta_r})$ for every r. Set $\theta_r \to \theta_0$, and there exists a constant $l_0 > 0$ and a set E with positive lower logarithmic density such that*

$$M(r,f)^{1/5} \leq \left|f\left(re^{i\theta}\right)\right| \tag{12}$$

for all $r \in E$ large enough and all θ such that $|\theta - \theta_0| < l_0$.

Proof. Since f is entire function, we know that $M(r, f)$ is nondecrease, $M(r, f) \to \infty$ as $r \to \infty$, and $|f(re^{i\theta})|$ is continuous on the circle $|z| = r$ for $z = re^{i\theta}$. Set $\theta_r \to \theta_0 \in [0, 2\pi)$ as $r \to \infty$. $A(u^a, u^b)$ denotes an annulus for $0 < a < b$ and sufficiently large u. Then, there exists a constant $4l_0$ $(< \pi)$ such that $|f(re^{i\theta})| > 1$ for $z = re^{i\theta} \in D$, where $D := \{(r,\theta) \mid r \in A(u^a, u^b), |\theta - \theta_0| < 4l_0\}$ for u sufficiently large. Then the function $h(z) := \log|f(z)|$ is a positive harmonic in D. So $H(t) = h(e^t)$ is a positive harmonic in the domain $S := \{t \mid a\log u < \Re(t) < b\log u, \theta_0 - 4l_0 < \Im(t) < \theta_0 + 4l_0\}$. Thus, if t_1 and t_2 satisfy $a\log u + 3l_0 < \Re(t_1) = \Re(t_2) < b\log u - 3l_0$ and $|\Im(t_1) - \Im(t_2)| < 2l_0$, where $\Im(t_j) \in (\theta_0 - l_0, \theta_0 + l_0)$, $j = 1, 2$, then $\{t \mid |t - t_1| < 3l_0\}$ and $|t_1 - t_2| < 2l_0 < 3l_0$. So

$$\frac{1}{5} = \frac{3l_0 - 2l_0}{3l_0 + 2l_0} \leq \frac{H(t_2)}{H(t_1)} \leq \frac{3l_0 + 2l_0}{3l_0 - 2l_0} = 5 \tag{13}$$

by Harnack's inequality; see [10, Theorem 1.3.1]. Therefore, if z_1 and z_2 are in the domain $D_1 := \{(r,\theta) \mid r \in A(u^a, u^b), |\theta - \theta_0| < l_0\}$, where u is sufficiently large, then

$$\frac{1}{5} \leq \frac{h(z_2)}{h(z_1)} \leq 5. \tag{14}$$

Set $m^*(r,f) = \min_{|z|=r,|\theta-\theta_0|<l_0}|f(re^{i\theta})|$. Then, we have $1/5\log M(r,f) \le \log m^*(r,f)$ for $z \in D_1$. If let $u \to \infty$, and then the set E of $r \in (u^a, u^b)$ is of positive lower logarithmic density. Thus, the conclusion of this lemma holds. □

Lemma 9. *Let $f(z)$ be an entire function with infinite order and let hyperorder $\sigma_2(f) \le 1$, $g(z)$ be an entire function with finite order $\sigma(g) < \infty$. For $r \in E$, where E is the infinite logarithmic measure set which is given in Lemma 8. Then, for any given ε_0,*

$$\left|\frac{g(z)}{f(z)}\right| < \varepsilon_0 \tag{15}$$

for all z such that $|z| = r \in E$ is sufficiently large and that $|f(z)| = M(r,f)$.

Proof. Since, for the entire function $g(z)$,

$$\sigma(g) = \limsup_{r\to\infty}\frac{\log\log M(r,g)}{\log r} \tag{16}$$

for any given ε, we have

$$|g(z)| \le \exp\left\{r^{\sigma(g)+\varepsilon}\right\} \tag{17}$$

for all r sufficiently large. Since the order of f is infinite, for $r \in E$, there exists a sufficiently large real number A such that

$$\sigma(f) = \limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r} > A. \tag{18}$$

Thus, for $r \in E$ is sufficiently large,

$$M(r,f) \ge \exp\left\{r^{A-\varepsilon}\right\}. \tag{19}$$

By (17) and (19), we conclude that

$$\begin{aligned}\left|\frac{g(z)}{f(z)}\right| &\le \frac{\exp\left\{r^{\sigma(g)+\varepsilon}\right\}}{M(r,f)} \\ &\le \frac{\exp\left\{r^{\sigma(g)+\varepsilon}\right\}}{\exp\left\{r^{A-\varepsilon}\right\}} \longrightarrow 0\end{aligned} \tag{20}$$

for all z satisfying $|f(z)| = M(r,f)$ such that $r \in E$ is sufficiently large. Thus, the conclusion holds. □

3. Proofs of Theorems

Proof of Theorem 1. Suppose that f is a solution of (4), and then f is an entire function.

Step 1. We prove that $\sigma_2(f) \le 1$. Since $\sigma(A_0; A_1; H) < 1$, set $\sigma(H) = \lambda < 1$. Then for any given ε satisfying $\varepsilon < 1-\lambda$, when r is sufficiently large, we have

$$\left|A_1e^{az}\right| \le \exp\left\{r^{1+\varepsilon}\right\}, \tag{21}$$

$$\left|A_0e^{bz}\right| \le \exp\left\{r^{1+\varepsilon}\right\}, \tag{22}$$

$$|H(z)| \le \exp\left\{r^{\lambda+\varepsilon}\right\}. \tag{23}$$

From the Wiman-Valiron theory, there is a set E_1 having finite logarithmic measure, such that

$$\begin{aligned}&\frac{f^{(j)}(z)}{f(z)} \\ &\quad = \left(\frac{\nu_f(r)}{z}\right)^j(1+O(1)) \quad (j=1,2)\end{aligned} \tag{24}$$

whenever $|f(z)| = M(r,f)$, $r \notin E_1$, where the $\nu_f(r)$ is the central index of $f(z)$, and we know that $\nu_f(r) \to \infty$ as $r \to \infty$. When r sufficiently large, we have $|f(z)| = M(r,f) > 1$. From (4) we have

$$\left|\frac{f''}{f}\right| \le \left|A_1e^{az}\right|\left|\frac{f'}{f}\right| + \left|A_0e^{bz}\right| + \left|\frac{H}{f}\right|. \tag{25}$$

Substituting (21), (22), (23), and (24) into (25), we obtain

$$\begin{aligned}&\left(\frac{\nu_f(r)}{|z|}\right)^2(1+O(1)) \\ &\quad \le \exp\left\{r^{1+\varepsilon}\right\}\frac{\nu_f(r)}{|z|}(1+O(1)) \\ &\qquad + \exp\left\{r^{1+\varepsilon}\right\} + \exp\left\{r^{\lambda+\varepsilon}\right\},\end{aligned} \tag{26}$$

where z satisfies $|z| = r \notin E_1$ and r sufficiently large. By (26) we get

$$\limsup_{r\to\infty}\frac{\log\log \nu_f(r)}{\log r} \le 1+\varepsilon. \tag{27}$$

Since ε is arbitrary, by (27) and Lemma 4, we have $\sigma_2(f) \le 1$.

Step 2. By Theorem B, we know that the order of f is infinite, and, by the first step, we clear that the hyperorder of f is less than one. Thus, by Lemma 9 and (23), we have

$$\left|\frac{H}{f}\right| < \varepsilon_0 \tag{28}$$

for all z satisfying $|f(z)| = M(r,f)$ such that $r \in E$ is sufficiently large, where E is of infinite logarithmic measure. Set $\sigma_2(f) = \alpha_0$, and we assert that $\alpha_0 = 1$. Now we assume that $\alpha_0 < 1$, and prove that $\sigma_2(f) = \alpha_0 < 1$ results in contradictions. E_2, E_3 are the sets in Lemmas 5 and 6, respectively.

Since $\lambda(E_1 \cup E_2 \cup E_3) < \infty$, we have that $\lambda(E \setminus (E_1 \cup E_2 \cup E_3))$ is infinite. Thus, by Lemma 5, we see that there exists a sequence of points $\{z_n = r_ne^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi)$, $r_n \in E\setminus(E_1\cup E_2\cup E_3)$, $r_n \to \infty$, and if $\sigma_2(f) = \alpha_0$ $(0 < \alpha_0 < \infty)$, then, for any given ε_1 $(0 < \varepsilon_1 < \min\{\alpha_0, 1-\alpha_0\})$,

$$\exp\left\{r_n^{\alpha_0-\varepsilon_1}\right\} < \nu_f(r_n) < \exp\left\{r_n^{\alpha_0+\varepsilon_1}\right\}; \tag{29}$$

if $\sigma_2(f) = 0$, then, for any given ε_2 $(0 < \varepsilon_2 < 1/2)$ and for any large M_1 (> 0),

$$r_n^{M_1} < \nu_f(r_n) < \exp\left\{r_n^{\varepsilon_2}\right\}. \tag{30}$$

Firstly, we prove the case when $\sigma_2(f) = \alpha_0$ (> 0). It can separate into three cases to discuss.

Case 1. First assume that $\delta(az,\theta_0) > 0$. From the continuity of $\delta(az,\theta_0)$, we have

$$\frac{1}{2}\delta(az,\theta_0) < \delta(az,\theta_n) < \frac{3}{2}\delta(az,\theta_0) \tag{31}$$

for sufficiently large n. From (9), we deduce that

$$\begin{aligned}&\exp\left\{\frac{1-\varepsilon}{2}\delta(az,\theta_0)r_n\right\}\\&\quad\le |A_1(z_n)e^{az_n}| \le \exp\left\{\frac{3(1+\varepsilon)}{2}\delta(az,\theta_0)r_n\right\}\end{aligned} \tag{32}$$

for all n sufficiently large.

From (4), we have

$$\begin{aligned}&\left|\frac{f'(z_n)}{f(z_n)} + \frac{A_0(z_n)}{A_1(z_n)}e^{(b-a)z_n}\right|\\&\quad\le \frac{1}{|A_1(z_n)e^{az_n}|}\left(\left|\frac{f''(z_n)}{f(z_n)}\right| + \left|\frac{H(z_n)}{f(z_n)}\right|\right).\end{aligned} \tag{33}$$

Subcase 1.1. We first assume that θ_0 satisfies $\eta := \delta((b-a)z,\theta_0) > 0$. From the continuity of $\delta((b-a)z,\theta)$, we also have

$$\begin{aligned}&\exp\left\{\frac{1-\varepsilon}{2}\eta r_n\right\}\\&\quad\le \left|\frac{A_0(z_n)}{A_1(z_n)}e^{(b-a)z_n}\right| \le \exp\left\{\frac{3(1+\varepsilon)}{2}\eta r_n\right\}\end{aligned} \tag{34}$$

for all n sufficiently large. From (33), we get

$$\begin{aligned}&\left|\frac{A_0(z_n)}{A_1(z_n)}e^{(b-a)z_n}\right|\\&\quad\le \left|\frac{f'(z_n)}{f(z_n)}\right| + \frac{1}{|A_1(z_n)e^{az_n}|}\left(\left|\frac{f''(z_n)}{f(z_n)}\right| + \left|\frac{H(z_n)}{f(z_n)}\right|\right).\end{aligned} \tag{35}$$

Substituting (24), (28), (29), (32), and (34) into (35), we obtain

$$\begin{aligned}&\exp\left\{\frac{1-\varepsilon}{2}\eta r_n\right\}\\&\quad\le \exp\{r_n^{\alpha_0+\varepsilon_1}\} r_n^{-1}(1+O(1))\\&\qquad+\exp\left\{-\frac{1-\varepsilon}{2}\delta(az,\theta_0)r_n\right\}\\&\qquad\times\left(\exp\{2r_n^{\alpha_0+\varepsilon_1}\} r_n^{-2}(1+O(1))+\varepsilon_0\right).\end{aligned} \tag{36}$$

Since $\alpha_0+\varepsilon_1 < 1$, we see that (36) are contradictory as $n \to \infty$.

Subcase 1.2. Next assume that $\eta := \delta((b-a)z,\theta_0) < 0$. Then, from (10), for n large enough, we deduce that

$$\begin{aligned}&\exp\left\{\frac{3(1+\varepsilon)}{2}\eta r_n\right\}\\&\quad\le \left|\frac{A_0(z_n)}{A_1(z_n)}e^{(b-a)z_n}\right| \le \exp\left\{\frac{1-\varepsilon}{2}\eta r_n\right\}.\end{aligned} \tag{37}$$

From (33), we get

$$\begin{aligned}\left|\frac{f'(z_n)}{f(z_n)}\right| &\le \left|\frac{A_0(z_n)}{A_1(z_n)}e^{(b-a)z_n}\right|\\&\quad+\frac{1}{|A_1(z_n)e^{az_n}|}\left(\left|\frac{f''(z_n)}{f(z_n)}\right| + \left|\frac{H(z_n)}{f(z_n)}\right|\right).\end{aligned} \tag{38}$$

Substituting (24), (28), (29), (32), and (37) into (38), we obtain

$$\begin{aligned}&\frac{\nu_f(r_n)}{r_n}(1+O(1))\\&\quad\le \exp\left\{\frac{1-\varepsilon}{2}\eta r_n\right\} + \exp\left\{-\frac{1-\varepsilon}{2}\delta(az,\theta_0)r_n\right\}\\&\qquad\times\left(\exp\{2r_n^{\alpha_0+\varepsilon_1}\} r_n^{-2}(1+O(1))+\varepsilon_0\right)\end{aligned} \tag{39}$$

as $n \to \infty$. Since $\alpha_0+\varepsilon_1 < 1$, this implies that $\nu_f(r) \to 0$, $n \to \infty$, which is impossible.

Subcase 1.3. Assume finally that $\eta := \delta((b-a)z,\theta_0) = 0$. Here, (12) may be used to construct another sequence of points $\{z_n^* = r_n e^{i\theta_n^*}\}$ with $\lim_{n\to\infty}\theta_n^* = \theta_0^*$, such that $\eta_1 := \delta((b-a)z,\theta_0^*) > 0$. Indeed, we may suppose, without less of generality, that

$$\begin{gathered}\eta := \delta((b-a)z,\theta) > 0,\\ \theta \in (\theta_0+2k\pi, \theta_0+(2k+1)\pi),\\ \eta := \delta((b-a)z,\theta) < 0,\\ \theta \in (\theta_0+(2k-1)\pi, \theta_0+2k\pi)\end{gathered} \tag{40}$$

with $k \in \mathbb{Z}$. When n is large enough, we have $|\theta_n-\theta_0| \le l_0$, where l_0 is a small constant. Choose now θ_n^* such that $l_0/2 \le \theta_n^*-\theta_n \le l_0$. Then $\theta_0+l_0/2 \le \theta_0^* \le \theta_0+l_0$. For sufficiently large n, we have (12) for z_n^* and $\eta_1 := \delta((b-a)z,\theta_0^*) > 0$. Therefore

$$\left|\frac{H(z_n^*)}{f(z_n^*)}\right| \le \frac{M(r_n,H)}{M(r_n,f)^{1/5}} \longrightarrow 0, \tag{41}$$

$$\begin{aligned}&\exp\left\{\frac{1-\varepsilon}{2}\eta_1 r_n\right\}\\&\quad\le \left|\frac{A_0(z_n^*)}{A_1(z_n^*)}e^{(b-a)z_n^*}\right| \le \exp\left\{\frac{3(1+\varepsilon)}{2}\eta_1 r_n\right\}\end{aligned} \tag{42}$$

for sufficiently large n. Taking now l_0 small enough, we have $\delta(az,\theta_0^*) > 0$, by the continuity of $\delta(az,\theta)$. This yields

$$\exp\left\{\frac{1-\varepsilon}{2}\delta\left(az,\theta_0^*\right)r_n\right\} \leq \left|A_1\left(z_n^*\right)e^{az_n^*}\right| \leq \exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta\left(az,\theta_0^*\right)r_n\right\}. \tag{43}$$

Similarly as (36), a contradiction easily follows.

Case 2. Suppose now that $\delta(az,\theta_0) < 0$. Then, from the continuity of $\delta(az,\theta)$ and (10), we have

$$\exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta\left(az,\theta_0\right)r_n\right\} \leq \left|A_1\left(z_n\right)e^{az_n}\right| \leq \exp\left\{\frac{1-\varepsilon}{2}\delta\left(az,\theta_0\right)r_n\right\} \tag{44}$$

for n large enough.

Subcase 2.1. Assume first that $\delta(bz,\theta_0) > 0$. From the continuity of $\delta(bz,\theta)$ and (9), we deduce that

$$\exp\left\{\frac{1-\varepsilon}{2}\delta\left(bz,\theta_0\right)r_n\right\} \leq \left|A_0\left(z_n\right)e^{bz_n}\right| \leq \exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta\left(bz,\theta_0\right)r_n\right\} \tag{45}$$

for n large enough. From (4), we have

$$\left|A_0e^{bz}\right| \leq \left|\frac{f''}{f}\right| + \left|A_1e^{az}\right|\left|\frac{f'}{f}\right| + \left|\frac{H}{f}\right|. \tag{46}$$

Substituting (24), (28), (29), (44), and (45) into (46), we obtain

$$\begin{aligned}\exp\left\{\frac{1-\varepsilon}{2}\delta\left(bz,\theta_0\right)r_n\right\} &\leq \exp\left\{2r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-2}\left(1+O\left(1\right)\right) \\ &\quad + \exp\left\{\frac{1-\varepsilon}{2}\delta\left(az,\theta_0\right)r_n\right\} \\ &\quad \times \exp\left\{r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-1}\left(1+O\left(1\right)\right)+\varepsilon_0.\end{aligned} \tag{47}$$

Since $\alpha_0+\varepsilon_1 < 1$, we see that (47) is contradictory as $n \to \infty$.

Subcase 2.2. Assume that $\delta(bz,\theta_0) < 0$. From the continuity of $\delta(bz,\theta)$ and (9), we deduce that

$$\exp\left\{\frac{3\left(1+\varepsilon\right)}{2}\delta\left(bz,\theta_0\right)r_n\right\} \leq \left|A_0\left(z_n\right)e^{bz_n}\right| \leq \exp\left\{\frac{1-\varepsilon}{2}\delta\left(bz,\theta_0\right)r_n\right\} \tag{48}$$

for n large enough. From (4), we have

$$\left|\frac{f''}{f}\right| \leq \left|A_0e^{bz}\right| + \left|A_1e^{az}\right|\left|\frac{f'}{f}\right| + \left|\frac{H}{f}\right|. \tag{49}$$

Substituting (24), (28), (29), (44), and (48) into (49), we obtain

$$\begin{aligned}\left(\frac{\nu_f\left(r_n\right)}{r_n}\right)^2\left(1+O\left(1\right)\right) &\leq \exp\left\{\frac{1-\varepsilon}{2}\delta\left(bz,\theta_0\right)r_n\right\} \\ &\quad + \exp\left\{\frac{1-\varepsilon}{2}\delta\left(az,\theta_0\right)r_n\right\} \\ &\quad \times \exp\left\{r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-1}\left(1+O\left(1\right)\right)+\varepsilon_0\end{aligned} \tag{50}$$

as $n \to \infty$. Since $\alpha_0+\varepsilon_1 < 1$, this implies that $\nu_f(r) \to 0$, $n \to \infty$, which is impossible.

Subcase 2.3. Assume that $\delta(bz,\theta_0) = 0$. Arguing similarly as in Subcase 1.3, we may again construct another sequence of points $\{z_n^* = r_ne^{i\theta_n^*}\}$ with $\lim_{n\to\infty}\theta_n^* = \theta_0^*$, such that $\delta(az,\theta_0^*) < 0 < \delta(bz,\theta_0^*)$. Replace $\delta(az,\theta_0)$ with $\delta(az,\theta_0^*)$ in (44) and $\delta(bz,\theta_0)$ with $\delta(bz,\theta_0^*)$ in (45), respectively. We obtain (44) and (45) for the sequence of $\{z_n^* = r_ne^{i\theta_n^*}\}$. Similarly as (47), we get a contradiction as $n \to \infty$.

Case 3. In this final case, we suppose that $\delta(az,\theta_0) = 0$. We discuss three subcases according to $\delta(bz,\theta_0)$ as follows.

Subcase 3.1. Suppose that $\delta(bz,\theta_0) > 0$. By an argument similar to that in Subcase 1.3, we can choose another sequence of points $\{z_n^* = r_ne^{i\theta_n^*}\}$ with $\lim_{n\to\infty}\theta_n^* = \theta_0^*$, and $l_0/2 \leq \theta_0^* - \theta_0 \leq l_0$, such that $\delta(az,\theta_0^*) < 0 < \delta(bz,\theta_0^*)$. Similarly as in Subcase 2.3, a contradiction follows as $n \to \infty$.

Subcase 3.2. Suppose that $\delta(bz,\theta_0) < 0$. By an argument similar to the Subcase 3.2 of the proof of Theorem 1.1 in [7], we can choose another sequence of points $\{z_n^* = r_ne^{i\theta_n^*}\}$, with $\lim_{n\to\infty}\theta_n^* = \theta_0^*$, such that $\delta(bz,\theta_0^*) < 0 < \delta(az,\theta_0^*)$. From (4), for $\{z_n^* = r_ne^{i\theta_n^*}\}$, we get

$$\left|\frac{f'\left(z_n^*\right)}{f\left(z_n^*\right)}\right| \leq \left|\frac{1}{A_1\left(z_n^*\right)e^{az_n^*}}\right| \times\left(\left|A_0\left(z_n^*\right)e^{bz_n^*}\right| + \left|\frac{f''\left(z_n^*\right)}{f\left(z_n^*\right)}\right| + \left|\frac{H\left(z_n^*\right)}{f\left(z_n^*\right)}\right|\right). \tag{51}$$

Replace $\delta(az,\theta_0)$ with $\delta(az,\theta_0^*)$ in (32) and $\delta(bz,\theta_0)$ with $\delta(bz,\theta_0^*)$ in (48), respectively. We obtain (32) and (48) for the sequence of $\{z_n^* = r_ne^{i\theta_n^*}\}$. Substituting them into (51), this implies that $\nu_f(r) \to 0, n \to \infty$, which is impossible.

Subcase 3.3. Finally, suppose that $\delta(bz,\theta_0) = 0$. We now have $a/b = c \in \mathbb{R}, c \neq 0, 1$, and so $az = cbz$, $(b-a)z = (1-c)bz$.

If $c < 0$, we may choose another sequence such that $\delta(bz,\theta) < 0 < \delta(az,\theta)$. By an argument similar to that in Subcase 3.2, we can get $\nu_f(r) \to 0, n \to \infty$, a contradiction.

If $0 < c < 1$, we similarly obtain $\delta((b-a)z,\theta) > 0$ and $\delta(az,\theta) > 0$ for another sequence. By an argument similar to that in Subcase 1.3, a contradiction follows.

Finally, if $c > 1$, we obtain $\delta((b-a)z,\theta) < 0 < \delta(az,\theta)$ for another sequence. Similarly as in Subcase 1.2, a contradiction again follows.

Thus, we complete the proof when $0 < \alpha_0 < 1$. When $\alpha_0 = 0$, we have (30). Similarly as the case when $0 < \alpha_0 < 1$, it results in contradiction. Hence, we get $\sigma_2(f) = \alpha_0 = 1$. □

Proof of Theorem 3. Suppose that f is a nontrivial solution of (5), and then f is an entire function. Since $\varrho = \max\{\sigma(D_0), \sigma(D_1)\} < 1$, we have

$$\left|D_j(z)\right| \leq \exp\left\{r^{\varrho+\varepsilon}\right\} \quad (j = 0, 1) \tag{52}$$

for any such that $0 < 3\varepsilon < 1 - \varrho$. Similarly as in the proof of Theorem 1, we may choose a sequence of points $\{z_n = r_n e^{i\theta_n}\}$ that satisfy $|f(z_n)| = M(r_n, f)$, with $\lim_{n\to\infty}\theta_n = \theta_0$, $r_n \in E \setminus (E_1 \cup E_2 \cup E_3)$, $r_n \to \infty$.

Step 1. We will prove that $\sigma_2(f) \leq 1$. Since $\sigma(A_0; A_1; H) < 1$, set $\sigma(H) = \lambda < 1$. Then for any given ε satisfying $\varepsilon < \min\{1-\lambda, (1-\varrho)/3\}$, when r is sufficiently large, we have

$$\begin{gathered}\left|A_1 e^{az} + D_1(z)\right| \leq \exp\left\{r^{1+\varepsilon}\right\}, \\ \left|A_0 e^{bz} + D_0(z)\right| \leq \exp\left\{r^{1+\varepsilon}\right\},\end{gathered} \tag{53}$$

$$|H(z)| \leq \exp\left\{r^{\lambda+\varepsilon}\right\}. \tag{54}$$

From the Wiman-Valiron theory, we have (24). By Theorem C, we know that $\sigma(f) = \infty$. So we have (28). From (5) we have

$$\frac{f''}{f} + \left(A_1 e^{az} + D_1\right)\frac{f'}{f} + \left(A_0 e^{bz} + D_0\right) = \frac{H}{f}. \tag{55}$$

Substituting (24), (28), and (53) into (55), we obtain (26) and (27); thus, we have $\sigma_2(f) \leq 1$.

Step 2. Set $\sigma_2(f) = \alpha_0$, and we assert that $\alpha_0 = 1$. Now we assume that $\alpha_0 < 1$, and prove that $\sigma_2(f) = \alpha_0 < 1$ results in contradiction. By Lemma 5, we have (29) and (30). Next we only prove the case $0 < \sigma_2(f) = \alpha_0 \leq 1$ by using (29). The case $\sigma_2(f) = \alpha_0 = 0$ also can be proved by the same method, a little different is that we use (30) instead of (29). Since $a = bc$, c (< 0) is a real number, there are three cases to be discussed, according to the signs of $\delta(az,\theta_0)$ and $\delta(bz,\theta_0)$.

Case 1. First assume that $\delta(bz,\theta_0) < 0 < \delta(az,\theta_0)$, so we have (32) and (48). Combining (52), (32), and (48), we deduce

$$\left|A_0(z_n)e^{bz_n} + D_0(z_n)\right| \leq \exp\left\{r^{\varrho+2\varepsilon}\right\}, \tag{56}$$

$$\left|A_1(z_n)e^{az_n} + D_1(z_n)\right| \geq \exp\left\{\frac{1-2\varepsilon}{2}\delta(az,\theta_0)r_n\right\} \tag{57}$$

provided that n is large enough. From (5), we have

$$\begin{aligned}&\left|\frac{f'(z_n)}{f(z_n)}\right| \\ &\quad\leq \frac{1}{\left|A_1(z_n)e^{az_n} + D_1(z_n)\right|} \\ &\qquad\times\left(\left|\frac{f''(z_n)}{f(z_n)}\right| + \left|A_0(z_n)e^{bz_n} + D_0(z_n)\right| + \left|\frac{H(z_n)}{f(z_n)}\right|\right).\end{aligned} \tag{58}$$

Substituting (24), (28), (56), and (57) into (58), we obtain

$$\begin{aligned}\frac{\nu_f(r_n)}{r_n}(1+O(1)) &\leq \exp\left\{-\frac{1-2\varepsilon}{2}\delta(az,\theta_0)r_n\right\} \\ &\quad\times\left(\exp\left\{2r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-2}(1+O(1))\right. \\ &\qquad\left.+\exp\left\{r_n^{\varrho+2\varepsilon}\right\}+\varepsilon_0\right)\end{aligned} \tag{59}$$

for n large enough. Since $\alpha_0 + \varepsilon_1 < 1$, this implies that $\nu_f(r_n) \to 0$, $n \to \infty$, which is impossible.

Case 2. Next, assume that $\delta(az,\theta_0) < 0 < \delta(bz,\theta_0)$, so we have (44) and (45). Combining (52), (44), and (45), we deduce

$$\begin{aligned}&\left|A_0(z_n)e^{bz_n} + D_0(z_n)\right| \\ &\quad\geq \exp\left\{\frac{1-2\varepsilon}{2}\delta(bz,\theta_0)r_n\right\},\end{aligned} \tag{60}$$

$$\left|A_1(z_n)e^{az_n} + D_1(z_n)\right| \leq \exp\left\{r_n^{\varrho+2\varepsilon}\right\} \tag{61}$$

for n is large enough. From (5), we have

$$\begin{aligned}&\left|A_0(z_n)e^{bz_n} + D_0(z_n)\right| \\ &\quad\leq \left|\frac{f''(z_n)}{f(z_n)}\right| + \left|A_1(z_n)e^{az_n} + D_1(z_n)\right| \\ &\qquad\times\left|\frac{f'(z_n)}{f(z_n)}\right| + \left|\frac{H(z_n)}{f(z_n)}\right|.\end{aligned} \tag{62}$$

Substituting (24), (28), (60), and (61) into (62), we obtain

$$\begin{aligned}&\exp\left\{\frac{1-2\varepsilon}{2}\delta(bz,\theta_0)r_n\right\} \\ &\quad\leq \exp\left\{2r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-2}(1+O(1)) \\ &\qquad+\exp\left\{r_n^{\varrho+2\varepsilon}\right\}\exp\left\{r_n^{\alpha_0+\varepsilon_1}\right\}r_n^{-1}(1+O(1))+\varepsilon_0\end{aligned} \tag{63}$$

for n large enough. Since $\alpha_0 + \varepsilon_1 < 1$, $\varrho + 2\varepsilon < 1$, this leads to a contradiction.

Case 3. Finally, we have to assume that $\delta(az,\theta_0) = \delta(bz,\theta_0) = 0$. Similarly as in Subcase 1.3 of the proof of Theorem 1, we may again construct a sequence of points $\{z_n^* = r_n e^{i\theta_n^*}\}$, with

$\lim_{n\to\infty}\theta_n^* = \theta_0^*$, such that $\delta(az, \theta_0^*) < 0$. Indeed, without loss of generality,

$$\begin{aligned} \delta(az, \theta) > 0, \quad & \theta \in (\theta_0 + 2k\pi, \theta_0 + (2k+1)\pi), \\ \delta(az, \theta) < 0, \quad & \theta \in (\theta_0 + (2k-1)\pi, \theta_0 + 2k\pi) \end{aligned} \tag{64}$$

for all $k \in \mathbb{Z}$. Provided that n is large enough, we have $|\theta_n - \theta_0| \le l_0$. Choosing now θ_0^* such that $l_0/2 \le \theta_n - \theta_n^* \le l_0$, then $l_0/2 \le \theta_0 - \theta_0^* \le l_0$, thus, $\theta_0 - l_0 \le \theta_0^* \le \theta_0 - l_0/2$, and $\delta(az, \theta_0^*) < 0$. Since now $\delta(bz, \theta_0^*) > 0$, a contradiction follows as in Case 2 above. □

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