

Research Article

Some New Gronwall-Bellman-Type Inequalities on Time Scales and Their Applications

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We establish some new Gronwall-Bellman-type inequalities on time scales. These inequalities are of new forms compared with other Gronwall-Bellman-type inequalities established so far in the literature. Based on them, new bounds for unknown functions are derived. For illustrating the validity of the inequalities established, we present some applications for them, in which the boundedness for solutions for some certain dynamic equations on time scales is researched.

1. Introduction

As is known, various integral and differential inequalities play an important role in the research of boundedness, global existence, and stability of solutions of differential and integral equations as well as difference equations. Among the investigations for inequalities, generalization of the Gronwall-Bellman-inequality [1, 2] is a hot topic, as such inequalities provide explicit bounds for unknown functions concerned. During the past decades, many Gronwall-Bellman-type inequalities have been discovered (e.g., see [3–18]). Recently, with the development of the theory of time scales [19], many integral inequalities on time scales have been established, for example, [20–32], which have proved to be very effective in the analysis of qualitative as well as quantitative analysis of solutions of dynamic equations. But for some certain dynamic equations, for example,

$$(u^p(t))^\Delta = F\left(t, u(t), u(t), \int_t^\infty W(\xi, u(\xi)) \Delta\xi\right), \quad (1)$$

or

$$u^p(t) = C + \int_t^\infty \widehat{F}\left(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta\xi\right) \Delta s$$

$$+ \int_T^\infty \widehat{F}\left(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta\xi\right) \Delta s, \quad (2)$$

it is inadequate to research the boundedness of their solutions by use of the existing results in the literature. So it is necessary to seek new approach to fulfill such analysis for them.

Based on the analysis above, in this paper, we establish some new Gronwall-Bellman-type inequalities on time scales, which are designed so as to be used as a handy tool to research the boundedness of the solutions of the equations mentioned above. By use of the established inequalities, some new bounds for the solutions for the two equations are derived.

In the rest of the paper, \mathbb{R} denotes the set of real numbers, and $\mathbb{R}_+ = [0, \infty)$. \mathbb{T} denotes an arbitrary time scale, and $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}$. On \mathbb{T} we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. The graininess $\mu \in (\mathbb{T}, \mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$. Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$, while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$. A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The set \mathbb{T}^κ is defined to be \mathbb{T} if \mathbb{T} does not have a left scattered maximum; otherwise it is \mathbb{T} without the left scattered maximum.

The following definitions and theorems in the theory of time scales are known to us.

Definition 1. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t)f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while \mathfrak{R} denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R}, 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 2. For some $t \in \mathbb{T}^\kappa$ and a function $f \in (\mathbb{T}, \mathbb{R})$, the delta derivative of f at t is denoted by $f^\Delta(t)$ (provided that it exists) with the property such that, for every $\varepsilon > 0$, there exists a neighborhood \mathfrak{U} of t satisfying

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in \mathfrak{U}. \quad (3)$$

Definition 3. If $F^\Delta(t) = f(t)$ and $t \in \mathbb{T}^\kappa$, then F is called an antiderivative of f , and the Cauchy integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a), \quad (4)$$

where $a, b \in \mathbb{T}^\kappa$.

Definition 4. For $p \in \mathfrak{R}$, the exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \quad \text{for } s, t \in \mathbb{T}. \quad (5)$$

Definition 5. If $\sup_{t \in \mathbb{T}} t = \infty$ and $p \in \mathfrak{R}$, we define

$$e_p(\infty, s) = \exp\left(\int_s^\infty \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \quad \text{for } t \in \mathbb{T}. \quad (6)$$

Theorem 6 (see [20, Theorem 2.1]). *If $f, g \in (\mathbb{T}, \mathbb{R})$ and $t \in \mathbb{T}^\kappa$, then*

(i)

$$f^\Delta(t) = \begin{cases} \frac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if } \mu(t) \neq 0, \\ \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} & \text{if } \mu(t) = 0. \end{cases} \quad (7)$$

(ii) *If f, g are delta differential at t , then fg is also delta differential at t , and*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t). \quad (8)$$

Theorem 7 (see [20, Theorem 2.2]). *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then one has the following:*

- (i) $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t;$
- (ii) $\int_a^b (\alpha f)(t) \Delta t = \alpha \int_a^b f(t) \Delta t;$

$$(iii) \int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t;$$

$$(iv) \int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t;$$

$$(v) \int_a^a f(t) \Delta t = 0;$$

$$(vi) \text{ if } f(t) \geq 0 \text{ for all } a \leq t \leq b, \text{ then } \int_a^b f(t) \Delta t \geq 0.$$

Theorem 8 (see [29, Theorem 5.2]). *If $p \in \mathfrak{R}$, then the following conclusions hold:*

$$(i) e_p(t, t) \equiv 1 \text{ and } e_0(t, s) \equiv 1;$$

$$(ii) e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s);$$

$$(iii) \text{ if } p \in \mathfrak{R}^+, \text{ then } e_p(t, s) > 0 \text{ for all } s, t \in \mathbb{T};$$

$$(iv) \text{ if } p \in \mathfrak{R}^+, \text{ then } \ominus p \in \mathfrak{R}^+;$$

$$(v) e_p(t, s) = 1/(e_{\ominus p}(s, t)) = e_{\ominus p}(s, t),$$

where $\ominus p = -p/(1 + \mu p)$.

Remark 9. If $s = \infty$, then Theorem 8 ((iii), (v)) still holds.

Theorem 10 (see [29, Theorem 5.1]). *If $p \in \mathfrak{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(t, t_0)$ is the unique solution of the following initial value problem:*

$$\begin{aligned} y^\Delta(t) &= p(t) y(t), \\ y(t_0) &= 1. \end{aligned} \quad (9)$$

2. Main Results

Lemma 11. *Suppose that $u, a, m \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, $\bar{m}(t) = -m(t)b(t)$, and $\bar{m} \in \mathfrak{R}_+$. Then*

$$u(t) \leq a(t) + b(t) \int_t^\infty m(s) u(s) \Delta s, \quad t \in \mathbb{T}_0, \quad (10)$$

implies

$$u(t) \leq a(t) + b(t) \int_t^\infty e_{\bar{m}}(t, \sigma(s)) m(s) a(s) \Delta s, \quad t \in \mathbb{T}_0. \quad (11)$$

Proof. Denote that $v(t) = \int_t^\infty m(s)u(s)\Delta s$. Then

$$u(t) \leq a(t) + b(t)v(t),$$

$$v^\Delta(t) = -m(t)u(t) \geq -m(t)b(t)v(t) \quad (12)$$

$$-m(t)a(t) = \bar{m}(t)v(t) - m(t)a(t).$$

Since $\bar{m} \in \mathfrak{R}^+$, then from Theorem 8 (iv), we have $\ominus \bar{m} \in \mathfrak{R}^+$, and furthermore from Theorem 8 (iii), we obtain $e_{\ominus \bar{m}}(t, \alpha) > 0$ for all $\alpha \in \mathbb{T}_0$.

According to Theorem 6 (ii),

$$[v(t)e_{\ominus \bar{m}}(t, \alpha)]^\Delta = [e_{\ominus \bar{m}}(t, \alpha)]^\Delta v(t) + e_{\ominus \bar{m}}(\sigma(t), \alpha) v^\Delta(t). \quad (13)$$

On the other hand, from Theorem 10, we have

$$[e_{\ominus\bar{m}}(t, \alpha)]^\Delta = (\ominus\bar{m})(t) e_{\ominus\bar{m}}(t, \alpha). \tag{14}$$

So combining (13), (14) and Theorem 8, it follows that

$$\begin{aligned} & [v(t) e_{\ominus\bar{m}}(t, \alpha)]^\Delta \\ &= (\ominus\bar{m})(t) e_{\ominus\bar{m}}(t, \alpha) v(t) + e_{\ominus\bar{m}}(\sigma(t), \alpha) v^\Delta(t) \\ &= e_{\ominus\bar{m}}(\sigma(t), \alpha) \left[\frac{(\ominus\bar{m})(t)}{1 + \mu(t) (\ominus\bar{m})(t)} v(t) + v^\Delta(t) \right] \\ &= e_{\ominus\bar{m}}(\sigma(t), \alpha) [v^\Delta(t) - \bar{m}(t) v(t)]. \end{aligned} \tag{15}$$

Substituting t with s , an integration for (15) with respect to s from α to ∞ yields

$$\begin{aligned} & v(\infty) e_{\ominus\bar{m}}(\infty, \alpha) - v(\alpha) e_{\ominus\bar{m}}(\alpha, \alpha) \\ &= \int_\alpha^\infty e_{\ominus\bar{m}}(\sigma(s), \alpha) [v^\Delta(s) - \bar{m}(s) v(s)] \Delta s. \end{aligned} \tag{16}$$

By $v(\infty) = 0$ and $e_{\ominus\bar{m}}(\alpha, \alpha) = 1$, it follows that

$$\begin{aligned} -v(\alpha) &\geq - \int_\alpha^\infty e_{\ominus\bar{m}}(\sigma(s), \alpha) m(s) a(s) \Delta s \\ &= - \int_\alpha^\infty e_{\bar{m}}(\alpha, \sigma(s)) m(s) a(s) \Delta s, \end{aligned} \tag{17}$$

which is followed by

$$v(\alpha) \leq \int_\alpha^\infty e_{\bar{m}}(\alpha, \sigma(s)) m(s) a(s) \Delta s. \tag{18}$$

Since $\alpha \in \mathbb{T}_0$ is arbitrary, after substituting α with t , we obtain the desired inequality. \square

Lemma 12. Under the conditions of Lemma 11, furthermore, if $b(t) \equiv 1$ and $a(t)$ is nonincreasing on \mathbb{T}_0 , then

$$u(t) \leq a(t) e_{-m}(t, \infty), \quad t \in \mathbb{T}_0. \tag{19}$$

Proof. Since $b(t) \equiv 1$ and $a(t)$ is nondecreasing on \mathbb{T}_0 , then $\bar{m} = m$ and

$$\begin{aligned} u(t) &\leq a(t) + \int_t^\infty e_{-m}(t, \sigma(s)) a(s) m(s) \Delta s \\ &\leq a(t) \left[1 + \int_t^\infty e_{-m}(t, \sigma(s)) m(s) \Delta s \right]. \end{aligned} \tag{20}$$

From [33, Theorems 2.39 and 2.36 (i)], we have

$$\begin{aligned} & \int_t^\infty e_{-m}(t, \sigma(s)) m(s) \Delta s \\ &= \lim_{x \rightarrow \infty} \int_x^t e_{-m}(t, \sigma(s)) (-m(s)) \Delta s \\ &= \lim_{x \rightarrow \infty} e_{-m}(t, x) - e_{-m}(t, t) = e_{-m}(t, \infty) - 1. \end{aligned} \tag{21}$$

Combining the above information, we can obtain the desired inequality. \square

Lemma 13 (see [4]). Assume that $a \geq 0, p \geq q \geq 0$, and $p \neq 0$; then for any $K > 0$, the following inequality holds:

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}. \tag{22}$$

First one will study the following Gronwall-Bellman-type inequality on time scales:

$$\begin{aligned} u^p(t) &\leq a(t) + b(t) \\ &\times \int_t^\infty \left[m(s) + f(s) u^p(s) \right. \\ &\quad \left. + g(s) u^q(s) + \int_s^\infty h(\xi) u^r(\xi) \Delta \xi \right] \Delta s. \end{aligned} \tag{23}$$

Theorem 14. Suppose that $u, f, g, h, m, a, b \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, a, b are nonincreasing, p, q , and r are constants, $p \geq q \geq 0, p \geq r \geq 0$, and $p \neq 0$. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies (23), then

$$u(t) \leq \left\{ [a(t) + b(t) H_2(t) e_{-H_3}(t, \infty)] H_1(t) \right\}^{1/p}, \quad t \in \mathbb{T}_0, \tag{24}$$

Provided that $\tilde{f}, -H_3 \in \mathfrak{R}_+$, where

$$\begin{aligned} H_1(t) &= 1 + b(t) \int_t^\infty e_{\tilde{f}}(t, \sigma(s)) f(s) \Delta s, \\ \tilde{f}(t) &= -f(t) b(t), \end{aligned} \tag{25}$$

$$\begin{aligned} H_2(t) &= \int_t^\infty \left\{ m(s) + g(s) \left[\frac{q}{p} K^{(q-p)/p} a(s) + \frac{p-q}{p} K^{q/p} \right] \right. \\ &\quad \times (H_1(s))^{q/p} \Delta s \\ &\quad \left. + \int_t^\infty \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} a(\xi) + \frac{p-r}{p} K^{r/p} \right] \right. \\ &\quad \left. \times (H_1(\xi))^{r/p} \Delta \xi \right\} \Delta s, \quad \forall K > 0, \end{aligned} \tag{26}$$

$$\begin{aligned} H_3(t) &= g(t) \frac{q}{p} K^{(q-p)/p} b(t) (H_1(t))^{q/p} \\ &\quad + \int_t^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} b(\xi) (H_1(\xi))^{r/p} \Delta \xi, \end{aligned} \tag{27}$$

$\forall K > 0.$

Proof. Let the right side of (23) be $v(t)$. Then

$$u(t) \leq v^{1/p}(t), \quad t \in \mathbb{T}_0, \tag{28}$$

$$\begin{aligned} v(t) &\leq a(t) + b(t) \\ &\times \int_t^\infty \left[m(s) + f(s) v(s) + g(s) v^{q/p}(s) \right. \\ &\quad \left. + \int_s^\infty h(\xi) v^{r/p}(\xi) \Delta \xi \right] \Delta s. \end{aligned} \tag{29}$$

Let

$$c(t) = a(t) + b(t) \times \int_t^\infty \left[m(s) + g(s) v^{q/p}(s) + \int_s^\infty h(\xi) v^{r/p}(\xi) \Delta\xi \right] \Delta s. \tag{30}$$

Then we have

$$v(t) \leq c(t) + b(t) \int_t^\infty f(s) v(s) \Delta s, \quad t \in \mathbb{T}_0. \tag{31}$$

From Lemma 11, considering $c(t)$ is nonincreasing on \mathbb{T}_0 , we can obtain

$$\begin{aligned} v(t) &\leq c(t) + b(t) \int_t^\infty e_{\tilde{f}}(t, \sigma(s)) c(s) f(s) \Delta s \\ &\leq c(t) \left[1 + b(t) \int_t^\infty e_{\tilde{f}}(t, \sigma(s)) f(s) \Delta s \right] \\ &= c(t) H_1(t), \quad t \in \mathbb{T}_0, \end{aligned} \tag{32}$$

where $H_1(t)$, $\tilde{f}(t)$ are defined in (25).

Let

$$y(t) = \int_t^\infty \left[m(s) + g(s) v^{q/p}(s) + \int_s^\infty h(\xi) v^{r/p}(\xi) \Delta\xi \right] \Delta s. \tag{33}$$

Then

$$c(t) = a(t) + b(t) y(t). \tag{34}$$

From Lemma 13, we have

$$\begin{aligned} &(a(t) + b(t) y(t))^{q/p} \\ &\leq \frac{q}{p} K^{(q-p)/p} (a(t) + b(t) y(t)) + \frac{p-q}{p} K^{q/p}, \\ &(a(t) + b(t) y(t))^{r/p} \\ &\leq \frac{r}{p} K^{(r-p)/p} (a(t) + b(t) y(t)) + \frac{p-r}{p} K^{r/p}, \quad \forall K > 0. \end{aligned} \tag{35}$$

A combination of (32), (33), (34), and (35) yields

$$\begin{aligned} y(t) &\leq \int_t^\infty \left[m(s) + g(s) (c(s) H_1(s))^{q/p} \right. \\ &\quad \left. + \int_{t_0}^s h(\xi) (c(\xi) H_1(\xi))^{r/p} \Delta\xi \right] \Delta s \\ &\leq \int_t^\infty \left\{ m(s) + g(s) [(a(s) + b(s) y(s)) H_1(s)]^{q/p} \right. \\ &\quad \left. + \int_s^\infty h(\xi) [(a(\xi) + b(\xi) y(\xi)) H_1(\xi)]^{r/p} \Delta\xi \right\} \Delta s \end{aligned}$$

$$\begin{aligned} &\leq \int_t^\infty \left\{ m(s) + g(s) \left[\frac{q}{p} K^{(q-p)/p} (a(s) + b(s) y(s)) \right. \right. \\ &\quad \left. \left. + \frac{p-q}{p} K^{q/p} \right] (H_1(s))^{q/p} \right\} \Delta s \\ &\quad + \int_t^\infty \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} (a(\xi) + b(\xi) y(\xi)) \right. \\ &\quad \left. + \frac{p-r}{p} K^{r/p} \right] (H_1(\xi))^{r/p} \Delta\xi \Delta s \\ &= \int_t^\infty \left\{ m(s) + g(s) \left[\frac{q}{p} K^{(q-p)/p} a(s) + \frac{p-q}{p} K^{q/p} \right] \right. \\ &\quad \times (H_1(s))^{q/p} \\ &\quad + \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} a(\xi) \right. \\ &\quad \left. + \frac{p-r}{p} K^{r/p} \right] (H_1(\xi))^{r/p} \Delta\xi \left. \right\} \Delta s \\ &\quad + \int_t^\infty \left[g(s) \frac{q}{p} K^{(q-p)/p} b(s) (H_1(s))^{q/p} y(s) \right. \\ &\quad \left. + \int_s^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} \right. \\ &\quad \left. \times b(\xi) (H_1(\xi))^{r/p} y(\xi) \Delta\xi \right] \Delta s \\ &\leq H_2(t) + \int_t^\infty H_3(s) y(s) \Delta s, \end{aligned} \tag{36}$$

where $H_2(t)$, $H_3(t)$ are defined in (26) and (27), respectively. By $H_2(t)$ being nonincreasing on \mathbb{T}_0 and $-H_3 \in \mathfrak{R}_+$, according to Lemma 12, we have

$$y(t) \leq H_2(t) e_{-H_3}(t, \infty), \quad t \in \mathbb{T}_0. \tag{37}$$

Combining (32), (34), and (37), we obtain

$$v(t) \leq [a(t) + b(t) H_2(t) e_{-H_3}(t, \infty)] H_1(t). \tag{38}$$

From (28), (38), we can obtain the desired inequality (13). \square

Based on Theorem 14, we will establish two Volterra-Fredholm type delay integral inequalities on time scales in the following two theorems.

Theorem 15. Suppose that $u, f, g, h, m, p, q,$ and r are defined as in Theorem 14 with $-f \in \mathfrak{R}_+, C > 0$ is a constant,

and $T \in \mathbb{T}_0$ is a fixed number. If for $t \in [T, \infty) \cap \mathbb{T}$, $u(t)$ satisfies the following inequality:

$$\begin{aligned}
 u^p(t) \leq & C + \int_t^\infty \left[m(s) + f(s)u^p(s) + g(s)u^q(s) \right. \\
 & \left. + \int_s^\infty h(\xi)u^r(\xi)\Delta\xi \right] \Delta s \\
 & + \int_T^\infty \left[m(s) + f(s)u^p(s) + g(s)u^q(s) \right. \\
 & \left. + \int_s^\infty h(\xi)u^r(\xi)\Delta\xi \right] \Delta s,
 \end{aligned} \tag{39}$$

and furthermore, $(1 + \tilde{H}_{21}(T)e_{\tilde{H}_3}(T, \infty))e_{-f}(T, \infty) < 2$, then for $t \in [T, \infty) \cap \mathbb{T}$,

$$\begin{aligned}
 u(t) \leq & \left\{ \left[\frac{C + \tilde{H}_{22}(T)e_{-\tilde{H}_3}(T, \infty)e_f(T, \infty)}{2 - (1 + \tilde{H}_{21}(T)e_{-\tilde{H}_3}(T, \infty))e_f(T, \infty)} \right] \right. \\
 & \times [1 + \tilde{H}_{21}(t)e_{-\tilde{H}_3}(t, \infty)] \\
 & \left. + \tilde{H}_{22}(t)e_{-\tilde{H}_3}(t, \infty) \right\} e_{-f}(t, \infty) \Bigg\}^{1/p},
 \end{aligned} \tag{40}$$

provided that $-\tilde{H}_3 \in \mathfrak{R}_+$, where

$$\tilde{H}_1(t) = 1 + \int_t^\infty e_{\tilde{f}}(t, \sigma(s))f(s)\Delta s, \quad \tilde{f}(t) = -f(t)b(t), \tag{41}$$

$$\begin{aligned}
 \tilde{H}_{21}(t) = & \int_t^\infty \left\{ g(s) \frac{q}{p} K^{(q-p)/p} (\tilde{H}_1(s))^{q/p} \right. \\
 & \left. + \int_s^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} (\tilde{H}_1(\xi))^{r/p} \Delta\xi \right\} \Delta s, \\
 & \forall K > 0,
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 \tilde{H}_{22}(t) = & \int_t^\infty \left\{ m(s) + g(s) \frac{p-q}{p} K^{q/p} (\tilde{H}_1(s))^{q/p} \right. \\
 & \left. + \int_s^\infty h(\xi) \frac{p-r}{p} K^{r/p} (\tilde{H}_1(\xi))^{r/p} \Delta\xi \right\} \Delta s, \\
 & \forall K > 0,
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 \tilde{H}_3(t) = & g(t) \frac{q}{p} K^{(q-p)/p} (\tilde{H}_1(t))^{q/p} \\
 & + \int_t^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} (\tilde{H}_1(\xi))^{r/p} \Delta\xi, \quad \forall K > 0.
 \end{aligned} \tag{44}$$

Proof. Let the right side of (39) be $v(t)$. Then

$$u(t) \leq v^{1/p}(t), \quad t \in [T, \infty) \cap \mathbb{T}. \tag{45}$$

Considering $v(\infty) = C + \int_T^\infty [m(s) + f(s)u^p(s) + g(s)u^q(s) + \int_s^\infty h(\xi)u^r(\xi)\Delta\xi]\Delta s$, it follows that

$$\begin{aligned}
 v(t) = & v(\infty) \\
 & + \int_t^\infty \left[m(s) + f(s)u^p(s) + g(s)u^q(s) \right. \\
 & \left. + \int_s^\infty h(\xi)u^r(\xi)\Delta\xi \right] \Delta s \\
 \leq & v(\infty) \\
 & + \int_t^\infty \left[m(s) + f(s)v(s) \right. \\
 & \left. + g(s)v^{q/p}(s) + \int_s^\infty h(\xi)v^{r/p}(\xi)\Delta\xi \right] \Delta s, \\
 & t \in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{46}$$

We notice that the structure of (46) is just similar to that of (29). So following the same manner as the process of (29)–(38) in Theorem 14 (i.e., $v(\infty)$ takes the place of $a(t)$ in Theorem 14, and let $b(t) \equiv 1$ in Theorem 14), considering $\tilde{H}_1(t) = 1 + \int_t^\infty e_{-f}(t, \sigma(s))f(s)\Delta s = e_{-f}(t, \infty)$, we can obtain

$$\begin{aligned}
 v(t) \leq & [v(\infty)(1 + \tilde{H}_{21}(t)e_{-\tilde{H}_3}(t, \infty)) \\
 & + \tilde{H}_{22}(t)e_{-\tilde{H}_3}(t, \infty)] e_{-f}(t, \infty), \\
 & t \in [T, \infty) \cap \mathbb{T},
 \end{aligned} \tag{47}$$

where $\tilde{H}_{21}(t)$, $\tilde{H}_{22}(t)$, and $\tilde{H}_3(t)$ are defined in (42), (43), and (44), respectively. Setting $t = T$ in (47), we obtain

$$\begin{aligned}
 v(T) \leq & [v(\infty)(1 + \tilde{H}_{21}(T)e_{-\tilde{H}_3}(T, \infty)) \\
 & + \tilde{H}_{22}(T)e_{-\tilde{H}_3}(T, \infty)] e_{-f}(T, \infty).
 \end{aligned} \tag{48}$$

As $2v(\infty) - C = v(T)$, it follows that

$$\begin{aligned}
 2v(\infty) - C = & v(T) \\
 \leq & [v(\infty)(1 + \tilde{H}_{21}(T)e_{-\tilde{H}_3}(T, \infty)) \\
 & + \tilde{H}_{22}(T)e_{-\tilde{H}_3}(T, \infty)] e_{-f}(T, \infty);
 \end{aligned} \tag{49}$$

that is,

$$v(\infty) \leq \frac{C + \tilde{H}_{22}(T)e_{-\tilde{H}_3}(T, \infty)e_{-f}(T, \infty)}{2 - (1 + \tilde{H}_{21}(T)e_{-\tilde{H}_3}(T, \infty))e_{-f}(T, \infty)}. \tag{50}$$

Combining (45), (47), and (50), we can obtain the desired inequality (39). \square

Theorem 16. Suppose that $u, f, h, p,$ and r are defined as in Theorem 14 with $-f \in \mathfrak{R}_+, C > 0$ is a constant, $T \in \mathbb{T}_0$ is a fixed number, $L \in C(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$, and $0 \leq L(s, x) - L(s, y) \leq$

$M(s, y)(x - y)$ for $x \geq y \geq 0$, where $M \in C(\mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+)$. If for $t \in [T, \infty) \cap \mathbb{T}$, $u(t)$ satisfies the following inequality:

$$\begin{aligned}
 u^p(t) &\leq C \\
 &+ \int_t^\infty \left[f(s) u^p(s) + L(s, u(s)) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) u^r(\xi) \Delta\xi \right] \Delta s \\
 &+ \int_T^\infty \left[f(s) u^p(s) + L(s, u(s)) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) u^r(\xi) \Delta\xi \right] \Delta s,
 \end{aligned} \tag{51}$$

with the initial condition (11), and furthermore, $e_{-\widehat{H}_2}(T, \infty)e_{-f}(T, \infty) < 2$, then

$$\begin{aligned}
 u(t) &\leq \left\{ \left[\frac{C + \widehat{H}_1(T) e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)}{2 - e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)} + \widehat{H}_1(t) \right] \right. \\
 &\quad \left. \times e_{-\widehat{H}_2}(t, \infty) e_{-f}(t, \infty) \right\}^{1/p}, \quad t \in [T, \infty) \cap \mathbb{T},
 \end{aligned} \tag{52}$$

provided that $-\widehat{H}_2 \in \mathfrak{R}_+$, where

$$\begin{aligned}
 \widehat{H}_1(t) &= \int_t^\infty L\left(s, \frac{p-1}{p} K^{1/p} (e_{-f}(s, \infty))^{1/p}\right) \Delta s \\
 &+ \int_t^\infty \int_s^\infty h(\xi) \frac{p-r}{p} K^{r/p} (e_{-f}(\xi, \infty))^{r/p} \Delta\xi \Delta s, \\
 &\quad \forall K > 0,
 \end{aligned} \tag{53}$$

$$\begin{aligned}
 \widehat{H}_2(t) &= M\left(t, \frac{p-1}{p} K^{1/p} (e_{-f}(t, \infty))^{1/p}\right) \\
 &\times \frac{1}{p} K^{(1-p)/p} (e_{-f}(t, \infty))^{1/p} \\
 &+ \int_t^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} (e_{-f}(\xi, \infty))^{r/p} \Delta\xi, \\
 &\quad \forall K > 0.
 \end{aligned} \tag{54}$$

Proof. Let the right side of (51) be $v(t)$. Then

$$u(t) \leq v^{1/p}(t), \quad t \in [T, \infty) \cap \mathbb{T}. \tag{55}$$

Furthermore, considering $v(\infty) = C + \int_T^\infty [f(s)u^p(s) + L(s, u(s)) + \int_s^\infty h(\xi)u^r(\xi)\Delta\xi]\Delta s$, we have

$$\begin{aligned}
 v(t) &= v(\infty) + \int_t^\infty \left[f(s) u^p(s) + L(s, u(s)) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) u^r(\xi) \Delta\xi \right] \Delta s \\
 &\leq v(\infty) + \int_t^\infty \left[f(s) v(s) + L(s, v^{1/p}(s)) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) v^{r/p}(\xi) \Delta\xi \right] \Delta s.
 \end{aligned} \tag{56}$$

Let

$$c(t) = \int_t^\infty \left[L(s, v^{1/p}(s)) + \int_s^\infty h(\xi) v^{r/p}(\xi) \Delta\xi \right] \Delta s. \tag{57}$$

Then

$$\begin{aligned}
 v(t) &\leq v(\infty) + c(t) + \int_t^\infty f(s) v(s) \Delta s, \\
 &\quad t \in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{58}$$

Considering $c(t)$ is nonincreasing on \mathbb{T}_0 , by Lemma 12 we obtain

$$\begin{aligned}
 v(t) &\leq (v(\infty) + c(t)) e_{-f}(t, \infty), \\
 &\quad t \in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{59}$$

Combining (57) and (59), it follows that

$$\begin{aligned}
 c(t) &\leq \int_t^\infty \left\{ L\left(s, ((v(\infty) + c(s)) e_{-f}(s, \infty))^{1/p}\right) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) [(v(\infty) + c(\xi)) e_{-f}(\xi, \infty)]^{r/p} \Delta\xi \right\} \Delta s, \\
 &\quad t \in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{60}$$

On the other hand, from Lemma 13 one can see that the following inequalities hold:

$$\begin{aligned}
 &(v(\infty) + c(t))^{1/p} \\
 &\leq \frac{1}{p} K^{(1-p)/p} (v(\infty) + c(t)) + \frac{p-1}{p} K^{1/p}, \\
 &(v(\infty) + c(t))^{r/p} \\
 &\leq \frac{r}{p} K^{(r-p)/p} (v(\infty) + c(t)) + \frac{p-r}{p} K^{r/p}, \quad \forall K > 0.
 \end{aligned} \tag{61}$$

So combining (60) and (61), we have

$$\begin{aligned}
 v(\infty) + c(t) &\leq v(\infty) \\
 &+ \int_t^\infty L \left(s, \left(\frac{1}{p} K^{(1-p)/p} (v(\infty) + c(s)) \right. \right. \\
 &\quad \left. \left. + \frac{p-1}{p} K^{1/p} \right) (e_{-f}(s, \infty))^{1/p} \right) \Delta s \\
 &+ \int_t^\infty \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} (v(\infty) + c(\xi)) \right. \\
 &\quad \left. + \frac{p-r}{p} K^{r/p} \right] \\
 &\quad \times (e_{-f}(\xi, \infty))^{r/p} \Delta \xi \Delta s \\
 &= v(\infty) + \int_t^\infty \left\{ L \left(s, \left(\frac{1}{p} K^{(1-p)/p} (v(\infty) + c(s)) \right. \right. \right. \\
 &\quad \left. \left. + \frac{p-1}{p} K^{1/p} \right) \right. \\
 &\quad \times (e_{-f}(s, \infty))^{1/p} \\
 &\quad \left. - L \left(s, \frac{p-1}{p} K^{1/p} (e_{-f}(s, \infty))^{1/p} \right) \right. \\
 &\quad \left. + L \left(s, \frac{p-1}{p} K^{1/p} (e_{-f}(s, \infty))^{1/p} \right) \right\} \Delta s \\
 &+ \int_t^\infty \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} (v(\infty) + c(\xi)) \right. \\
 &\quad \left. + \frac{p-r}{p} K^{r/p} \right] \\
 &\quad \times (e_{-f}(\xi, \infty))^{r/p} \Delta \xi \Delta s \\
 &\leq v(\infty) + \int_t^\infty \left\{ M \left(s, \frac{p-1}{p} K^{1/p} (e_{-f}(s, \infty))^{1/p} \right) \right. \\
 &\quad \times \left[\frac{1}{p} K^{(1-p)/p} (v(\infty) + c(s)) \right. \\
 &\quad \left. \times (e_{-f}(s, \infty))^{1/p} \right] \\
 &\quad \left. + L \left(s, \frac{p-1}{p} K^{1/p} \right. \right. \\
 &\quad \left. \left. \times (e_{-f}(s, \infty))^{1/p} \right) \right\} \Delta s \\
 &+ \int_t^\infty \left[\int_s^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} (e_{-f}(\xi, \infty))^{r/p} \Delta \xi \right]
 \end{aligned}$$

$$\begin{aligned}
 &\times (v(\infty) + c(s)) \Delta s \\
 &+ \int_t^\infty \int_s^\infty h(\xi) \frac{p-r}{p} K^{r/p} (e_{-f}(\xi, \infty))^{r/p} \Delta \xi \Delta s \\
 &= v(\infty) + \widehat{H}_1(t) + \int_t^\infty \widehat{H}_2(s) (v(\infty) + c(s)) \Delta s,
 \end{aligned} \tag{62}$$

where $\widehat{H}_1(t)$, $\widehat{H}_2(t)$ are defined in (53) and (54), respectively.

Considering $\widehat{H}_1(t)$ is nonincreasing on \mathbb{T}_0 , then by Lemma 12, we have

$$\begin{aligned}
 v(\infty) + c(t) &\leq [v(\infty) + \widehat{H}_1(t)] e_{-\widehat{H}_2}(t, \infty), \\
 t &\in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{63}$$

Combining (59) and (63), we obtain

$$\begin{aligned}
 v(t) &\leq [v(\infty) + \widehat{H}_1(t)] e_{-\widehat{H}_2}(t, \infty) e_{-f}(t, \infty), \\
 t &\in [T, \infty) \cap \mathbb{T}.
 \end{aligned} \tag{64}$$

Setting $t = T$ in (64), considering $2v(\infty) - C = v(T)$, we obtain

$$\begin{aligned}
 2v(\infty) - C &= v(T) \\
 &\leq [v(\infty) + \widehat{H}_1(T)] e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty),
 \end{aligned} \tag{65}$$

which is followed by

$$v(\infty) \leq \frac{C + \widehat{H}_1(T) e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)}{2 - e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)}. \tag{66}$$

Then combining (55), (64), and (66), we can obtain the desired inequality (52). \square

3. Applications

In this section, we apply the results established above to analysis of boundedness of solutions for certain dynamic equations.

Example 17. Consider the following dynamic equation:

$$(u^p(t))^\Delta = F \left(t, u(t), u(t), \int_t^\infty W(\xi, u(\xi)) \Delta \xi \right), \quad t \in \mathbb{T}_0, \tag{67}$$

where $u \in C_{rd}(\mathbb{T}_0, \mathbb{R})$, C is a constant with $C = u^p(\infty)$, and $p > 0$ is a constant.

Theorem 18. Suppose that $u(t)$ is a solution of (66)-(67) and assume that $|F(t, x, y, z)| \leq f(t)|x|^p + g(t)|y|^q + |z|$ and $|W(t, x)| \leq h(t)|x|^r$, where f, g, h, q , and r are defined the same as in Theorem 14; then the following inequality holds:

$$|u(t)| \leq \left\{ [C + H_2(t) e_{-H_3}(t, \infty)] H_1(t) \right\}^{1/p}, \quad t \in \mathbb{T}_0, \tag{68}$$

where

$$\begin{aligned}
 H_1(t) &= 1 + \int_t^\infty e_{-f}(t, \sigma(s)) f(s) \Delta s, \\
 H_2(t) &= \int_t^\infty \left\{ g(s) \left[\frac{q}{p} K^{(q-p)/p} |C| + \frac{p-q}{p} K^{q/p} \right] \right. \\
 &\quad \times (H_1(s))^{q/p} \\
 &\quad + \int_s^\infty h(\xi) \left[\frac{r}{p} K^{(r-p)/p} |C| + \frac{p-r}{p} K^{r/p} \right] \\
 &\quad \left. \times (H_1(\xi))^{r/p} \Delta \xi \right\} \Delta s, \quad \forall K > 0, \\
 H_3(t) &= g(t) \frac{q}{p} K^{(q-p)/p} (H_1(t))^{q/p} \\
 &\quad + \int_t^\infty h(\xi) \frac{r}{p} K^{(r-p)/p} (H_1(\xi))^{r/p} \Delta \xi, \\
 &\quad \forall K > 0.
 \end{aligned} \tag{69}$$

Proof. The equivalent integral equation of (56) can be denoted by

$$u^p(t) = C + \int_t^\infty F(s, u(s), u(s), \int_s^\infty W(\xi, u(\xi)) \Delta \xi) \Delta s. \tag{70}$$

Then we have

$$\begin{aligned}
 |u^p(t)| &\leq |C| + \int_t^\infty \left| F(s, u(s), u(s), \int_s^\infty W(\xi, u(\xi)) \Delta \xi) \right| \Delta s \\
 &\leq |C| + \int_t^\infty \left[f(s) |u(s)|^p + g(s) |u(s)|^q \right. \\
 &\quad \left. + \left| \int_s^\infty W(\xi, u(\xi)) \Delta \xi \right| \right] \Delta s \\
 &\leq |C| + \int_t^\infty \left[f(s) |u(s)|^p + g(s) |u(s)|^q \right. \\
 &\quad \left. + \int_s^\infty h(\xi) |u(\xi)|^r \Delta \xi \right] \Delta s.
 \end{aligned} \tag{71}$$

A suitable application of Theorem 14 (i.e., $|C|$ takes the place of $a(t)$ in Theorem 14, and $b(t) \equiv 1$ in Theorem 14) yields (68). \square

Remark 19. Under the conditions of Theorem 18, considering $H_1(t) = 1 + \int_t^\infty e_{-f}(t, \sigma(s)) f(s) \Delta s = e_{-f}(t, \infty)$, furthermore we have the following estimate:

$$|u(t)| \leq \left\{ \left[|C| + H_2(t) e_{-H_3}(t, \infty) \right] e_{-f}(t, \infty) \right\}^{1/p}, \quad t \in \mathbb{T}_0. \tag{72}$$

Example 20. Consider the following dynamic equation:

$$\begin{aligned}
 u^p(t) &= C + \int_t^\infty \widehat{F}(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi) \Delta s \\
 &\quad + \int_T^\infty \widehat{F}(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi) \Delta s, \\
 &\quad t \in [T, \infty) \cap \mathbb{T},
 \end{aligned} \tag{73}$$

where $u \in C_{rd}(\mathbb{T}_0, \mathbb{R})$, $T \in \mathbb{T}_0$, C is a constant with $C = u^p(\infty)$, and $p > 0$ is a constant.

Theorem 21. Suppose that $u(t)$ is a solution of (73) and assume that $|\widehat{F}(t, x, y, z)| \leq f(t)|x|^p + L(t, |y|) + |z|$ and $|\widehat{W}(t, x)| \leq h(t)|x|^r$, where f , h , r , and L are defined the same as in Theorem 16. Then the following inequality holds:

$$\begin{aligned}
 u(t) &\leq \left[\frac{|C| + \widehat{H}_1(T) e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)}{2 - e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty)} + \widehat{H}_1(t) \right] \\
 &\quad \times e_{-\widehat{H}_2}(t, \infty) e_{-f}(t, \infty), \quad t \in [T, \infty) \cap \mathbb{T},
 \end{aligned} \tag{74}$$

provided that $e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty) < 2$ and $-f, -\widehat{H}_2 \in \mathfrak{R}^+$, where $\widehat{H}_1(t), \widehat{H}_2(t)$ are defined as in Theorem 16.

Proof. From (73), we have

$$\begin{aligned}
 |u^p(t)| &\leq |C| + \int_t^\infty \left| \widehat{F}(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi) \right| \Delta s \\
 &\quad + \int_T^\infty \left| \widehat{F}(s, u(s), u(s), \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi) \right| \Delta s \\
 &\leq |C| + \int_t^\infty \left[f(s) |u(s)|^p + L(s, |u(s)|) \right. \\
 &\quad \left. + \left| \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi \right| \right] \Delta s \\
 &\quad + \int_T^\infty \left[f(s) |u(s)|^p + L(s, |u(s)|) \right. \\
 &\quad \left. + \left| \int_s^\infty \widehat{W}(\xi, u(\xi)) \Delta \xi \right| \right] \Delta s \\
 &\leq |C| + \int_t^\infty \left[f(s) |u(s)|^p + L(s, |u(s)|) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) |u(\xi)|^r \right] \Delta s \\
 &\quad + \int_T^\infty \left[f(s) |u(s)|^p + L(s, |u(s)|) \right. \\
 &\quad \left. + \int_s^\infty h(\xi) |u(\xi)|^r \right] \Delta s.
 \end{aligned} \tag{75}$$

Then under the condition $e_{-\widehat{H}_2}(T, \infty) e_{-f}(T, \infty) < 2$, a suitable application of Theorem 16 yields (74). \square

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