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Research Article

Strong Convergence Iterative Algorithms for Equilibrium Problems and Fixed Point Problems in Banach Spaces

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We first introduce the concept of Bregman asymptotically quasinonexpansive mappings and prove that the fixed point set of this kind of mappings is closed and convex. Then we construct an iterative scheme to find a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a countable family of Bregman asymptotically quasinonexpansive mappings in reflexive Banach spaces and prove strong convergence theorems. Our results extend the recent ones of some others.

1. Introduction

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the dual space of E equipped with the inducted norm $\|\cdot\|_*$. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function and the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in E\}. \tag{1}$$

We denote by dom f the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$.

Let C be a nonempty, closed, and convex subset of E and $T : C \to C$ a nonlinear mapping. The fixed points set of T is denoted by

$$F(T) = \{x \in C : x = Tx\}. \tag{2}$$

Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if, for each $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|. \tag{3}$$

Nakajo-Takahashi [1] introduced the following hybrid method which is the so-called *CQ-method* for a nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered} x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{gathered} \tag{4}$$

where $\{\alpha_n\} \subset [0,1]$ and P_K is the metric projection from H onto a closed and convex subset K of H. They proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a fixed point of T under suitable conditions.

Takahashi et al. [2] introduced a new hybrid iterative scheme called the *shrinking projection method* for a nonexpansive mapping T in a Hilbert space H as follows:

$$\begin{gathered} x_0 \in H, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \end{gathered}$$

$$\begin{gathered}
y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\
C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}} x, \quad \forall n \ge 1,
\end{gathered} \tag{5}$$

where $\{\alpha_n\} \subset [0,1]$, and they proved that the sequence $\{x_n\}$ generated by (5) converges strongly to a fixed point of T under suitable conditions.

In 2010, Reich and Sabach [3] introduced the following two hybrid iterative schemes for Bregman strongly nonexpansive mappings $T_i : E \to E\,(i = 1,2,\ldots,N)$ in a reflexive Banach space E:

$$\begin{gathered}
x_0 \in E, \\
y_n^i = T_i\left(x_n + e_n^i\right), \\
C_n^i = \left\{z \in E : D_f\left(z, y_n^i\right) \le D_f\left(z, x_n + e_n^i\right)\right\}, \\
C_n := \bigcap_{i=1}^{N} C_n^i, \\
Q_n = \{z \in E : \langle z - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \le 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}^f (x_0), \quad \forall n \ge 0,
\end{gathered} \tag{6}$$

$$\begin{gathered}
x_0 \in E, \\
C_0^i = E, \quad i = 1,2,\ldots,N, \\
y_n^i = T_i\left(x_n + e_n^i\right), \\
C_{n+1}^i = \left\{z \in C_n^i : D_f\left(z, y_n^i\right) \le D_f\left(z, x_n + e_n^i\right)\right\}, \\
C_{n+1} := \bigcap_{i=1}^{N} C_{n+1}^i, \\
x_{n+1} = P_{C_n \cap Q_n}^f (x_0), \quad \forall n \ge 0,
\end{gathered} \tag{7}$$

where P_K^f is the Bregman projection with respect to f from E onto a closed and convex subset K of E. They proved that the sequence $\{x_n\}$ generated by both (6) and (7) converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

The construction of fixed points for Bregman-type mappings via iterative processes has been investigated in, for example, [4–8].

In this paper, we design a new hybrid iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a countable family of Bregman asymptotically quasinonexpansive mappings in reflexive Banach spaces and prove some strong convergence theorems. Our results extend the recent one of Reich and Sabach [3].

2. Preliminaries

Let E be a real Banach space. For any $x \in \operatorname{int} \operatorname{dom} f$ and $y \in E$, we define the *right-hand derivative* of f at x in the direction y by

$$f^o(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

The function f is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0^+}((f(x+ty) - f(x))/t)$ exists for any y. In this case, $f^o(x,y)$ coincides with $\nabla f(x)$, the value of the *gradient* ∇f of f at x. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let E be a reflexive Banach space. The Legendre function is defined from a general Banach space E into $(-\infty, +\infty]$ (see [9]). According to [9], the function f is *Legendre* if and only if it satisfies the following conditions

(L1) The interior of the domain of f (denoted by $\operatorname{int}\operatorname{dom} f$) is nonempty; f^* is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f$ and $\operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$.

(L2) The interior of the domain f^* (denoted by $\operatorname{int}\operatorname{dom} f^*$) is nonempty; f^* is Gâteaux differentiable on $\operatorname{int}\operatorname{dom} f^*$ and $\operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$.

Since E is reflexive, we always have $(\partial f)^{-1} = \partial f^*$ (see [10]). This fact, when combined with conditions (L1) and (L2), implies the following equalities:

$$\begin{gathered}
\nabla f = (\nabla f^*)^{-1}, \\
\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*, \\
\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f.
\end{gathered} \tag{9}$$

Also, conditions (L1) and (L2), in conjunction with [9], imply that the functions f and f^* are strictly convex on the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [9, 11]. Especially, the functions $(1/s)\|\cdot\|^s$ with $s \in (1,\infty)$ are Legendre, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined as

$$D_f(y, x) := f(y) - f(x) - \langle y - x, \nabla f(x)\rangle \tag{10}$$

is called the *Bregman distance* with respect to f [12].

By the definition, we know the following property (the three point identity): for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$\begin{aligned}
&D_f(x,y) + D_f(y,z) - D_f(x,z) \\
&\quad = \langle \nabla f(z) - \nabla f(y), x - y\rangle.
\end{aligned} \tag{11}$$

Recall that the *Bregman projection* [13] of $x \in \operatorname{int} \operatorname{dom} f$ onto the nonempty, closed, and convex subset C of $\operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_C^f(x) \in C$ satisfying

$$D_f\left(\operatorname{proj}_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \quad (12)$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function f is said to be *totally convex* at $x \in \operatorname{int} \operatorname{dom} f$ if its modulus of total convexity at x, that is, the function $\nu_f : \operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\left\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\right\}, \quad (13)$$

is positive whenever $t > 0$. The function f is said to be *totally convex* when it is totally convex at every point $x \in \operatorname{int} \operatorname{dom} f$. In addition, the function f is said to be *totally convex on bounded sets* if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int} \operatorname{dom} f \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\left\{\nu_f(x, t) : x \in B \cap \operatorname{dom} f\right\}. \quad (14)$$

Some examples of the totally convex functions can be found in [14, 15].

Recall that the function f is said to be *sequentially consistent* [15] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0. \quad (15)$$

Let C be a nonempty, closed, and convex subset of E and $g : C \times C \to \mathbb{R}$ a bifunction that satisfies the following conditions:

(C1) $g(x, x) = 0$ for all $x \in C$;

(C2) g is monotone, that is, $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$;

(C3) $\limsup_{t\downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)$ for all $x, y, z \in C$;

(C4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

The equilibrium problem with respect to g is as follows: find $\overline{x} \in C$ such that

$$g(\overline{x}, y) \geq 0, \quad \forall y \in C. \quad (16)$$

The set of all solutions of (16) is denoted by $\mathrm{EP}(g)$. The resolvent of a bifunction $g : C \times C \to \mathbb{R}$ [16] is the operator $\operatorname{Res}_g^f : E \to 2^C$ denoted by

$$\begin{aligned}\operatorname{Res}_g^f(x) = \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \\ \geq 0, \forall y \in C\}.\end{aligned} \quad (17)$$

For any $x \in E$, there exists $z \in C$ such that $z = \operatorname{Res}_g^f(x)$; see [3].

Let K be a convex subset of $\operatorname{int} \operatorname{dom} f$ and $T : K \to K$ a mapping. A point p in the closure of K is said to be an *asymptotic fixed point* of T [17, 18] if K contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. The mapping T is called *Bregman quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$D_f(v, x) \leq D_f(v, x), \quad \forall v \in F(T), \ x \in K. \quad (18)$$

T is said to be *Bregman (quasi)-strongly nonexpansive* [6] with respcet to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x), \quad (19)$$

for all $p \in \widehat{F}(T)$ and $x \in K$, and if whenever $\{x_n\} \subset K$ is bounded, $p \in \widehat{F}(T)$, and

$$\lim_{n\to\infty}\left(D_f(p, x_n) - D_f(p, Tx_n)\right) = 0, \quad (20)$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0. \quad (21)$$

The mapping T is called *Bregman firmly nonexpansive* if

$$\begin{aligned}&\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\quad \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle\end{aligned} \quad (22)$$

for all $x, y \in K$.

Next, we introduce a new mapping that is called Bregman asymptotically quasinonexpansive mapping which is a natural extension of Bregman quasinonexpansive mapping introduced by Reich and Sabach [3]. The mapping $T : K \to K$ is said to be *Bregman asymptotically quasi-nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that, for every $n \geq 1$,

$$D_f(v, T^n x) \leq k_n D_f(v, x), \quad \forall v \in F(T), \ x \in K. \quad (23)$$

Obviously, every Bregman quasinonexpansive mapping is a Bregman asymptotically quasi-nonexpansive one with $k_n = 1$.

Let E be a Banach space and C a nonempty subset of E. The mapping $T : C \to C$ is said to be *uniformly asymptotically regular* on C if

$$\lim_{n\to\infty}\left(\sup_{x\in C}\left\|T^{n+1}x - T^n x\right\|\right) = 0. \quad (24)$$

The mapping T is said to be *closed* if, for any sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, $Tx_0 = y_0$.

The following is an important result which will be used in the next section.

Lemma 1. *Let E be a reflexive Banach space and $f : E \to (-\infty, +\infty)$ a Gâteaux differentiable and Legendre function* *which is totally convex on bounded sets. Let K be a nonempty, closed and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $T : K \to K$ a closed Bergman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\} \subset [1, +\infty)$ such that $k_n \to 1$ as $n \to \infty$. Then $F(T)$ is closed and convex.*

Proof. The closedness of $F(T)$ comes directly from the closedness of T. Now, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0,1)$, put $p_3 = tp_1 + (1-t)p_2$. We prove that $Tp_3 = p_3$. Indeed, from the definition of D_f, we see that

$$\begin{aligned}
D_f(p_3, T^n p_3) &= f(p_3) - f(T^n p_3) \\
&\quad - \langle \nabla f(T^n p_3), p_3 - T^n p_3 \rangle \\
&= f(p_3) - f(T^n p_3) \\
&\quad - \langle \nabla f(T^n p_3), tp_1 + (1-t)p_2 - T^n p_3 \rangle \\
&= f(p_3) - f(T^n p_3) \\
&\quad - t\langle \nabla f(T^n p_3), p_1 - T^n p_3 \rangle \\
&\quad - (1-t)\langle \nabla f(T^n p_3), p_2 - T^n p_3 \rangle \\
&= f(p_3) + tD_f(p_1, T^n p_3) \\
&\quad + (1-t)D_f(p_2, T^n p_3) - tf(p_1) \\
&\quad - (1-t)f(p_2) \\
&\le f(p_3) + tk_n D_f(p_1, p_3) \\
&\quad + (1-t)k_n D_f(p_2, p_3) \\
&\quad - tf(p_1) - (1-t)f(p_2) \\
&\quad = f(p_3) + k_n [t(f(p_1) - f(p_3) \\
&\qquad - \langle \nabla f(p_3), p_1 - p_3 \rangle) \\
&\qquad + (1-t) \\
&\qquad \times (f(p_2) - f(p_3) \\
&\qquad - \langle \nabla f(p_3), p_2 - p_3 \rangle)] \\
&\quad - tf(p_1) - (1-t)f(p_2) \\
&= (1-k_n)f(p_3) \\
&\quad + k_n(tf(p_1) + (1-t)f(p_2)) \\
&\quad - tf(p_1) - (1-t)f(p_2) \\
&= (k_n - 1)(tf(p_1) + (1-t)f(p_2) - f(p_3)).
\end{aligned} \tag{25}$$

This implies that $\lim_{n\to\infty} D_f(p_3, T^n p_3) = 0$. It follows from Lemma 3 below that

$$\lim_{n\to\infty} \|p_3 - T^n p_3\| = 0, \tag{26}$$

that is, $TT^n p_3 - p_3 \to 0$ as $n \to \infty$. In view the closedness of T, we can obtain the desired conclusion. This completes the proof. □

Finally, we state some lemmas that will used in the proof of main results in next section.

Lemma 2 (see [7]). *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 3 (see [14]). *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Lemma 4 (see [15]). *Suppose that f is Gâteaux differentiable and totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in$* $\operatorname{int}\operatorname{dom} f$ *and C a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$. *If $\hat{x} \in C$, then the following conditions are equivalent.*

(i) *The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f.*

(ii) *The vector $\hat{x}$ is the unique solution of the variational inequality.*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \ge 0, \quad \forall y \in C. \tag{27}$$

(iii) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \le D_f(y, x), \quad \forall y \in C. \tag{28}$$

Lemma 5 (see [6]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 6 (see [3]). *Let $f : E \to (-\infty, +\infty)$ be a coercive (i.e., $\lim_{\|x\|\to\infty}(f(x)/\|x\|) = +\infty$) and Gâteaux differentiable function. Let C be a closed and convex subset of E. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions* (C1)–(C4), *then*

(1) Res_g^f *is single-valued;*

(2) Res_g^f *is a Bregman firmly nonexpansive mapping;*

(3) *the set of fixed points of* Res_g^f *is the solution set of the equilibrium problem, that is,* $F(\mathrm{Res}_g^f) = \mathrm{EP}(g)$;

(4) $\mathrm{EP}(g)$ *is a closed and convex subset of C;*

(5) *for all $x \in E$ and $u \in F(\mathrm{Res}_g^f)$, one has*

$$\begin{aligned}
&D_f\left(u, \mathrm{Res}_g^f(x)\right) + D_f\left(\mathrm{Res}_g^f(x), x\right) \\
&\quad \le D_f(u, x).
\end{aligned} \tag{29}$$

3. Main Results

Now, we give our main theorems.

Theorem 7. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let K be a nonempty, closed, and convex subset of* $\operatorname{int}\operatorname{dom} f$ *and $\{T_i\}_{i=1}^{\infty} : K \to K$ a countable family of closed Bregman asymptotically quasi-nonexpansive mappings with the sequences $\{k_{i,n}\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_{i,n} = 1$ for every $i \geq 1$. Let $k_n = \sup\{k_{i,n} : i \geq 1\}$ and suppose that $\lim_{n\to\infty} k_n = 1$. Let $g : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions (C1)–(C4). Assume that each $T_i (i \geq 1)$ is uniformly asymptotically regular and $\Omega = [\bigcap_{i=1}^{\infty} F(T_i)] \cap \mathrm{EP}(g)$ is nonempty and bounded. Let $\{\alpha_{i,n}\}$ be a real sequence in $(0,1)$ with $\sum_{i=1}^{n} \alpha_{i,n} = 1$ for every $n \geq 1$ and $\liminf_{n\to\infty} \alpha_{i,n} > 0$ for every $i \geq 1$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x_1 = x \in K \text{ chosen arbitrarily}, \\ u_{i,n} \in K \text{ such that} \\ g\left(u_{i,n}, y\right) + \left\langle \nabla f\left(u_{i,n}\right) - \nabla f\left(T_i^n x_n\right), y - u_{i,n} \right\rangle \geq 0, \\ \forall y \in K, \; i = 1, \ldots, n, \\ C_n = \left\{ z \in K : \sum_{i=1}^{n} \alpha_{i,n} D_f\left(z, u_{i,n}\right) \leq D_f\left(z, x_n\right) + \left(k_n - 1\right) M_n \right\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = \operatorname{proj}_{D_n}^{f} x, \quad n = 1, 2, \ldots, \end{gathered} \tag{30}$$

where $M_n = \sup\{D_f(v, x_n) : v \in \Omega\}$ for each $n \geq 1$. Then, $\{x_n\}$ defined by (30) *converges strongly to* $\operatorname{proj}_{\Omega}^{f} x$ *as $n \to \infty$.*

Proof. First, we prove that the sequence $\{x_n\}$ is well defined. Note that

$$\sum_{i=1}^{n} \alpha_{i,n} D_f\left(z, u_{i,n}\right) \leq D_f\left(z, x_n\right) + \left(k_n - 1\right) M_n \tag{31}$$

is

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_{i,n} \left(f(z) - f\left(u_{i,n}\right) - \left\langle \nabla f\left(u_{i,n}\right), z - u_{i,n} \right\rangle \right) \\ &\quad \leq f(z) - f\left(x_n\right) - \left\langle \nabla f\left(x_n\right), z - x_n \right\rangle + \left(k_n - 1\right) M_n, \end{aligned} \tag{32}$$

that is,

$$\begin{aligned} &f\left(x_n\right) - \sum_{i=1}^{n} \alpha_{i,n} f\left(u_{i,n}\right) + \left\langle \nabla f\left(x_n\right), z - x_n \right\rangle \\ &\quad \leq \sum_{i=1}^{n} \alpha_{i,n} \left\langle \nabla f\left(u_{i,n}\right), z - u_{i,n} \right\rangle + \left(k_n - 1\right) M_n. \end{aligned} \tag{33}$$

This shows that C_n is closed and convex for every $n \geq 1$. From the definition of D_n, it is easy to see that D_n is closed and convex for every $n \geq 1$. For every $i \geq 1$ and $n \geq 1$, Lemma 6 shows that $u_{i,n} = \mathrm{Res}_g^f T_i^n x_n$ and $D_f(v, \mathrm{Res}_g^f y) \leq D_f(v, y)$ for any $v \in \Omega$ and $y \in E$. Hence,

$$\begin{aligned} D_f\left(v, u_{i,n}\right) &= D_f\left(v, \mathrm{Res}_g^f T_i^n x_n\right) \\ &\leq D_f\left(v, T_i^n x_n\right) \\ &\leq k_{i,n} D_f\left(v, x_n\right) \\ &\leq k_n D_f\left(v, x_n\right) \\ &= D_f\left(v, x_n\right) + \left(k_n - 1\right) D_f\left(v, x_n\right) \\ &\leq D_f\left(v, x_n\right) + \left(k_n - 1\right) M_n. \end{aligned} \tag{34}$$

Since $\sum_{i=1}^{n} \alpha_{i,n} = 1$ for every $n \geq 1$, we have

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_{i,n} D_f\left(v, u_{i,n}\right) \\ &\quad \leq \sum_{i=1}^{n} \alpha_{i,n} \left(D_f\left(v, x_n\right) + \left(k_n - 1\right) M_n \right) \\ &\quad = D_f\left(v, x_n\right) + \left(k_n - 1\right) M_n. \end{aligned} \tag{35}$$

This shows that $v \in C_n$ for every $n \geq 1$. Thus $\Omega \subset C_n$ for every $n \geq 1$. Further, we have $\Omega \subset D_n$ for every $n \geq 1$. Thus the sequence $\{x_n\}$ is well defined.

From $\operatorname{proj}_{D_n}^{f} x = x_{n+1}$, by Lemma 4(iii) we have

$$\begin{aligned} D_f\left(x_{n+1}, x\right) &= D_f\left(\operatorname{proj}_{D_n}^{f} x, x\right) \\ &\leq D_f(v, x) - D_f\left(v, \operatorname{proj}_{D_n}^{f} x\right) \\ &\leq D_f(v, x) \end{aligned} \tag{36}$$

for any $v \in \Omega$. Hence the sequence $D_f(x_n, x)$ is bounded. Therefore by Lemma 5 the sequence $\{x_n\}$ is bounded.

On the other hand, in view of $x_{n+1} = \operatorname{proj}_{D_n}^{f} x$ and $x_{n+2} = \operatorname{proj}_{D_{n+1}}^{f} x \in D_{n+1} \subset D_n$, from Lemma 4(iii) we have

$$D_f\left(x_{n+2}, \operatorname{proj}_{D_n}^{f} x\right) + D_f\left(\operatorname{proj}_{D_n}^{f} x, x\right) \leq D_f\left(x_{n+2}, x\right), \tag{37}$$

that is,

$$D_f\left(x_{n+2}, x_{n+1}\right) + D_f\left(x_{n+1}, x\right) \leq D_f\left(x_{n+2}, x\right). \tag{38}$$

Therefore the sequence $\{D_f(x_n, x)\}$ is increasing, and since it is also bounded, $\lim_{n\to\infty} D_f(x_n, x)$ exists. By the construction of D_n, we have that $D_m \subset D_n$ and $x_m = \operatorname{proj}_{D_{m-1}}^{f} x \in D_{m-1} \subset D_{n-1}$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} D_f\left(x_m, x_n\right) &= D_f\left(x_m, \operatorname{proj}_{D_{n-1}}^{f} x\right) \\ &\leq D_f\left(x_m, x\right) - D_f\left(\operatorname{proj}_{D_{n-1}}^{f} x, x\right) \\ &= D_f\left(x_m, x\right) - D_f\left(x_n, x\right). \end{aligned} \tag{39}$$

Letting $m, n \to \infty$ in (39), we see that $D_f(x_m, x_n) \to 0$. It follows from Lemma 3 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and K is closed and convex, we can assume that

$$\lim_{n\to\infty} x_n = x^* \in K. \tag{40}$$

By taking $m = n + 1$ in (39), we see that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{41}$$

Lemma 3 implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{42}$$

Since $x_{n+1} = \operatorname{proj}^f_{D_n} x \in D_n \subset C_n$, we have

$$\sum_{i=1}^{n} \alpha_{i,n} D_f(x_{n+1}, u_{i,n}) \le D_f(x_{n+1}, x_n) + (k_n - 1) M_n. \tag{43}$$

Then (41) implies that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_{i,n} D_f(x_{n+1}, u_{i,n}) = 0. \tag{44}$$

Note that $\alpha_{i,n} D_f(x_{n+1}, u_{i,n}) \le \sum_{i=1}^{n} \alpha_{i,n} D_f(x_{n+1}, u_{i,n})$ and $\liminf_{n\to\infty} \alpha_{i,n} > 0$, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, u_{i,n}) = 0 \tag{45}$$

for every $i \ge 1$. It follows from Lemma 3 that

$$\lim_{n\to\infty} \|x_{n+1} - u_{i,n}\| = 0 \tag{46}$$

for every $i \ge 1$. Note that

$$\|u_{i,n} - x_n\| \le \|u_{i,n} - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{47}$$

Combining (42) with (46), we see that

$$\lim_{n\to\infty} \|u_{i,n} - x_n\| = 0 \tag{48}$$

for every $i \ge 1$. This means that the sequence $\{u_{i,n}\}$ is bounded. Since f is uniformly Fréchet differentiable, it follows from Lemma 2 that

$$\lim_{n\to\infty} \|\nabla f(u_{i,n}) - \nabla f(x_n)\|_* = 0. \tag{49}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous (see [19, Theorem 1.8, p.13]) and therefore

$$\lim_{n\to\infty} \|f(u_{i,n}) - f(x_n)\| = 0. \tag{50}$$

From the definition of the Bregman distance, we obtain that for every

$$\begin{aligned}
&D_f(v, x_n) - D_f(v, u_{i,n}) \\
&\quad = [f(v) - f(x_n) - \langle \nabla f(x_n), v - x_n\rangle] \\
&\qquad - [f(v) - f(u_{i,n}) - \langle \nabla f(u_n), v - u_{i,n}\rangle] \\
&\quad = f(u_{i,n}) - f(x_n) + \langle \nabla f(u_{i,n}), v - u_{i,n}\rangle \\
&\qquad - \langle \nabla f(x_n), v - x_n\rangle \\
&\quad = f(u_{i,n}) - f(x_n) \\
&\qquad + \langle \nabla f(u_{i,n}), x_n - u_{i,n}\rangle \\
&\qquad + \langle \nabla f(u_{i,n}) - \nabla f(x_n), v - x_n\rangle
\end{aligned} \tag{51}$$

for any $v \in \Omega$. Since every sequence $\{u_{i,n}\}$ is bounded, $\{\nabla f(u_{i,n})\}$ is also bounded for every $i \ge 1$. Now from (48)–(50), we have

$$\lim_{n\to\infty} D_f(v, x_n) - D_f(v, u_{i,n}) = 0 \tag{52}$$

for any $v \in \Omega$ and for every $i \ge 1$.

In view of $u_{i,n} = \operatorname{Res}^f_g T_i^n x_n$, by Lemma 6 (5) we have

$$\begin{aligned}
D_f(u_{i,n}, T_i^n x_n) &= D_f(\operatorname{Res}^f_g T_i^n x_n, T_i^n x_n) \\
&\le D_f(v, T^n x_n) - D_f(v, \operatorname{Res}^f_g T_i^n x_n) \\
&\le k_n D_f(v, x_n) - D_f(v, \operatorname{Res}^f_g T_i^n x_n) \\
&\le D_f(v, x_n) + (k_n - 1) M_n - D_f(v, u_{i,n}).
\end{aligned} \tag{53}$$

Note that M_n is bounded and $k_n \to 1$ as $n \to \infty$. It follows from (52) that

$$\lim_{n\to\infty} D_f(u_{i,n}, T_i^n x_n) = 0 \tag{54}$$

for every $i \ge 1$. Lemma 3 shows that

$$\lim_{n\to\infty} \|u_{i,n} - T_i^n x_n\| = 0. \tag{55}$$

Note that $\|T_i^n x_n - x_n\| \le \|T_i^n x_n - u_{i,n}\| + \|u_{i,n} - x_n\|$. From (48) and (55) we get

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0 \tag{56}$$

for every $i \ge 1$. Note that

$$\|T_i^n x_n - x^*\| \le \|T_i^n x_n - x_n\| + \|x_n - x^*\|. \tag{57}$$

It follows from (40) and (56) that

$$\lim_{n\to\infty} \|T_i^n x_n - x^*\| = 0 \tag{58}$$

for every $i \ge 1$. On the other hand, we have

$$\|T_i^{n+1} x_n - x^*\| \le \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - x^*\|. \tag{59}$$

Since every T_i is uniformly asymptotically regular and (58), we obtain that, for every $i \geq 1$,

$$\lim_{n\to\infty} \left\| T_i^{n+1} x_n - x^* \right\| = 0, \tag{60}$$

that is, $T_i T_i^n x_n \to x^*$ as $n \to \infty$. From the closedness of T_i, we see that $x^* \in F(T_i)$ for every $i \geq 1$. Thus $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$.

Next we prove that $x^* \in \mathrm{EP}(g)$ for every $i \geq 1$. Since f is uniformly Fréchet differentiable, ∇f is uniformly continuous. Thus, by (55) we have

$$\lim_{n\to\infty} \left(\nabla f(u_{i,n}) - \nabla f(T_i^n x_n)\right) = 0. \tag{61}$$

Since $u_{i,n} = \mathrm{Res}_g^f T_i^n x_n$, we have

$$\begin{aligned} &g(u_{i,n}, y) + \langle \nabla f(u_{i,n}) - \nabla f(T_i^n x_n), y - u_{i,n} \rangle \\ &\quad \geq 0, \quad \forall y \in K. \end{aligned} \tag{62}$$

We have from (C2) that

$$\begin{aligned} &\langle \nabla f(u_{i,n}) - \nabla f(T_i^n x_n), y - u_{i,n} \rangle \\ &\quad \geq -g(u_{i,n}, y) \\ &\quad \geq g(y, u_{i,n}), \quad \forall y \in K. \end{aligned} \tag{63}$$

Letting $n \to \infty$, we have from (61) and (C4) that

$$g(y, x^*) \leq 0, \quad \forall y \in K. \tag{64}$$

For t with $0 < t \leq 1$ and $y \in K$, let $y_t = ty + (1-t)x^*$. Since $y \in K$ and $x^* \in K$, we have $y_t \in K$ and hence $g(y_t, x^*) \leq 0$. So, from (C1) we have

$$\begin{aligned} 0 &= g(y_t, y_t) \\ &\leq t g(y_t, y) + (1-t) g(y_t, x^*) \\ &\leq t g(y_t, y). \end{aligned} \tag{65}$$

Dividing by t, we have

$$g(y_t, y) \geq 0, \quad \forall y \in K. \tag{66}$$

Letting $t \downarrow 0$, from (C3) we have

$$g(x^*, y) \geq 0, \quad \forall y \in K. \tag{67}$$

Therefore, $x^* \in \mathrm{EP}(g)$. Thus $x^* \in \cap_{i=1}^{\infty} \mathrm{EP}(g)$.

Finally, we show that $x^* = \mathrm{proj}_{\Omega} x$. Since $\Omega \subset D_n$ for every $n \geq 1$, by Lemma 4(ii) we arrive at

$$\langle x_n - v, \nabla f(x) - \nabla f(x_n) \rangle \geq 0, \quad \forall v \in \Omega. \tag{68}$$

Taking the limit as $n \to \infty$ in (68), we obtain that

$$\langle x^* - v, \nabla f(x) - \nabla f(x^*) \rangle \geq 0, \quad \forall v \in \Omega \tag{69}$$

and hence $x^* = \mathrm{proj}_{\Omega} x$ by Lemma 4(ii). This completes the proof. □

Corollary 8. *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let K be a nonempty, closed, and convex subset of* int dom f *and $T : K \to K$ a closed Bregman asymptotically quasi-nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $g : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions (C1)–(C4). Assume that T is uniformly asymptotically regular and $\Omega = F(T) \cap \mathrm{EP}(g)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x \in K \text{ chosen arbitrarily,} \\ u_n \in K \text{ such that} \\ g(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n x_n), y - u_n \rangle \\ \geq 0, \quad \forall y \in K, \\ C_n = \left\{ z \in K : D_f(z, u_n) \leq D_f(z, x_n) + (k_n - 1) M_n \right\}, \\ D_n = \bigcap_{i=1}^{n} C_i, \\ x_{n+1} = \mathrm{proj}_{D_n}^f x, \quad n = 1, 2, \ldots, \end{gathered} \tag{70}$$

where $M_n = \sup\{D_f(v, x_n) : v \in \Omega\}$ for each $n \geq 1$. Then, $\{x_n\}$ defined by (70) *converges strongly to $\mathrm{proj}_{\Omega}^f x$ as $n \to \infty$.*

Since every Bregman quasi-nonexpansive mapping is Bregman quasi-asymptotically nonexpansive, we have the following results.

Corollary 9. *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let K be a nonempty, closed, and convex subset of* int dom f. *Let $\{T_i\}_{i=1}^{\infty} : K \to K$ be a countable family of closed Bregman quasi-nonexpansive mappings and $g : K \times K \to \mathbb{R}$ a bifunction satisfying conditions (C1)–(C4). Assume that $\Omega = [\cap_{i=1}^{\infty} F(T_i)] \bigcap \mathrm{EP}(g) \neq \emptyset$. Let $\{\alpha_{i,n}\}$ be a real sequence in* $(0, 1)$ *with $\sum_{i=1}^{n} \alpha_{i,n} = 1$ and $\liminf_{n\to\infty} \alpha_{i,n} > 0$ for every $i \geq 1$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x \in K \text{ chosen arbitrarily,} \\ u_{i,n} \in K \text{ such that} \\ g(u_{i,n}, y) + \langle \nabla f(u_{i,n}) - \nabla f(T_i^n x_n), y - u_{i,n} \rangle \\ \geq 0, \quad \forall y \in K, \ i = 1, \ldots, n, \\ C_n = \left\{ z \in K : \sum_{i=1}^{n} \alpha_{i,n} D_f(z, u_{i,n}) \leq D_f(z, x_n) \right\}, \end{gathered}$$

$$D_n = \bigcap_{i=1}^{n} C_i,$$
$$x_{n+1} = \text{proj}_{D_n}^f x, \quad n = 1, 2, \ldots. \tag{71}$$

Then, $\{x_n\}$ defined by (71) *converges strongly to* $\text{proj}_\Omega^f x$ *as* $n \to \infty$.

Corollary 10. *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let K be a nonempty, closed, and convex subset of* int dom f. *Let $T : K \to K$ be a closed Bregman quasi-nonexpansive mapping and $g : K \times K \to \mathbb{R}$ a bifunction satisfying conditions (C1)–(C4). Assume that $\Omega = F(T) \cap \text{EP}(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x \in K \text{ chosen arbitrarily,}$$
$$u_n \in K \text{ such that}$$
$$g(u_n, y) + \langle \nabla f(u_n) - \nabla f(T^n x_n), y - u_n \rangle \geq 0, \quad \forall y \in K,$$
$$C_n = \{z \in K : D_f(z, u_n) \leq D_f(z, x_n)\},$$
$$D_n = \bigcap_{i=1}^{n} C_i,$$
$$x_{n+1} = \text{proj}_{D_n}^f x, \quad n = 1, 2, \ldots. \tag{72}$$

Then, $\{x_n\}$ defined by (72) *converges strongly to* $\text{proj}_\Omega^f x$ *as* $n \to \infty$

Remark 11. Set $\alpha_{n,i} = 1/i(i+1) + 1/n(n+1)$ for each $n \geq 1$ and $i = 1, 2, \ldots, n$ and $k_{i,n} = 1 + 1/in$ for each $n \geq 1$ and $i \geq 1$. Then $\sum_{i=1}^{n} \alpha_{i,n} = 1$ and $\liminf_{n\to\infty} \alpha_{i,n} = 1/i(i+1) > 0$. Also, $k_n = \sup\{k_{i,n} : i \geq 1\} = 1$ for every $n \geq 1$. Hence, $\{\alpha_{i,n}\}$ and $\{k_{i,n}\}$ satisfy the conditions of Theorem 7.

Remark 12. It needs to notice that Corollaries 9 and 10 still hold if we replace the closedness of the mappings with $\widehat{F}(T) = F(T)$.

In the equilibrium problem, the bifunction g is usually required to satisfy conditions (C1)–(C4). But, if the condition (C3) is replaced with the following condition:

(C3′) for every fixed $y \in C$, $g(\cdot, y)$ is continuous, then we have the following result:

Lemma 13. *Let $f : E \to (-\infty, +\infty)$ be a coercive (i.e., $\lim_{\|x\|\to\infty}(f(x)/\|x\|) = +\infty$) and Gâteaux differentiable function. Let C be a closed and convex subset of E. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (C1), (C2), (C3′), and (C4), then the mapping Res_g^f defined by (2.2) is closed.*

Proof. Let $\{x_n\} \subset E$ converge to x' and $\{\text{Res}_g^f x_n\}$ to $\widehat{x}$. To end the conclusion, we need to prove that $\text{Res}_g^f x' = \widehat{x}$. Indeed, for each x_n, Lemma 6 shows that there exists a unique $z_n \in C$ such that $z_n = \text{Res}_g^f x_n$, that is,

$$g(z_n, y) + \langle \nabla f(z_n) - \nabla f(x_n), y - z_n \rangle \geq 0, \quad \forall y \in C. \tag{73}$$

Since f is uniformly Fréchet differentiable, ∇f is uniformly continuous. So, taking the limit as $n \to \infty$ in (73), by using (C3′) we get

$$g(\widehat{x}, y) + \langle \nabla f(\widehat{x}) - \nabla f(x'), y - \widehat{x} \rangle \geq 0, \quad \forall y \in C, \tag{74}$$

which implies that $\text{Res}_g^f x' = \widehat{x}$. This completes the proof. □

If the bifunction g satisfies conditions (C1), (C2), (C3′), and (C4) instead of (C1)–(C4), then we have a simple method to prove that $x^* \in \text{EP}(g)$ in the proof of Theorem 7. Indeed, from the proof of Theorem 7, we see that

$$u_{i,n} - T_i^n x_n \longrightarrow 0, \quad \text{that is, } \text{Res}_g^f T_i^n x_n - T_i^n x_n \longrightarrow 0,$$
$$T_i^n x_n - x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \ \forall i \geq 1. \tag{75}$$

Note that $x_n \to x^*$ as $n \to \infty$. This shows that $T_i^n x_n \to x^*$ as $n \to \infty$ for every $i \geq 1$. It follows from the closedness of Res_g^f that $x^* \in F(\text{Res}_g^f)$. Lemma 6 shows that $x^* \in \text{EP}(g)$.

Remark 14. Obviously, the proof process of $x^* \in \text{EP}(g)$ is simple if we replace condition (C3) with (C3′) which is such that Res_g^f is closed. In fact, although condition (C3′) is stronger than (C3), it is not easier to verify condition (C3) than to verify the condition (C3′). Hence, from this viewpoint, the condition (C3′) is acceptable.

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