Research Article

H_∞ Control for Linear Positive Discrete-Time Systems

Youmei Zhang,[1,2] Qingling Zhang,[1] and Tamaki Tanaka[2]

[1] *Institute of Systems Science, Northeastern University, Shenyang 110819, China*
[2] *Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan*

Correspondence should be addressed to Qingling Zhang; qlzhangneu1@163.com

This paper is concerned with H_∞ control for linear positive discrete-time systems. Positive systems are characterized by nonnegative restriction on systems' variables. This restriction results in some remarkable results which are available only for linear positive systems. One of them is the celebrated diagonal positive definite matrix solutions to some existed well-known results for linear systems without nonnegative restriction. We provide an alternative proof for criterion of H_∞ norm by using separating hyperplane theorem and Perron-Frobenius theorem for nonnegative matrices. We also consider H_∞ control problem for linear positive discrete-time systems via state feedback. Necessary and sufficient conditions for such problem are presented under controller gain with and without nonnegative restriction, and then the desired controller gains can be obtained from the feasible solutions.

1. Introduction

Positive systems are widespread in many practical systems, such as economic systems [1], biology systems [2], and age-structured population models [3], whose variables are required to be nonnegative and have no meaning with negative values. The explicit definition of a positive system is that its state and output are always nonnegative for any nonnegative initial state and any nonnegative input. Due to the nonnegative restriction on systems' variables, positive systems are defined on cones rather than linear space. Hence, there are excellent and remarkable outcomes which are available only for positive systems. One of them is the existence of diagonal positive definite matrix solutions to some celebrated results for linear systems without nonnegative restriction. Therefore, the investigation of positive systems is interesting and challenging and developed a new branch in systems theory. Positive systems have been of great interest to many researchers over several decades. A great number of results have been reported in the literature; see, for instance, [3–16].

It is worth noting that convex optimization is a powerful tool for analysis of positive systems. In [9–14], some remarkable results for positive systems are studied using convex optimization. On the other hand, the problem of H_∞ control has been a topic of recurring interest for several decades. A great number of results on H_∞ control have been

obtained, and different approaches have been proposed. In recent years, increasing attention has been paid to H_∞ norm analysis for positive systems. In [12], the KYP lemma for linear positive continuous-time systems is proved based on the semidefinite programming duality. The alternative proofs along the line of the rank-one separable property are given to several remarkable and peculiar results for positive systems in [13]. In [14], the KYP lemma for linear positive discrete-time systems is studied using a theorem of alternatives on the feasibility of linear matrix inequalities (LMIs).

This paper is organized as follows. Preliminaries are introduced in Section 2. Main results on H_∞ control for linear positive discrete-time systems are presented and proved in Section 3. Section 4 is devoted to illustrate the effectiveness of the obtained results by numerical examples. Section 5 concludes this paper.

2. Preliminaries

In this section, we introduce terminology, positive systems, various other definitions, and lemmas, which will be essentially used for proving our main results.

At first, the following notations will be used throughout this paper.

$\mathbb{Z}_+$ denotes the nonnegative integers. $x \geq 0$ denotes the vector $x \in \mathbb{R}^n$ with nonnegative entries. $A \geq 0$ denotes

the matrix $A \in \mathbb{R}^{n \times m}$ with nonnegative entries. $\mathbb{R}_+^n$ denotes the set of all vectors $x \in \mathbb{R}^n$ with $x \geq 0$. $\mathbb{R}_+^{n \times m}$ denotes the set of all matrices $A \in \mathbb{R}^{n \times m}$ with $A \geq 0$. S^n denotes the set of all symmetric matrices. $D_+^{n \times n}$ denotes the set of all diagonal positive definite matrices. $A \succeq 0$, $A \succ 0$, and $A \prec 0$ mean that A is a positive semidefinite, positive definite, and negative definite matrix, respectively. A_{ij} denotes the ijth entry of matrix A. x_i denotes the ith entry of vector x. For two matrices $A, B \in \mathbb{R}^{n \times m}$, $A \geq B$ means $A_{ij} \geq B_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$, $i = 1, 2, \ldots, n$. $\rho(A)$ denotes the spectral radius of matrix A which is defined as $\rho(A) := \max_{1 \leq i \leq n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. $D(A)$ denotes the vector which is composed of the diagonal entries of $A \in \mathbb{R}^{n \times n}$. $\langle X, Y \rangle = \text{trace}(XY)$ is the inner product on S^n.

Consider the following linear discrete-time system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k) + Du(k),
\end{aligned}
\tag{1}
$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the input, and $y(k) \in \mathbb{R}^p$ is the output. $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are known matrices.

Definition 1 (see [3]). System (1) is said to be positive if and only if $x(k) \geq 0$, $y(k) \geq 0$, $k \in \mathbb{Z}_+$ for any $x(0) \geq 0$ and for any $u(k) \geq 0$, $k \in \mathbb{Z}_+$.

Definition 2 (see [3]). System (1) is said to be asymptotically stable if $\rho(A) < 1$.

Definition 3 (see [13]). For a given matrix $H \in S^n$ with $H \succeq 0$, we define $h \in \mathbb{R}_+^n$ by

$$
h_i := \sqrt{H_{ii}}, \quad i = 1, 2, \ldots, n.
\tag{2}
$$

Lemma 4 (see [3], Perron-Frobenius theorem for nonnegative matrices). *Let $A \in \mathbb{R}_+^{n \times n}$; then $\rho(A)$ is an eigenvalue of A and A has a nonnegative eigenvector v corresponding to $\rho(A)$.*

A matrix $A \in \mathbb{R}^{n \times n}$ is called a Metzler matrix if $A_{ij} \geq 0$, for all i, j with $i \neq j$.

Lemma 5 (see [13]). *For a given Metzler matrix $A \in \mathbb{R}^{n \times n}$ and $H \in S^n$ with $H \succeq 0$, the following conditions hold:*

  (i) $(hh^T)_{ii} = H_{ii}$, $(hh^T)_{ij} \geq H_{ij}$, $i \neq j$,

  (ii) $D(hh^T A) \geq D(HA)$,

where $h \in \mathbb{R}_+^n$ is defined from H as in Definition 3.

Lemma 6 (see [3]). *System (1) is positive if and only if $A \geq 0$, $B \geq 0$, $C \geq 0$, and $D \geq 0$.*

Lemma 7 (see [17], separating hyperplane theorem). *Suppose that C_1 and C_2 are two convex sets that do not intersect; that is, $C_1 \cap C_2 = \varnothing$. Then, there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C_1$ and $a^T x \geq b$ for all $x \in C_2$.*

For $A \in \mathbb{R}_+^{n \times n}$, we consider $B := sI - A$. If $s > 0$ and $s \geq \rho(A)$, then B is called an M-matrix. If $s > \rho(A)$, then B is a nonsingular M-matrix.

Lemma 8 (see [4]). *A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is an M-matrix if and only if $A^{-1} \geq 0$.*

Lemma 9 (see [18], Schur complement). *Given any real matrices Q, S, and R with $Q = Q^T$ and $R = R^T$, the following statement holds:*

$$
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \prec 0
\tag{3}
$$

if and only if

$$
R < 0, \quad Q - SR^{-1}S^T < 0,
\tag{4}
$$

or, equivalently,

$$
Q < 0, \quad R - S^T Q^{-1} S < 0.
\tag{5}
$$

The transfer function matrix of system (1) is given by

$$
G(z) = C(zI - A)^{-1}B + D,
\tag{6}
$$

and its H_∞ norm is defined as

$$
\|G\|_\infty := \sup_{\theta \in (-\pi, \pi]} \overline{\sigma}\left(G\left(e^{j\theta}\right)\right),
\tag{7}
$$

where $\overline{\sigma}(G(e^{j\theta}))$ denotes the maximum singular value of $G(e^{j\theta})$. In [14], it has been pointed out that $\|G\|_\infty = \|G(1)\|$, where $\|G(1)\| = \overline{\sigma}(G(1))$, if system (1) is positive and asymptotically stable.

3. H_∞ Control

In this section, we give an alternative proof for the existed result of H_∞ norm for positive discrete-time systems and investigate the H_∞ control under state feedback.

At first, we propose the following theorem which is helpful for the alternative proof.

Theorem 10. *Suppose that system (1) is positive; the following conditions are equivalent:*

  (i) *There exists a nonzero $h \in \mathbb{R}_+^n$ such that $(A - I)h \geq 0$.*

  (ii) *System (1) is not asymptotically stable.*

Proof. (i)$\Rightarrow$(ii). Since $A \geq 0$, from Perron-Frobenius theorem for nonnegative matrices, it follows that $A^T v = \rho(A)v \geq 0$, where $\rho(A) \geq 0$ is the spectral radius of matrix A^T and $v \geq 0$ is an eigenvector corresponding to $\rho(A)$. Then it is obtained from condition (i) that

$$
(\rho(A) - 1) v^T h \geq 0,
\tag{8}
$$

which implies $\rho(A) \geq 1$; namely, system (1) is not asymptotically stable.

(ii)$\Rightarrow$(i). System (1) is not asymptotically stable; that is, $\rho(A) \geq 1$. From Perron-Frobenius theorem, we immediately obtain the following result:

$$\rho(A)h = Ah \geq h, \quad (9)$$

where $h \geq 0$ is an eigenvector corresponding to $\rho(A)$. This completes the proof. $\square$

The following theorem was firstly presented and proved in the light of alternatives on the feasibility of LMIs by Federico Najson in the literature [14]. Now we will give another proof using separating hyperplane theorem and Theorem 10.

Theorem 11. *Suppose that system* (1) *is positive; the following conditions are equivalent.*

(i) *System* (1) *is asymptotically stable and* $\|G\|_\infty < 1$.

(ii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} A^T PA - P + C^T C & A^T PB + C^T D \\ B^T PA + D^T C & B^T PB + D^T D - I \end{bmatrix} \prec 0. \quad (10)$$

(iii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} -P & 0 & A^T P & C^T \\ 0 & -I & B^T P & D^T \\ PA & PB & -P & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0. \quad (11)$$

Proof. We only prove (i)$\Rightarrow$(ii) since the implication (ii)$\Rightarrow$(i) is obvious from the existed criterion for linear discrete-time systems, and the equivalence between (ii) and (iii) is immediately obtained using Schur complement.

To the contrary, suppose that condition (10) does not hold for any diagonal positive definite matrix P. Define the following two sets:

$$C_1 := \left(\begin{bmatrix} A^T PA - P + C^T C & A^T PB + C^T D \\ B^T PA + D^T C & B^T PB + D^T D - I \end{bmatrix} \mid P \in D_+^{n \times n} \right),$$

$$C_2 := \{Q \mid Q \prec 0, Q \in S^{n+m}\}; \quad (12)$$

then it is easy to check that sets C_1 and C_2 are nonempty and convex. By the assumption, we have $C_1 \cap C_2 = \varnothing$. Then from the separating hyperplane theorem, there exists a nonzero $H \in S^{n+m}$ such that

$$\langle H, S \rangle \geq 0, \quad \forall S \in C_1, \quad (13)$$

$$\langle H, S \rangle \leq 0, \quad \forall S \in C_2. \quad (14)$$

By condition (14), we can conclude that $\langle H, S \rangle = \text{trace}(HS) \leq 0$, for all $S \prec 0$, from which it is easy to verify that $H \succeq 0$. Thus it follows from condition (13) that there exists a nonzero $H \succeq 0$ such that

$$\text{trace}\left(H \begin{bmatrix} A^T PA - P + C^T C & A^T PB + C^T D \\ B^T PA + D^T C & B^T PB + D^T D - I \end{bmatrix} \right) \geq 0,$$

$$\forall P \in D_+^{n \times n}, \quad (15)$$

which is equivalent to

$$\text{trace}\left(H \begin{bmatrix} A^T PA - P & A^T PB \\ B^T PA & B^T PB \end{bmatrix} \right)$$
$$+ \text{trace}\left(H \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n \times n}. \quad (16)$$

Let $H := \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$, $H_{11} \succeq 0$, $H_{22} \succeq 0$; then the above condition can be rewritten as

$$\text{trace}\left(H_{11}(A^T PA - P) + H_{12} B^T PA + H_{12}^T A^T PB + H_{22} B^T PB \right)$$
$$+ \text{trace}\left(H \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n \times n}, \quad (17)$$

or, equivalently,

$$\text{trace}\left(P(AH_{11}A^T - H_{11} + AH_{12}B^T + BH_{12}^T A^T + BH_{22}B^T) \right)$$
$$+ \text{trace}\left(H \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n \times n}, \quad (18)$$

which implies that

(a) $D(AH_{11}A^T - H_{11} + AH_{12}B^T + BH_{12}^T A^T + BH_{22}B^T) \geq 0$,

(b) $\text{trace}(H \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix}) \geq 0$.

From condition (a), it follows that

$$D\left(AH_{11}A^T + AH_{12}B^T + BH_{12}^T A^T + BH_{22}B^T \right)$$
$$= D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right) \quad (19)$$
$$\geq D\left(\begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Since system (1) is positive, then

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \quad (20)$$

are nonnegative matrix and Metzler matrix, respectively. Define the nonzero $h \in \mathbb{R}_+^n$ from H as in Definition 3 then from Lemma 5, we obtain

$$D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} hh^T \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right)$$
$$\geq D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right) \quad (21)$$
$$\geq D\left(\begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$\text{trace}\left(hh^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \right)$$
$$\geq \text{trace}\left(H \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - I \end{bmatrix} \right) \geq 0. \quad (22)$$

Set $h := \begin{bmatrix} h_1^T & h_2^T \end{bmatrix}^T$, $h_1 \in \mathbb{R}_+^n$, $h_2 \in \mathbb{R}_+^m$; conditions (21) and (22) can be rewritten as

$$D\left(Ah_1h_1^TA^T + Ah_1h_2^TB^T + Bh_2h_1^TA^T + Bh_2h_2^TB^T\right) \tag{23}$$
$$\geq D\left(h_1h_1^T\right),$$

$$h_1^TC^TCh_1 + h_2^TD^TCh_1 + h_1^TC^TDh_2 + h_2^TD^TDh_2 \tag{24}$$
$$= \left(Ch_1 + Dh_2\right)^T\left(Ch_1 + Dh_2\right) \geq h_2^Th_2.$$

Since H is nonzero, then we have the following three cases.

(1) Consider that $h_1 = 0, h_2 \neq 0$. By condition (24), $h_2^TD^TDh_2 \geq h_2^Th_2$, which contradicts $\|G\|_\infty < 1$.

(2) Consider that $h_1 \neq 0, h_2 = 0$. We observe from (23) that $D(Ah_1h_1^TA^T) \geq D(h_1h_1^T)$, which implies that $Ah_1 \geq h_1$.

From Theorem 10, this is a contradiction.

(3) Consider that $h_1 \neq 0, h_2 \neq 0$. From condition (24), it yields that $Ch_1 + Dh_2 \neq 0$. Define matrix

$$\Delta = \frac{h_2(Ch_1 + Dh_2)^T}{(Ch_1 + Dh_2)^T(Ch_1 + Dh_2)}, \tag{25}$$

which is well-defined and satisfies $\bar{\sigma}(\Delta) \leq 1$. Note that $h_2 = \Delta(Ch_1 + Dh_2)$. On the other hand, $\bar{\sigma}(D) < 1$; otherwise, this contradicts $\|G\|_\infty < 1$. It is known that for matrices $M \in \mathbb{C}^{l \times m}$ and $N \in \mathbb{C}^{m \times l}$ with $\bar{\sigma}(M) \leq 1$ and $\bar{\sigma}(N) < 1$, $\det(I - MN) \neq 0$. Thus, $I - \Delta D$ is invertible and $(I - \Delta D)^{-1} \geq 0$ since $I - \Delta D$ is a nonsingular M-matrix. Therefore, $h_2 = (I - \Delta D)^{-1}\Delta Ch_1$ holds. Then from (23) we have

$$D\left(Ah_1h_1^TA^T + Ah_1h_2^TB^T + Bh_2h_1^TA^T + Bh_2h_2^TB^T\right)$$
$$= D\left(Ah_1h_1^TA^T + Ah_1h_1^TC^T\Delta^T(I - \Delta D)^{-T}B^T\right.$$
$$\left. + B(I - \Delta D)^{-1}\Delta Ch_1h_1^TA^T\right.$$
$$\left. + B(I - \Delta D)^{-1}\Delta Ch_1h_1^TC^T\Delta^T(I - \Delta D)^{-T}B^T\right)$$
$$= D\left(\left(A + B(I - \Delta D)^{-1}\Delta C\right)h_1h_1^T\left(A + B(I - \Delta D)^{-1}\Delta C\right)^T\right)$$
$$= D\left(\widetilde{A}h_1h_1^T\widetilde{A}^T\right)$$
$$\geq D\left(h_1h_1^T\right), \tag{26}$$

where $\widetilde{A} := A + B(I - \Delta D)^{-1}\Delta C \geq 0$. From Theorem 10, $\rho(\widetilde{A}) \geq 1$, which contradicts $\|G\|_\infty < 1$. $\qquad\square$

Corollary 12. *Suppose that system (1) is positive; then the following conditions are equivalent.*

(i) *System (1) is asymptotically stable and $\|G\|_\infty < \gamma$.*

(ii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \gamma^2I \end{bmatrix} < 0. \tag{27}$$

(iii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} -P & 0 & A^TP & C^T \\ 0 & -\gamma^2I & B^TP & D^T \\ PA & PB & -P & 0 \\ C & D & 0 & -I \end{bmatrix} < 0. \tag{28}$$

Now, our purpose is to design a state feedback controller given by

$$u(k) = Kx(k) + v(k), \tag{29}$$

where $K \in \mathbb{R}^{m \times n}$ is the controller gain to be designed, and $v(k) \in \mathbb{R}_+^n$, such that the closed-loop system described as

$$x(k+1) = (A + BK)x(k) + Bv(k),$$
$$y(k) = (C + DK)x(k) + Dv(k), \tag{30}$$

is positive, asymptotically stable, and $\|\overline{G}\|_\infty < 1$, where $\overline{G}(z) = (C + DK)(zI - (A + BK))^{-1} + D$.

At first, we focus on nonnegative control gain, as it has practical importance in many cases. For instance, for a chemical system whose variables represent concentrations of reactants and reaction speed is impacted by concentrations, in order to improve the speed of reaction, it is natural to consider such controller for increasing concentrations.

Theorem 13. *For the given positive system (1), there exists a nonnegative controller of the form in (29) such that the closed-loop system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$ if and only if there exist $X \in D_+^{n \times n}$ and $Y \geq 0$ satisfying*

$$\begin{bmatrix} -X & 0 & XA^T + Y^TB^T & XC^T + Y^TD^T \\ 0 & -I & B^T & D^T \\ AX + BY & B & -X & 0 \\ CX + DY & D & 0 & -I \end{bmatrix} < 0. \tag{31}$$

Under the above condition, the desired nonnegative controller gain is obtained as

$$K = YX^{-1}. \tag{32}$$

Proof. Necessity. From Theorem 11, there exists a diagonal positive definite matrix P such that

$$\begin{bmatrix} -P & 0 & (A + BK)^TP & (C + DK)^T \\ 0 & -I & B^TP & D^T \\ P(A + BK) & PB & -P & 0 \\ C + DK & D & 0 & -I \end{bmatrix} < 0. \tag{33}$$

Multiplying on both sides of inequality (33) by $T = \mathrm{diag}(P^{-1}, I, P^{-1}, I)$, it follows that

$$\begin{bmatrix} -P^{-1} & 0 & P^{-1}A^T + P^{-1}K^TB^T & P^{-1}C^T + P^{-1}K^TD^T \\ 0 & -I & B^T & D^T \\ AP^{-1} + BKP^{-1} & B & -P^{-1} & 0 \\ CP^{-1} + DKP^{-1} & D & 0 & -I \end{bmatrix}$$
$$< 0. \tag{34}$$

By defining $X = P^{-1}$, $Y = KP^{-1}$, inequality (31) is immediately obtained. On the other hand, since $X \in D_+^{n \times n}$ and $Y \geq 0$, it is easy to see that $K \geq 0$.

Sufficiency. Positivity of the closed-loop system (30) is obvious. From (32), $Y = KX$; substituting it into inequality (31) leads to

$$
\begin{bmatrix}
-X & 0 & XA^T + XK^TB^T & XC^T + XK^TD^T \\
0 & -I & B^T & D^T \\
AX + BKX & B & -X & 0 \\
CX + DKX & D & 0 & -I
\end{bmatrix} \prec 0.
$$
(35)

Multiplying on both sides of inequality (35) by $T = \text{diag}(X^{-1}, I, X^{-1}, I)$, we have

$$
\begin{bmatrix}
-X^{-1} & 0 & (A+BK)^TX^{-1} & (C+DK)^T \\
0 & -I & B^TX^{-1} & D^T \\
X^{-1}(A+BK) & X^{-1}B & -X^{-1} & 0 \\
C+DK & D & 0 & -I
\end{bmatrix} \prec 0.
$$
(36)

Therefore, from Theorem 11, the closed-loop system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$. □

Remark 14. Under the assumption that system (1) is positive, it is worth noting that there does not exist any $X \in D_+^{n \times n}$ or $Y \geq 0$ satisfying condition (31) if there exists $A_{ii} > 1$, $i = 1, 2, \ldots, n$. It is easy to verify in the light of the following facts.

(1) A linear positive discrete-time system is unstable if at least one diagonal entry of matrix A is greater than 1 which is presented in the literature [3].

(2) $\rho(B) \leq \rho(A)$ if $0 \leq B \leq A$ which has been pointed out in the literature [19].

On the other hand, it is known that the maximal eigenvalue $\rho(A)$ of $A \geq 0$ belongs to the interval

$$
\max\{\min c_i, \min r_i\} \leq \rho(A) \leq \min\{\max c_i, \max r_i\}, \quad (37)
$$

where c_i and r_i denote the sum of the elements of the ith column and the ith row of matrix A, respectively. Therefore, there also does not exist $X \in D_+^{n \times n}$ or $Y \geq 0$ satisfying condition (31) if $\max\{\min c_i, \min r_i\} \geq 1$.

Remark 15. If $0 \leq B \leq A$, then $\overline{\sigma}(A) \geq \overline{\sigma}(B)$ which is obtained immediately due to the fact (2) in Remark 14. Suppose that system (1) is positive, $\|G\|_\infty \geq 1$, and there exists a nonnegative controller (29) such that system (30) is asymptotically stable; it is obvious that system (1) is also asymptotically stable. Thus,

$$
\begin{aligned}
\|\overline{G}\|_\infty &= \|\overline{G}(1)\| \\
&= \|(C + DK)(I - (A + BK))^{-1}B + D\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| (C + DK)\left(\sum_{i=0}^{\infty}(A+BK)^i\right)B + D \right\| \\
&= \overline{\sigma}\left((C+DK)\left(\sum_{i=0}^{\infty}(A+BK)^i\right)B + D\right) \\
&\geq \overline{\sigma}\left(C\left(\sum_{i=0}^{\infty}A^i\right)B + D\right) = \|G\|_\infty \geq 1.
\end{aligned}
$$
(38)

Therefore, if system (1) is positive, asymptotically stable, but $\|G\|_\infty \geq 1$, there does not exist any nonnegative state feedback (29) such that system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$.

Corollary 16. *For the given positive system (1), there exists a nonnegative controller of the form in (29) such that the closed-loop system (30) is asymptotically stable and $\|\overline{G}\|_\infty < \gamma$ if and only if there exist $X \in D_+^{n \times n}$ and $Y \geq 0$ satisfying*

$$
\begin{bmatrix}
-X & 0 & XA^T + Y^TB^T & XC^T + Y^TD^T \\
0 & -\gamma^2 I & B^T & D^T \\
AX + BY & B & -X & 0 \\
CX + DY & D & 0 & -I
\end{bmatrix} \prec 0.
$$
(39)

Under the above condition, the desired controller gain is obtained as (32).

From Remarks 14 and 15, for some positive systems, there is no nonnegative state feedback (29) such that system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$. Hence, we are obligated to pay attention to state feedback without nonnegative restriction. It is also natural to consider such controller. For example, for an ecosystem whose variables represent population of animals in a forest; population cannot exceed the ecological capacity of the forest, otherwise, the ecosystem may be destroyed. Therefore, we must decrease the number of animals by means of harvesting or other methods when population of certain animals exceeds their ecological capacity.

Now we are looking for a state feedback without nonnegative restriction having form in (29) such that the closed-loop system (30) is positive, asymptotically stable, and $\|\overline{G}\|_\infty < 1$.

Theorem 17. *For the given positive system (1), there exists a controller of the form in (29) such that the closed-loop system (30) is positive, asymptotically stable, and $\|\overline{G}\|_\infty < 1$ if and only if there exist $X \in D_+^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$ satisfying (39) and*

$$
AX + BY \geq 0, \tag{40}
$$

$$
CX + DY \geq 0. \tag{41}
$$

Under the above conditions, the desired controller gain is given by (32).

Proof. We only prove conditions (40) and (41).

Necessity. The close-loop system (30) is positive then it follows that

$$A + BK \geq 0, \qquad C + DK \geq 0. \tag{42}$$

Since X is a diagonal positive definite matrix, then it is easy to check that

$$AX + BKX \geq 0, \qquad CX + DKX \geq 0. \tag{43}$$

Conditions (40) and (41) are immediately obtained.

Sufficiency. The proof is similar to necessity. □

Corollary 18. *For the given positive system* (1)*, there exists a controller of the form in* (29) *such that the closed-loop system* (30) *is positive, asymptotically stable, and* $\|\overline{G}\|_\infty < \gamma$ *if and only if there exist* $X \in D_+^{n \times n}$ *and* $Y \in \mathbb{R}^{m \times n}$ *satisfying conditions* (39)*,* (40)*, and* (41)*. Under these conditions, the desired controller gain is obtained as* (32)*.*

4. Examples

In this section, we give some examples to illustrate the validity of the obtained results.

Example 1. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 0.6048 & 0 \\ 0.0861 & 0.4187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0952 & 0 \\ 0.0457 & 0.1813 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.0921 & 0 \\ 0 & 0.1286 \end{bmatrix}. \tag{44}$$

Solving the conditions in Theorem 13, one feasible solution is obtained as

$$X = \begin{bmatrix} 0.3033 & 0 \\ 0 & 0.4477 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0.0854 & 0.1142 \\ 0.0743 & 0.1150 \end{bmatrix}. \tag{45}$$

Then the desired controller gain is given by

$$K = \begin{bmatrix} 0.2816 & 0.2551 \\ 0.2551 & 0.2551 \end{bmatrix}. \tag{46}$$

Example 2. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 1.2148 & 0 \\ 0.0861 & 0.9187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1702 \\ 0.0457 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.816 & 0 \end{bmatrix}, \qquad D = 0.3157. \tag{47}$$

We observe that $A_{11} > 1$ and check the condition in Theorem 13; as pointed out in Remark 14, there is no feasible solution.

Then solving the conditions in Theorem 17, one feasible solution is obtained as

$$X = \begin{bmatrix} 2.1747 & 0 \\ 0 & 810.9260 \end{bmatrix}, \qquad Y = \begin{bmatrix} -4.0062 & 5.0132 \end{bmatrix}. \tag{48}$$

The desired controller gain is

$$K = \begin{bmatrix} -1.8421 & 0.0062 \end{bmatrix}. \tag{49}$$

Example 3. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 0.9187 & 0.1295 \\ 0.0861 & 0.9187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1702 \\ 0.1570 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5700 & 0.3850 \end{bmatrix}, \qquad D = 0.2103. \tag{50}$$

By calculating, $\max\{\min c_i, \min r_i\} = 1.0048$. We check the condition in Theorem 13; there is no feasible solution.

Solving the conditions in Theorem 17, one feasible solution is obtained as

$$X = \begin{bmatrix} 0.4315 & 0 \\ 0 & 1.6751 \end{bmatrix}, \qquad Y = \begin{bmatrix} -0.2126 & -1.2241 \end{bmatrix}. \tag{51}$$

Then the controller gain is given by

$$K = \begin{bmatrix} -0.4926 & -0.7308 \end{bmatrix}. \tag{52}$$

5. Conclusion

In this paper, we are interested in H_∞ control for linear positive discrete-time systems. We present a necessary and sufficient condition to check the stability of linear positive discrete-time systems using Perron-Frobenius theorem for nonnegative matrices, which is the key point for the alternative proof. We believe that it is useful for checking the existence of diagonal positive definite matrices to some other results given for linear discrete-time systems without nonnegative restriction. The alternative proof is along the line of separating hyperplane theorem and Theorem 10 given in this paper. In addition, we investigate the H_∞ control for such systems under state feedback. Necessary and sufficient conditions for such problem are presented under controller gain with and without nonnegative restriction, and then the desired controller gains can be obtained from the feasible solutions. However, in this paper, we have restricted our attention to the case of state feedback, but in practice, it is not always possible to have access to all of the state variables. The case of static and dynamic output feedback is left for future research. Robust H_∞ analysis and synthesis for positive uncertain systems are also open problems.

Acknowledgments

References

[1] B. Cantó, C. Coll, and E. Sánchez, "Positive solutions of a discrete-time descriptor system," *International Journal of Systems Science*, vol. 39, no. 1, pp. 81–88, 2008.

[2] E. Klipp, R. Herwig, A. Kowald, C. Wierlig, and H. Lehrach, *Systems Biology in Practice: Concepts, Implementation and Application*, Wiley-VCH, Weiheim, Germany, 2005.

[3] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley, New York, NY, USA, 2000.

[4] T. Kaczorek, *Positive 1D and 2D Systems*, Springer, Berlin, Germany, 2001.

[5] L. Caccetta and V. G. Rumchev, "A survey of reachability and controllability for positive linear systems," *Annals of Operations Research*, vol. 98, pp. 101–122, 2000.

[6] P. De Leenheer and D. Aeyels, "Stabilization of positive linear systems," *Systems & Control Letters*, vol. 44, no. 4, pp. 259–271, 2001.

[7] J. Feng, J. Lam, P. Li, and Z. Shu, "Decay rate constrained stabilization of positive systems using static output feedback," *International Journal of Robust and Nonlinear Control*, vol. 21, no. 1, pp. 44–54, 2011.

[8] M. A. Rami and F. Tadeo, "Controller synthesis for positive linear systems with bounded controls," *IEEE Transactions on Circuits and Systems II*, vol. 54, no. 2, pp. 151–155, 2007.

[9] O. Mason and R. Shorten, "On linear copositive Lyapunov functions and the stability of switched positive linear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1346–1349, 2007.

[10] R. Shorten, O. Mason, and C. King, "An alternative proof of the Barker, Berman, Plemmons (BBP) result on diagonal stability and extensions," *Linear Algebra and its Applications*, vol. 430, no. 1, pp. 34–40, 2009.

[11] O. Mason, "Diagonal Riccati stability and positive time-delay systems," *Systems & Control Letters*, vol. 61, no. 1, pp. 6–10, 2012.

[12] T. Tanaka and C. Langbort, "KYP lemma for internally positive systems and a tractable class of distributed H_∞ control problems," in *Proceedings of the American Control Conference (ACC '10)*, pp. 6238–6243, July 2010.

[13] Y. Ebihara, D. Peaucelle, D. Arzelier, T. Hagiwara, and Y. Oishi, "Dual LMI approach to linear positive system analysis," in *Proceedings of the 12th International Conference on Control, Automation and Systems (ICCAS '12)*, pp. 887–891, 2012.

[14] F. Najson, "On the Kalman-Yakubovich-Popov Lemma for discrete-time positive linear systems: a novel simple proof and some related results," *International Journal of Control*, 2013.

[15] X. Chen, J. Lam, P. Li, and Z. Shu, "ℓ1-induced norm and controller synthesis of positive systems," *Automatica*, vol. 49, no. 5, pp. 1377–1385, 2013.

[16] C. Wang and T. Huang, "Static output feedback control for positive linear continuous-time systems," *International Journal of Robust and Nonlinear Control*, vol. 23, pp. 1537–1544, 2013.

[17] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.

[18] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 1994.

[19] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, NY, USA, 1979.