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Research Article

Viscosity Method for Hierarchical Fixed Point Problems with an Infinite Family of Nonexpansive Nonself-Mappings

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A viscosity method for hierarchical fixed point problems is presented to solve variational inequalities, where the involved mappings are nonexpansive nonself-mappings. Solutions are sought in the set of the common fixed points of an infinite family of nonexpansive nonself-mappings. The results generalize and improve the recent results announced by many other authors.

1. Introduction and Preliminaries

Let X a real Banach space and J be the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\} \quad (1)$$

for all $x \in X$, where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between X and X^*. If $X = H$ is a Hilbert space, then J becomes the identity mapping on H. A point $x \in C$ is a fixed point of $T : C \subset X \to X$ provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Let X be a normed linear space with dim $X \geq 2$. The modulus of smoothness of X is the function $\rho_X : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_X(\tau) := \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}. \quad (2)$$

The space X is said to be smooth if $\rho_X(\tau) > 0$, for all $\tau > 0$. It is well known that if X is smooth then J is single valued. A Banach space X is said to be strictly convex if $\|x\| = \|y\| = 1$, $x \neq y$, implies $\|x+y\|/2 < 1$.

Let C be a nonempty closed convex subset of a real Banach space X. Recall the following concepts.

Definition 1. (i) A mapping $f : C \to C$ is a ρ-contraction if $\rho \in [0, 1)$ and if the following property is satisfied

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \quad (3)$$

(ii) A mapping $T : C \to E$ is nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (4)$$

(iii) A mapping $S : C \to X$ is

(a) accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y) \rangle \geq 0; \quad (5)$$

(b) β-strongly accretive if for any $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y) \rangle \geq \beta \|x - y\|^2, \quad (6)$$

for some real constant $\beta > 0$.

Noting that if $S : C \to X$ is nonexpansive, then $I - S$ is accretive; if $f : C \to C$ is a ρ-contraction, then $I - f$ is $(1-\rho)$-strongly accretive. particulary, if $X = H$ is a Hilbert space, then (strongly) accretive mappings become (strongly) monotone mappings.

Definition 2. Let C and D be nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$.

(i) A mapping $Q : C \to D$ is called sunny, if $Q(Qx + t(x - Qx)) = Qx$ for each $x \in C$ and $t \geq 0$ with $Q(Qx + t(x - Qx)) \in C$.

(ii) A mapping $Q : C \to D$ is called a retraction from C to D if Q is continuous and $F(Q) = D$.

(iii) A subset D of $C \subset E$ is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction Q of C onto D. For details, see [1–3].

Note that if $X = H$ is a Hilbert space, Q becomes the projection on C, denoted by P_C.

Let $P : C \to C$ a nonexpansive self-mapping on C and $\{T_n\}$ be a countable family of nonexpansive nonself-mappings of C into X such that $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then we consider the following problem: find hierarchically a common fixed point of the infinite family $\{T_n\}$ with respect to a nonexpansive mapping P; namely, find $x^* \in \mathcal{F}$, such that

$$\langle x^* - Px^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{7}$$

Particularly, if $\{T_n\}$ is a finite family of nonexpansive nonself-mappings, problem (7) has been studied by Ceng and Petruşel [4]. If $X = H$ and $\{T_n\}$ is an infinite family of nonexpansive self-mappings, Problem (7) reduces to the following problem: find hierarchically a common fixed point of $\{T_n\}$ with respect to a nonexpansive mapping P, namely, find $x^* \in \mathcal{F}$, such that

$$\langle x^* - Px^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathcal{F}, \tag{8}$$

which was studied by Zhang et al. [5]. If $X = H$ is a Hilbert space and $T_n = T$, for all $n \geq 1$, where T is a nonexpansive mapping on C, then problem (7) reduces to the following problem: finding hierarchically a fixed point of T with respect to another nonexpansive mapping P; namely, find $x^* \in F(T)$ such that

$$\langle x^* - Px^*, x - x^*\rangle \geq 0, \quad \forall x \in F(T). \tag{9}$$

Problem (7) includes many problems as special cases, so it is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [6–9]).

In 2007, Moudafi [10] introduced the following Krasnoselski-Mann's algorithm in Hilbert spaces:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n (\sigma_n P x_n + (1 - \sigma_n) T x_n), \quad \forall n \geq 0, \tag{10}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two real sequences in (0,1) and T and P are two nonexpansive mappings of C into itself. Furthermore, he established a weak convergence result for Algorithm (10) for solving problem (9).

Subsequently, Yao and Liou [11] derived a weak convergence result of algorithm (10) under the restrictions on parameters weaker than those in [10, Theorem 2.1].

Recently, Marino and Xu [12] introduced the following explicit hierarchical fixed point algorithm in Hilbert spaces:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)(\alpha_n V x_n + (1 - \alpha_n) T x_n), \quad \forall n \geq 0, \tag{11}$$

where f is a contraction on C and V, T are two nonexpansive mappings of C into itself and proved that the sequence $\{x_n\}$ generated by (11) converges strongly to a solution of problem (9).

Very recently, Zhang et al. [5] introduced the following iterative algorithm in order to find hierarchically a fixed point of Problem (8):

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \\ y_n = \beta_n P(x_n) + (1 - \beta_n) T x_n, \end{gathered} \tag{12}$$

where $f : C \to C$ is a contraction, $P : C \to C$ is a nonexpansive mapping, $\{T_n\} : C \to C$ is a countable family of nonexpansive mappings, and $T : C \to C$ is a mapping defined by

$$T = \sum_{n=1}^{\infty} \lambda_n T_n, \quad \lambda_n \geq 0 \ (n = 1, 2, \ldots) \text{ with } \sum_{n=1}^{\infty} \lambda_n = 1. \tag{13}$$

Under suitable conditions on parameters $\{\alpha_n\}$ and $\{\beta_n\}$, they established some strong and weak convergence theorems. Note that, in [5], $\{T_n\}$ is an infinite family of self-mappings and P is also a self-mapping. And they obtained the results in the setting of Hilbert spaces.

Motivated and inspired by the above researches, in a reflexive Banach space which admits a weakly sequentially continuous duality mapping J, we propose and analyze an iteration process for a countable family of nonexpansive nonself-mappings $\{T_n\} : C \to X$ and $S : C \to X$ is a nonexpansive nonself-mapping as follows:

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = Q(\alpha_n f(x_n) + (1 - \alpha_n) y_n), \\ y_n = \beta_n S x_n + (1 - \beta_n) T x_n, \quad n \geq 0, \end{gathered} \tag{14}$$

where Q is a sunny nonexpansive retraction of X onto C and establishes a convergence theorem. particularly, if $X = H$ is a Hilbert space, we obtain some convergence results.

To prove the main results, we need the following lemmas.

Lemma 3 (see [1]). *Let C be a nonempty and convex subset of a smooth Banach space X, $D \subset C$, $J : X \to X^*$ the normalized duality mapping of X, and $Q : C \to D$ a retraction. Then the following conditions are equivalent:*

(i) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, *for all $x \in C$ and $y \in D$;*

(ii) *Q is both sunny and nonexpansive.*

Lemma 4 (see [13, Lemma 3.1, 3.3]). *Let X be a real smooth and strictly convex Banach space and C a nonempty closed and* *convex subset of X which is also a sunny nonexpansive retract of X. Assuming that $T : C \to X$ is a nonexpansive mapping and Q is a sunny nonexpansive retraction of X onto C, then $F(T) = F(QT)$.*

Lemma 5 (see [1]). *Let X be a real Banach space and $J : X \to 2^{X^*}$ the normalized duality mapping. Then for any $x, y \in X$, the following hold:*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$, *for all $j(x + y) \in J(x + y)$;*

(ii) $\|x\|^2 + 2\langle y, j(x)\rangle \le \|x + y\|^2$, *for all $j(x) \in J(x)$.*

Lemma 6 (see [14]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty, \qquad a_{n+1} \le a_n + b_n, \quad n = 0, 1, 2, \ldots. \tag{15}$$

Then $\lim_{n\to\infty} a_n$ exists.

Lemma 7 (see [15]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 - \lambda_n) a_n + \lambda_n b_n + c_n, \quad \forall n \ge 0, \tag{16}$$

where $\{\lambda_n\}$, $\{b_n\}$ and $\{c_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \lambda_n) = 0$;

(ii) $\limsup_{n\to\infty} b_n \le 0$;

(iii) $c_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

If Banach space X admits sequentially continuous duality mapping J from weak topology to weak $*$ topology, then by [16, Lemma 1] we get that duality mapping J is single-valued. In this case, duality mapping J is also said to be weakly sequentially continuous, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup Jx$ [16, 17].

Recall that a Banach space X is said to be satisfying Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ $(n \to \infty)$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E, \text{ with } y \ne x. \tag{17}$$

By [16, Lemma 1], we know that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition.

In the sequel, we also need the following lemmas.

Lemma 8 (see [17]). *Let C be a nonempty, closed and convex subset of a reflexive Banach space X which satisfies Opial's condition and $T : C \to X$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed at zero, that is,*

$$x_n \rightharpoonup x, \qquad x_n - Tx_n \longrightarrow 0 \qquad \text{imply } x = Tx. \tag{18}$$

Let C be a nonempty and convex subset of a Banach space X. Then for $x \in C$, one defines the inward set $I_C(x)$ as follows [2, 3]:

$$I_C(x) = \{y \in X : y = x + \lambda(z - x), z \in C, \lambda \ge 0\}. \tag{19}$$

A mapping $T : C \to X$ is said to satisfy the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to satisfy the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set if C does.

Lemma 9 (see [18, Theorem 2.4]). *Let X be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a nonempty closed convex subset of X which is also a sunny nonexpansive retract of X, and $T : C \to X$ is a nonexpansive mapping satisfying the weakly inward condition and $F(T) \ne \emptyset$. Let $\{u_n\}$ be defined by*

$$u_0 \in C, \qquad u_{n+1} = Q(\alpha_n f(u_n) + (1 - \alpha_n) Tu_n), \tag{20}$$

where Q is a sunny nonexpansive retract of X onto C and $\alpha_n \in (0, 1)$ satisfy the following conditions:

(i) $\alpha_n \to 0$, *as* $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} (\alpha_n/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ converges strongly to a fixed point p of T such that p is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f) p, j(p - u)\rangle \le 0, \quad \forall u \in F(T). \tag{21}$$

Remark 10. If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, from Lemma 1 of [16] it follows that X is smooth. So for Lemma 9, if X is a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J, by Lemma 4, the weakly inward condition of T can be removed.

2. Main Results

Theorem 11. *Let X be a reflexive and strictly convex Banach space which admits a weakly sequentially continuous duality mapping $J : X \to X^*$ and C a nonempty, closed and convex subset of X which is also a sunny nonexpansive retract of X. Let $S : C \to X$ be a nonexpansive nonself-mapping, $f : C \to C$*

a contractive mapping with a contractive constant $\rho \in (0,1)$ *and* $T_i : C \to X$ $(i = \{1,2,\ldots\})$ *an infinite family of nonexpansive nonself-mappings such that* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $T : C \to X$ *be defined by* (13) *and* Q *a sunny nonexpansive retraction of* X *onto* C. *Let* $\{x_n\}$ *be the sequence generated by* (14), *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *the sequences in (0,1) satisfying the following conditions:*

(i) $\alpha_n \to 0$ $(n \to \infty)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ *converges strongly to some point* $x^* \in F(T) = \bigcap_{i=1}^{\infty} F(T_i)$, *which is the unique solution to the following variational inequality:*

$$\langle (I-f)x^*, J(x-x^*)\rangle \geq 0, \quad \forall x \in F(T). \tag{22}$$

Proof. From condition (ii), without loss of generality, we can assume that $\beta_n \leq \alpha_n$, for all $n \geq 0$.

First we prove that the sequence $\{x_n\}$ is bounded.

In fact, for any $u \in F(T)$, we have

$$\begin{aligned}
&\|x_{n+1} - u\| \\
&= \|Q(\alpha_n f(x_n) + (1-\alpha_n) y_n) - Qu\| \\
&\leq \alpha_n \|f(x_n) - u\| + (1-\alpha_n)\|\beta_n S x_n + (1-\beta_n) T x_n - u\| \\
&\leq \alpha_n (\rho \|x_n - u\| + \|f(u) - u\|) \\
&\quad + (1-\alpha_n)(\beta_n \|Sx_n - u\| + (1-\beta_n)\|Tx_n - u\|) \\
&\leq (1-\alpha_n(1-\rho))\|x_n - u\| + \alpha_n \|f(u) - u\| \\
&\quad + (1-\alpha_n)\beta_n \|Su - u\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n - u\| \\
&\quad + \alpha_n (\|f(u) - u\| + \|Su - u\|) \\
&\leq \max\left\{\|x_n - u\|, \frac{\|f(u) - u\| + \|Su - u\|}{1-\rho}\right\}.
\end{aligned} \tag{23}$$

By induction,

$$\|x_{n+1} - u\| \leq \max\left\{\|x_0 - u\|, \frac{\|f(u) - u\| + \|Su - u\|}{1-\rho}\right\}. \tag{24}$$

Thus $\{x_n\}$ is bounded, so $\{Sx_n\}$ and $\{Tx_n\}$ are also bounded.

Next we prove that $\|x_n - u_n\| \to 0$, as $n \to \infty$, where the sequence $\{u_n\}$ is defined by

$$\begin{aligned}
u_0 &= x_0 \in C, \\
u_{n+1} &= Q(\alpha_n f(u_n) + (1-\alpha_n) T u_n).
\end{aligned} \tag{25}$$

By Lemma 9 and Remark 10, $\{u_n\}$ converges strongly to some point $x^* \in F(T)$, which is the unique solution to the following variational inequality:

$$\langle (I-f)x^*, j(x^* - x)\rangle \leq 0, \quad \forall x \in F(T). \tag{26}$$

Furthermore, we obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| \\
&\leq \|Q(\alpha_n f(x_n) + (1-\alpha_n) y_n) \\
&\qquad - Q(\alpha_n f(u_n) + (1-\alpha_n) T u_n)\| \\
&\leq \|\alpha_n (f(x_n) - f(u_n)) + (1-\alpha_n)(y_n - Tu_n)\| \\
&\leq \alpha_n \rho \|x_n - u_n\| + (1-\alpha_n) \\
&\qquad \times (\beta_n \|Sx_n - Tu_n\| + (1-\beta_n)\|Tx_n - Tu_n\|) \\
&\leq (1-\alpha_n(1-\rho))\|x_n - u_n\| + (1-\alpha_n)\beta_n M \\
&\leq (1-\alpha_n(1-\rho))\|x_n - u_n\| + \beta_n M,
\end{aligned} \tag{27}$$

where $M = \sup_{n\geq 0}\|Sx_n - Tu_n\|$. It follows from conditions (i)-(ii) and Lemma 7 we have $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since as $n \to \infty$, $u_n \to x^* \in F(T)$, we get $x_n \to x^* (n \to \infty)$, which is the unique solution to the variational inequality (22). □

Remark 12. Theorem 11 extends Theorem 2.1 in [5] from the following aspects: (i) from Hilbert spaces to reflexive and strictly convex Banach spaces which admits a weakly sequentially continuous duality mapping; (ii) for the infinite family of mappings $\{T_i\}$ from self-mappings to nonself-mappings. In addition, the existence of the sunny nonexpansive retraction has been proved in [19, Theorem 3.10].

Remark 13. If we take

$$\begin{aligned}
\alpha_n &= \frac{1}{(1+n)^{\alpha}}, \\
\beta_n &= \frac{1}{(1+n)^{\beta}}, \\
0 &< \alpha < \beta < 1,
\end{aligned} \tag{28}$$

then since $|\alpha_{n+1} - \alpha_n| \approx 1/n^{\alpha+1}$ and $|\beta_{n+1} - \beta_n| \approx 1/n^{\beta+1}$ (as $n \to \infty$), it is not hard to find that the conditions (i)–(iii) are satisfied. For details, see [12, Remark 3.2].

In the sequel, we consider the result in the setting of Hilbert spaces.

Theorem 14. *Let* H *be a Hilbert space and* C *a nonempty, closed and convex subset of* H. *Let* $S : C \to H$ *be a nonexpansive nonself-mapping,* $f : C \to C$ *a contractive mapping with a contractive constant* $\rho \in (0,1)$, *and* $T_i : C \to H$ $(i = \{1,2,\ldots\})$ *an infinite family of nonexpansive nonself-mappings such that* $F(T) = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by* (14) *and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *the sequences in* $(0,1)$ *satisfying the following conditions:*

(i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} (\beta_n/\alpha_n) = \tau \in (0,+\infty)$;

(iii) $\lim_{n\to\infty}((|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|)/\alpha_n\beta_n) = 0$;

(iv) *there exists a constant* $K > 0$ *such that* $1/\alpha_n \,|(1/\beta_n) - (1/\beta_{n-1})| \leq K$ *for all* $n > 0$.

Then $\{x_n\}$ *converges strongly to some point* $x^* \in F(T)$*, which is the unique solution to the following variational inequality:*

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T). \tag{29}$$

Proof. By condition (ii), without loss of generality, we can assume that $\beta_n \leq (\tau+1)\alpha_n$, for all $n \geq 0$. Similar to the proof of (24), for any $u \in F(T)$, we have

$$\begin{aligned}\|x_{n+1} - u\| \\ &\leq \max\left\{\|x_0 - u\|, \frac{(\tau+1)(\|f(u) - u\| + \|Su - u\|)}{1-\rho}\right\}.\end{aligned} \tag{30}$$

Thus $\{x_n\}$ is bounded. Furthermore, $\{f(x_n)\}, \{Tx_n\}, \{y_n\}, \{Sx_n\}$ are all bounded. Put $u_n = \alpha_n f(x_n) + (1-\alpha_n)y_n$ and $M = \sup_{n\geq 0}\{\|f(x_n)\| + \|y_n\|, \|Tx_n\| + \|Sx_n\|\}$. So $\{u_n\}$ and $\{P_C(u_n)\}$ are also bounded.

Step 1. We prove that $\|x_{n+1} - x_n\| \to 0$ $(n \to \infty)$.

From (14), we obtain

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\quad = \|P_C(u_n) - P_C(u_{n-1})\| \leq \|u_n - u_{n-1}\| \\ &\quad \leq \alpha_n \|f(x_n) - f(x_{n-1})\| + (1-\alpha_n)\|y_n - y_{n-1}\| \\ &\qquad + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|y_{n-1}\|) \\ &\quad \leq \alpha_n\rho\|x_n - x_{n-1}\| + (1-\alpha_n) \\ &\qquad \times \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| M,\end{aligned} \tag{31}$$

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &\quad \leq \beta_n\|Sx_n - Sx_{n-1}\| + (1-\beta_n)\|Tx_n - Tx_{n-1}\| \\ &\qquad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|Tx_{n-1}\|) \\ &\quad \leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| M.\end{aligned} \tag{32}$$

Substituting (32) into (31), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 - \alpha_n(1-\rho))\|x_n - x_{n-1}\| \\ &\quad + \alpha_n \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\alpha_n}.\end{aligned} \tag{33}$$

By conditions (i), (iii), and Lemma 7, we have $\|x_{n+1} - x_n\| \to 0$ $(n \to \infty)$.

Step 2. We prove that $\omega_w(x_n) \subset F(T)$, where $\omega_w(x_n)$ is the ω-limit point set of $\{x_n\}$ in the weak topology:

$$\begin{aligned}&\|x_{n+1} - QTx_n\| \\ &\quad \leq \alpha_n\|f(x_n)\| + \beta_n\|Sx_n\| + (\alpha_n + \beta_n + \alpha_n\beta_n)\|Tx_n\|.\end{aligned} \tag{34}$$

Noting that $\alpha_n \to 0$ and $\beta_n \to 0$, we have $\|x_{n+1} - QTx_n\| \to 0$ $(n \to \infty)$. Then from Step 1 we have $\|x_n - QTx_n\| \to 0$ $(n \to \infty)$. Furthermore, it follows from Lemmas 4 and 8 that $\omega_w(x_n) \subset F(QT) = F(T)$, where $Q = P_C$.

Step 3. We show that $\|x_{n+1} - x_n\|/\beta_n \to 0$ $(n \to \infty)$.

It follows from (31) and (33) that

$$\begin{aligned}&\frac{\|x_{n+1} - x_n\|}{\beta_n} \\ &\quad \leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \leq (1 - \alpha_n(1-\rho))\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\qquad + (1 - \alpha_n(1-\rho))\|x_n - x_{n-1}\|\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \\ &\qquad + \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\beta_n} \\ &\quad \leq (1 - \alpha_n(1-\rho))\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} \\ &\qquad + \alpha_n\|x_n - x_{n-1}\| K \\ &\qquad + \frac{(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|)M}{\alpha_n\beta_n}\alpha_n.\end{aligned} \tag{35}$$

By conditions (i) and (iii), $\|x_n - x_{n-1}\| \to 0$ $(n \to \infty)$, and Lemma 7, we have

$$\frac{\|x_{n+1} - x_n\|}{\beta_n} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{36}$$

Thus from (35), we get

$$\frac{\|u_n - u_{n-1}\|}{\beta_n} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{37}$$

Step 4. We show that $\{x_n\}$ converges strongly to some point $x' \in F(T)$, which is the unique solution of (29).

Setting $W_n = \beta_n S + (1-\beta_n)T$, we have

$$x_{n+1} = P_C(u_n) - u_n + \alpha_n f(x_n) + (1-\alpha_n)W_n x_n. \tag{38}$$

Then

$$\begin{aligned}&x_n - x_{n+1} \\ &\quad = u_n - P_C(u_n) + \alpha_n(I-f)x_n + (1-\alpha_n)(I-W_n)x_n.\end{aligned} \tag{39}$$

Letting $v_n = (x_n - x_{n+1})/(1-\alpha_n)\beta_n$, from condition (i) and (36), we have $v_n \to 0$ $(n \to \infty)$. Noting that $I - W_n$ is monotone and $I - f$ is $(1-\rho)$-strongly monotone, for any $x^* \in F(T)$, from Lemma 3 we obtain

$$\begin{aligned}
&\langle v_n, x_n - x^* \rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x_n, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I-W_n)x_n, x_n - x^* \rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x_n, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I-W_n)x_n - (I-W_n)x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I-W_n)x^*, x_n - x^* \rangle \\
&= \frac{1}{(1-\alpha_n)\beta_n} \langle u_n - P_C(u_n), P_C(u_n) - x^* \rangle \\
&\quad + \frac{1}{(1-\alpha_n)\beta_n} \\
&\quad \times \langle u_n - P_C(u_n), -(P_C(u_n) - x^*) + (P_C(u_{n-1}) - x^*) \rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x_n - (I-f)x^*, x_n - x^* \rangle \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I-W_n)x_n - (I-W_n)x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{\beta_n} \langle (I-W_n)x^*, x_n - x^* \rangle \\
&\geq \frac{1}{(1-\alpha_n)\beta_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
&\quad + \frac{\alpha_n(1-\rho)}{(1-\alpha_n)\beta_n} \|x_n - x^*\|^2 \\
&\quad + \frac{\alpha_n}{(1-\alpha_n)\beta_n} \langle (I-f)x^*, x_n - x^* \rangle \\
&\quad + \langle (I-S)x^*, x_n - x^* \rangle .
\end{aligned} \tag{40}$$

Thus we have

$$\begin{aligned}
&\|x_n - x^*\|^2 \\
&\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)} \langle v_n, x_n - x^* \rangle \\
&\quad - \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)} \langle (I-S)x^*, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n(1-\rho)} \langle u_n - P_C(u_n), P_C(u_{n-1}) - P_C(u_n) \rangle \\
&\quad - \frac{1}{(1-\rho)} \langle (I-f)x^*, x_n - x^* \rangle \\
&\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)} \|v_n\| \|x_n - x^*\| \\
&\quad - \frac{(1-\alpha_n)\beta_n}{\alpha_n(1-\rho)} \langle (I-S)x^*, x_n - x^* \rangle \\
&\quad + \frac{1}{(1-\rho)} \|u_n - P_C(u_n)\| \left\| \frac{u_{n-1} - u_n}{\alpha_n} \right\| \\
&\quad - \frac{1}{(1-\rho)} \langle (I-f)x^*, x_n - x^* \rangle .
\end{aligned} \tag{41}$$

Since $\beta_n \leq (\tau+1)\alpha_n$, by (37) we have

$$\frac{\|u_n - u_{n-1}\|}{\alpha_n} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{42}$$

Combining condition (ii), $v_n \to 0$ $(n \to \infty)$, (41), and (42), every weak cluster point of $\{x_n\}$ is also a strong cluster point. From (40), we obtain

$$\begin{aligned}
&\langle (I-f)x_n, x_n - x^* \rangle \\
&= \frac{(1-\alpha_n)\beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), x_n - x^* \rangle \\
&\quad - \frac{(1-\alpha_n)}{\alpha_n} \langle (I-W_n)x_n, x_n - x^* \rangle \\
&= \frac{(1-\alpha_n)\beta_n}{\alpha_n} \langle v_n, x_n - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_n) - x^* \rangle \\
&\quad - \frac{1}{\alpha_n} \langle u_n - P_C(u_n), P_C(u_{n-1}) \\
&\qquad - P_C(u_n) \rangle - \frac{(1-\alpha_n)}{\alpha_n} \\
&\quad \times \langle (I-W_n)x_n - (I-W_n)x^*, x_n - x^* \rangle \\
&\quad - \frac{(1-\alpha_n)}{\alpha_n} \langle (I-W_n)x^*, x_n - x^* \rangle
\end{aligned}$$

$$\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n}\|v_n\|\|x_n - x^*\| + \frac{1}{\alpha_n}\|u_n - P_C(u_n)\| P_C(u_{n-1}) - P_C(u_n)\| - \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S)x^*, x_n - x^*\rangle$$
$$\leq \frac{(1-\alpha_n)\beta_n}{\alpha_n}\|v_n\|\|x_n - x^*\| + \frac{\|u_{n-1} - u_n\|}{\alpha_n}\|u_n - P_C(u_n)\| - \frac{(1-\alpha_n)\beta_n}{\alpha_n}\langle (I-S)x^*, x_n - x^*\rangle. \quad (43)$$

Note that the sequence $\{x_n\}$ is bounded; thus there exists a subsequence $\{x_{n_j}\}$ converging to a point $x' \in H$. From Step 2, we have $x' \in F(T)$. Then it follows from the above inequality, (42), and $v_n \to 0$ $(n \to \infty)$ that

$$\left\langle (I-f)x', x' - x^*\right\rangle \leq -\tau\left\langle (I-S)x^*, x' - x^*\right\rangle, \quad \forall x^* \in F(T). \quad (44)$$

Replacing x^* with $x' + \mu(x^* - x')$, where $\mu \in (0,1)$ and $x^* \in F(T)$, we have

$$\langle (I-f)x', x' - x^*\rangle \leq -\tau\langle (I-S)\left(x' + \mu\left(x^* - x'\right)\right), x' - x^*\rangle, \quad \forall x^* \in F(T). \quad (45)$$

Letting $\mu \to 0$, we have

$$\left\langle (I-f)x', x' - x^*\right\rangle \leq -\tau\left\langle (I-S)x', x' - x^*\right\rangle, \quad \forall x^* \in F(T). \quad (46)$$

If there exists another subsequence $\{x'_{n_j}\}$ of $\{x_n\}$ converging to a point $x'' \in H$. From Step 2, we also have $x'' \in F(T)$. Then from (46) we obtain

$$\langle (I-f)x', x' - x''\rangle \leq -\tau\left\langle (I-S)x', x' - x''\right\rangle \quad (47)$$

and, via interchanging x' and x'',

$$\left\langle (I-f)x'', x'' - x'\right\rangle \leq -\tau\left\langle (I-S)x'', x'' - x'\right\rangle. \quad (48)$$

Adding up these two inequalities yields

$$(1-\rho)\left\|x' - x''\right\|^2 \leq \left\langle (I-f)x' - (I-f)x'', x' - x''\right\rangle \leq 0, \quad (49)$$

which implies $x' = x''$. Then $\{x_n\}$ converges strongly to $x' \in F(T)$, which is the solution to the following variational inequality:

$$\left\langle \frac{1}{\tau}(I-f)x' + (I-S)x', x - x'\right\rangle \geq 0, \quad \forall x \in F(T). \quad (50)$$

Since $I-f$ is $(1-\rho)$-strongly monotone and $I-S$ is monotone, it is easy to see that the above variational inequality has a unique solution. □

Remark 15. Theorem 14 extends Theorem 3.2 in [12] from the following aspects: (i) from a nonexpansive mapping T to an infinite family of nonexpansive mappings $\{T_i\}$; (ii) from self-mappings to nonself-mappings.

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