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Research Article

Some Equivalences between Cone b-Metric Spaces and b-Metric Spaces

Poom Kumam,[1] Nguyen Van Dung,[2] and Vo Thi Le Hang[3]

[1] *Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
[2] *Faculty of Mathematics and Information Technology Teacher Education, Dong Thap University, Cao Lanh City, Dong Thap Province 871200, Vietnam*
[3] *Journal of Science, Dong Thap University, Cao Lanh City, Dong Thap Province 871200, Vietnam*

Correspondence should be addressed to Poom Kumam; poom.kum@kmutt.ac.th

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We introduce a b-metric on the cone b-metric space and then prove some equivalences between them. As applications, we show that fixed point theorems on cone b-metric spaces can be obtained from fixed point theorems on b-metric spaces.

1. Introduction and Preliminaries

The fixed point theory in b-metric spaces was investigated by Bakhtin [1], Czerwik [2], Akkouchi [3], Olatinwo and Imoru [4], and Păcurar [5]. A b-metric space was also called a *metric-type space* in [6]. The fixed point theory in metric-type spaces was investigated in [6, 7]. Recently, Hussain and Shah introduced the notion of a cone b-metric as a generalization of a b-metric in [8]. Some fixed point theorems on cone b-metric spaces were stated in [8–10].

Note that the relation between a cone b-metric and a b-metric is likely the relation between a cone metric [11] and a metric. Some authors have proved that fixed point theorems on cone metric spaces are, essentially, fixed point theorems on metric space; see [12–16] for example. Very recently, Du used the method in [12] to introduce a b-metric on a cone b-metric space and stated some relations between fixed point theorems on cone b-metric spaces and on b-metric spaces [17].

In this paper, we use the method in [13] to introduce another b-metric on the cone b-metric space and then prove some equivalences between them. As applications, we show that fixed point theorems on cone b-metric spaces can be obtained from fixed point theorems on b-metric spaces.

Now, we recall some definitions and lemmas.

Definition 1 (see [1]). Let X be a nonempty set and $d : X \times X \to [0, +\infty)$. Then, d is called a b-metric on X if

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

The pair (X, d) is called a *b-metric space*. A sequence $\{x_n\}$ is called *convergent* to x in X, written $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ is called a *Cauchy sequence* if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. The b-metric space (X, d) is called *complete* if every Cauchy sequence in X is a convergent sequence.

Remark 2. On a b-metric space (X, d), we consider a topology induced by its convergence. For results concerning b-metric spaces, readers are invited to consult papers [1, 2].

Remark 3. Let (X, d) be a b-metric space. For each $r > 0$ and $x \in X$, we set

$$B(x, r) = \{y \in X : d(x, y) < r\}. \tag{1}$$

In [3], Akkouchi claimed that the topology $\mathscr{T}(d)$ on X associated with d is given by setting $U \in \mathscr{T}(d)$ if and only if, for each $x \in U$, there exists some $r > 0$ such that $B(x, r) \subset U$ and the convergence of $\{x_n\}_n$ in the b-metric space (X, d) and that in the topological space $(X, \mathcal{T}(d))$ are equivalent. Unfortunately, this claim is not true in general; see Example 13. Note that; on a b-metric space, we always consider the topology induced by its convergence. Most of concepts and results obtained for metric spaces can be extended to the case of b-metric spaces. For results concerning b-metric spaces, readers are invited to consult papers [1, 2].

In what follows, let E be a real Banach space, P a subset of E, θ the zero element of E, and $\operatorname{int} P$ the interior of P. We define a partially ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We also write $x < y$ to indicate that $x \leq y$ and $x \neq y$ and write $x \ll y$ to indicate that $y - x \in \operatorname{int} P$. Let $\|\cdot\|$ denote the norm on E.

Definition 4 (see [11]). P is called a *cone* if and only if

(1) P is closed and nonempty and $P \neq \{\theta\}$;

(2) $a, b \in \mathbb{R}$; $a, b \geq 0$; $x, y \in P$ imply that $ax + by \in P$;

(3) $P \cap (-P) = \{\theta\}$.

The cone P is called *normal* if there exists $K \geq 1$ such that, for all $x, y \in E$, we have $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying the above is called the *normal constant* of P.

Definition 5 (see [11, Definition 1]). Let X be a nonempty set and $d : X \times X \to E$ satisfy

(1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space.*

Definition 6 (see [8, Definition 2.1]). Let X be a nonempty set and $d : X \times X \to P$ satisfy

(1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for some $s \geq 1$ and all $x, y, z \in X$.

Then d is called a *cone b-metric* with coefficient s on X and (X, d) is called a *cone b-metric space* with coefficient s.

Definition 7 (see [8, Definition 2.4]). Let (X, d) be a cone b-metric space and $\{x_n\}$ a sequence in X.

(1) $\{x_n\}$ is called *convergent* to x, written $\lim_{n\to\infty} x_n = x$, if for each $c \in E$ with $\theta \ll c$, there exists n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

(2) $\{x_n\}$ is called a *Cauchy sequence* if for each $c \in E$ with $\theta \ll c$ there exists n_0 such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.

(3) (X, d) is called *complete* if every Cauchy sequence in X is a convergent sequence.

Lemma 8 (see [8, Proposition 2.5]). *Let (X, d) be a cone b-metric space, P a normal cone with normal constant K, $x \in X$, and $\{x_n\}$ a sequence in X. Then one has the following.*

(1) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} d(x_n, x) = \theta$.

(2) *The limit point of a convergent sequence is unique.*

(3) *Every convergent sequence is a Cauchy sequence.*

(4) $\{x_n\}$ *is a Cauchy sequence if* $\lim_{n,m\to\infty} d(x_n, x_m) = \theta$.

Lemma 9 (see [8, Remark 2.6]). *Let (X, d) be a cone b-metric space over an ordered real Banach space E with a cone P. Then one has the following.*

(1) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*

(2) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

(3) *If $\theta \leq u \ll c$ for all $c \in \operatorname{int} P$, then $u = \theta$.*

(4) *If $c \in \operatorname{int} P$, $\theta \leq a_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \theta$, then there exists n_0 such that $a_n \ll c$ for all $n \geq n_0$.*

(5) *If $\theta \ll c$, $\theta \leq d(x_n, x) \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = \theta$, then $d(x_n, x) \ll c$ eventually.*

(6) *If $\theta \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, then $a \leq b$.*

(7) *If $a \in P$, $0 \leq \lambda < 1$, and $a \leq \lambda \cdot a$, then $a = \theta$.*

(8) *For each $\alpha > 0$, one has $\alpha \cdot \operatorname{int} P \subset \operatorname{int} P$.*

(9) *For each $\delta > 0$ and $x \in \operatorname{int} P$, there exists $0 < \gamma < 1$ such that $\|\gamma \cdot x\| < \delta$.*

(10) *For each $\theta \ll c_1$ and $c_2 \in P$, there exists $\theta \ll d$ such that $c_1 \ll d$ and $c_2 \ll d$.*

(11) *For each $\theta \ll c_1$ and $\theta \ll c_2$, there exists $\theta \ll e$ such that $e \ll c_1$ and $e \ll c_2$.*

Remark 10 (see [10, Remark 1.3]). Every cone metric space is a cone b-metric space. Moreover, cone b-metric spaces generalize cone metric spaces, b-metric spaces, and metric spaces.

Example 11 (see [10, Example 2.2]). Let

$$E = C^1_{\mathbb{R}}[0, 1], \qquad P = \{\varphi \in E : \varphi \geq 0\}, \qquad X = [1, +\infty), \tag{2}$$

and $d(x, y)(t) = |x - y|^2 e^t$ for all $x, y \in X$ and $t \in [0, 1]$. Then (X, d) is a cone b-metric space with coefficient $s = 2$, but it is not a cone metric space.

Example 12 (see [10, Example 2.3]). Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |u(x)|^2 dx < +\infty$, $E = C_{\mathbb{R}}[0, 1]$, $P = \{\varphi \in E : \varphi \geq 0\}$. Define $d : X \times X \to E$ as

$$d(u(t), v(t)) = e^t \int_0^1 |u(s) - v(s)|^2 ds, \tag{3}$$

for all $u, v \in X$ and $t \in [0, 1]$. Then (X, d) is a cone b-metric space with coefficient $s = 2$, but it is not a cone metric space.

2. Main Results

The following example shows that the family of all balls $B(x, r)$ does not form a base for any topology on a b-metric space (X, d).

Example 13. Let $X = \{0, 1, 1/2, \ldots, 1/n, \ldots\}$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0, 1\} \\ |x - y| & \text{if } x \neq y \in \left\{0, \dfrac{1}{2n}, \dfrac{1}{2m}\right\} \\ 4 & \text{otherwise.} \end{cases} \quad (4)$$

Then we have the following.

(1) d is a b-metric on X with coefficient $s = 8/3$.

(2) $0 \in B(1, 2)$ but $B(0, r) \not\subset B(1, 2)$ for all $r > 0$.

Proof. (1) For all $x, y \in X$, we have $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$.

If $d(x, y) = d(0, 1) = 1$, then

$$d(x,z) + d(z,y) = \begin{cases} d\left(0, \dfrac{1}{2n}\right) + d\left(\dfrac{1}{2n}, 1\right) = \dfrac{1}{2n} + 4 & \text{if } z = \dfrac{1}{2n} \\ d\left(0, \dfrac{1}{2n+1}\right) + d\left(\dfrac{1}{2n+1}, 1\right) & \\ \quad = 4 + 4 & \text{if } z = \dfrac{1}{2n+1}. \end{cases} \quad (5)$$

If $d(x, y) = d(0, 1/2n) = 1/2n$, then

$$d(x,z) + d(z,y) = \begin{cases} d\left(0, \dfrac{1}{2m}\right) + d\left(\dfrac{1}{2m}, \dfrac{1}{2n}\right) & \\ \quad = \dfrac{1}{2m} + \left|\dfrac{1}{2m} - \dfrac{1}{2n}\right| & \text{if } z = \dfrac{1}{2m} \\ d\left(0, \dfrac{1}{2m+1}\right) & \\ \quad + d\left(\dfrac{1}{2m+1}, \dfrac{1}{2n}\right) = 4 + 4 & \text{if } z = \dfrac{1}{2m+1} \neq 1 \\ d(0, 1) + d\left(1, \dfrac{1}{2n}\right) = 1 + 4 & \text{if } z = 1. \end{cases} \quad (6)$$

If $d(x, y) = d(1/2k, 1/2n) = |1/2k - 1/2n|$, then

$$d(x,z) + d(z,y) = \begin{cases} d\left(\dfrac{1}{2k}, \dfrac{1}{2m}\right) + d\left(\dfrac{1}{2m}, \dfrac{1}{2n}\right) & \\ \quad = \left|\dfrac{1}{2k} - \dfrac{1}{2m}\right| + \left|\dfrac{1}{2m} - \dfrac{1}{2n}\right| & \text{if } z = \dfrac{1}{2m} \\ d\left(\dfrac{1}{2k}, \dfrac{1}{2m+1}\right) & \\ \quad + d\left(\dfrac{1}{2m+1}, \dfrac{1}{2n}\right) = 4 + 4 & \text{if } z = \dfrac{1}{2m+1} \\ d\left(\dfrac{1}{2k}, 0\right) + d\left(0, \dfrac{1}{2n}\right) & \\ \quad = \dfrac{1}{2k} + \dfrac{1}{2n} & \text{if } z = 0. \end{cases} \quad (7)$$

If $d(x, y) = d(1/2k, 1/(2n + 1)) = 4$ with $1/(2n + 1) \neq 1$, then

$$d(x,z) + d(z,y) = \begin{cases} d\left(\dfrac{1}{2k}, 0\right) + d\left(0, \dfrac{1}{2n+1}\right) = \dfrac{1}{2k} + 4 & \text{if } z = 0 \\ d\left(\dfrac{1}{2k}, \dfrac{1}{2m}\right) + d\left(\dfrac{1}{2m}, \dfrac{1}{2n+1}\right) & \\ \quad = \left|\dfrac{1}{2k} - \dfrac{1}{2m}\right| + 4 & \text{if } z = \dfrac{1}{2m} \\ d\left(\dfrac{1}{2k}, \dfrac{1}{2m+1}\right) & \\ \quad + d\left(\dfrac{1}{2m+1}, \dfrac{1}{2n+1}\right) = 4 + 4 & \text{if } z = \dfrac{1}{2m+1}. \end{cases} \quad (8)$$

If $d(x, y) = d(1/(2k + 1), 1/(2n + 1)) = 4$ with $1/(2k + 1) \neq 1$ and $1/(2n + 1) \neq 1$, then

$$d(x,z) + d(z,y) = \begin{cases} d\left(\dfrac{1}{2k+1}, 0\right) & \\ \quad + d\left(0, \dfrac{1}{2n+1}\right) = 4 + 4 & \text{if } z = 0 \\ d\left(\dfrac{1}{2k+1}, \dfrac{1}{2m}\right) & \\ \quad + d\left(\dfrac{1}{2m}, \dfrac{1}{2n+1}\right) = 4 + 4 & \text{if } z = \dfrac{1}{2m} \\ d\left(\dfrac{1}{2k+1}, \dfrac{1}{2m+1}\right) & \\ \quad + d\left(\dfrac{1}{2m+1}, \dfrac{1}{2n+1}\right) = 4 + 4 & \text{if } z = \dfrac{1}{2m+1}. \end{cases} \quad (9)$$

If $d(x,y)=d(1/2k,1)=4$, then

$$d(x,z)+d(z,y) = \begin{cases} d\left(\frac{1}{2k},0\right)+d(0,1)=\frac{1}{2k}+1 & \text{if } z=0 \\ d\left(\frac{1}{2k},\frac{1}{2m}\right)+d\left(\frac{1}{2m},1\right) & \\ \quad =\left|\frac{1}{2k}-\frac{1}{2m}\right|+4 & \text{if } z=\frac{1}{2m} \\ d\left(\frac{1}{2k},\frac{1}{2m+1}\right) & \\ \quad +d\left(\frac{1}{2m+1},1\right)=4+4 & \text{if } z=\frac{1}{2m+1}\neq 1. \end{cases} \tag{10}$$

If $d(x,y)=d(1/(2k+1),1)=4$, then

$$d(x,z)+d(z,y) = \begin{cases} d\left(\frac{1}{2k+1},0\right)+d(0,1)=4+1 & \text{if } z=0 \\ d\left(\frac{1}{2k+1},\frac{1}{2m}\right)+d\left(\frac{1}{2m},1\right) & \\ \quad =4+4 & \text{if } z=\frac{1}{2m} \\ d\left(\frac{1}{2k+1},\frac{1}{2m+1}\right) & \\ \quad +d\left(\frac{1}{2m+1},1\right)=4+4 & \text{if } z=\frac{1}{2m+1}\neq 1. \end{cases} \tag{11}$$

If $d(x,y)=d(1/(2k+1),0)=4$, then

$$d(x,z)+d(z,y) = \begin{cases} d\left(\frac{1}{2k+1},1\right)+d(1,0)=4+1 & \text{if } z=1 \\ d\left(\frac{1}{2k+1},\frac{1}{2m}\right)+d\left(\frac{1}{2m},0\right) & \\ \quad =4+\frac{1}{2m} & \text{if } z=\frac{1}{2m} \\ d\left(\frac{1}{2k+1},\frac{1}{2m+1}\right) & \\ \quad +d\left(\frac{1}{2m+1},0\right)=4+4 & \text{if } z=\frac{1}{2m+1}\neq 1. \end{cases} \tag{12}$$

By the previous calculations, we get $d(x,y)\leq(8/3)[d(x,z)+d(z,y)]$ for all $x,y,z\in X$. This proves that d is a b-metric on X with $s=8/3$.

(2) We have $B(1,2)=\{x\in X: d(x,1)<2\}=\{1,0\}$. Then $0\in B(1,2)$.

For each $r>0$, since $d(0,1/2n)=1/2n$, we have $1/2n\in B(0,r)$ for n being large enough. Note that $d(1,1/2n)=4$, so $1/2n\notin B(1,2)$ for all $n\in\mathbb{N}$. This proves that $B(0,r)\not\subset B(1,2)$. □

We introduce a b-metric on the cone b-metric space and then prove some equivalences between them as follows.

Theorem 14. *Let (X,d) be a cone b-metric space with coefficient s and*

$$D(x,y)=\inf\left\{\|u\|: u\in P,\ u\geq\frac{1}{s}d(x,y)\right\}, \tag{13}$$

for all $x,y\in X$. Then one has the following.

(1) *D is a b-metric on X.*

(2) *$\lim_{n\to\infty}x_n=x$ in the cone b-metric space (X,d) if and only if $\lim_{n\to\infty}x_n=x$ in the b-metric space (X,D).*

(3) *$\{x_n\}$ is a Cauchy sequence in the cone b-metric space (X,d) if and only if $\{x_n\}$ is a Cauchy sequence in the b-metric space (X,D).*

(4) *The cone b-metric space (X,d) is complete if and only if the b-metric space (X,D) is complete.*

Proof. (1) For all $x,y\in X$, it is obvious that $D(x,y)\geq 0$ and $D(x,y)=D(y,x)$.

If $x=y$, then $D(x,y)=\inf\{\|u\|: u\in P,\ u\geq\theta\}=0$.

If $D(x,y)=\inf\{\|u\|: u\in P,\ u\geq(1/s)d(x,y)\}=0$, then, for each $n\in\mathbb{N}$, there exists $u_n\in P$ such that $u_n\geq(1/s)d(x,y)$ and $\|u_n\|<1/n$. Then $\lim_{n\to\infty}u_n=\theta$, and by Lemma 9(6), we have $d(x,y)\leq\theta$. It implies that $d(x,y)\in P\cap(-P)$. Therefore, $d(x,y)=\theta$; that is, $x=y$.

For each $x,y,z\in X$, we have

$$\begin{gathered} D(x,z)=\inf\left\{\|u_1\|: u_1\in P,\ u_1\geq\frac{1}{s}d(x,z)\right\}, \\ D(x,y)=\inf\left\{\|u_2\|: u_2\in P,\ u_2\geq\frac{1}{s}d(x,y)\right\}, \\ D(y,z)=\inf\left\{\|u_3\|: u_3\in P,\ u_3\geq\frac{1}{s}d(y,z)\right\}. \end{gathered} \tag{14}$$

Since $u_2,u_3\in P$ and $u_2\geq(1/s)d(x,y)$, $u_3\geq(1/s)d(y,z)$, we have

$$s(u_2+u_3)\geq d(x,y)+d(y,z)\geq\frac{1}{s}d(x,z). \tag{15}$$

Then we have

$$\begin{aligned} &\left\{u_1\in P: u_1\geq\frac{1}{s}d(x,z)\right\} \\ &\quad\supset\left\{s(u_2+u_3)\in P:\right. \\ &\qquad\left. u_2\geq\frac{1}{s}d(x,y),\ u_3\frac{1}{s}\geq d(y,z)\right\}. \end{aligned} \tag{16}$$

It implies that

$$\inf\left\{\|s(u_2+u_3)\| : u_2,u_3\in P,\ u_2\geq\frac{1}{s}d(x,y),\ u_3\geq\frac{1}{s}d(y,z)\right\}\geq\inf\left\{\|u_1\| : u_1\in P,\ u_1\geq\frac{1}{s}d(x,z)\right\}. \tag{17}$$

Now, we have

$$\begin{aligned}D(x,z)&=\inf\left\{\|u_1\| : u_1\in P,\ u_1\geq\frac{1}{s}d(x,z)\right\}\\ &\leq\inf\left\{\|s(u_2+u_3)\| : u_2,u_3\in P,\ u_2\geq\frac{1}{s}d(x,y),\ u_3\geq\frac{1}{s}d(y,z)\right\}\\ &=s\inf\left\{\|u_2+u_3\| : u_2,u_3\in P,\ u_2\geq\frac{1}{s}d(x,y),\ u_3\geq\frac{1}{s}d(y,z)\right\}\\ &\leq s\inf\left\{\|u_2\|+\|u_3\| : u_2,u_3\in P,\ u_2\geq\frac{1}{s}d(x,y),\ u_3\geq\frac{1}{s}d(y,z)\right\}\\ &=s\inf\left\{\|u_2\| : u_2\in P,\ u_2\geq\frac{1}{s}d(x,y)\right\}+s\inf\left\{\|u_3\| : u_3\in P,\ u_3\geq\frac{1}{s}d(y,z)\right\}\\ &=s\left[D(x,y)+D(y,z)\right].\end{aligned} \tag{18}$$

By the previously metioned, D is a b-metric on X.

(2) *Necessity.* Let $\lim_{n\to\infty}x_n = x$ in the cone b-metric space (X,d). For each $\varepsilon>0$, by Lemma 9(8), if $\theta\ll c$, then $\theta\ll s\cdot\varepsilon\cdot(c/\|c\|)$. Then, for each $c\in E$ with $\theta\ll c$, there exists n_0 such that $d(x_n,x)\ll s\cdot\varepsilon\cdot(c/\|c\|)$ for all $n\geq n_0$. Using Lemma 9(8) again, we get $(1/s)d(x_n,x)\ll\varepsilon\cdot(c/\|c\|)$. It implies that

$$\begin{aligned}D(x_n,x)&=\inf\left\{\|u\| : u\in P,\ u\geq\frac{1}{s}d(x_n,x)\right\}\\ &\leq\varepsilon\cdot\left\|\frac{c}{\|c\|}\right\|=\varepsilon,\end{aligned} \tag{19}$$

for all $n\geq n_0$. This proves that $\lim_{n\to\infty}D(x_n,x)=0$; that is, $\lim_{n\to\infty}x_n=x$ in the b-metric space (X,D).

Sufficiency. Let $\lim_{n\to\infty}x_n = x$ in the b-metric space (X,D). For each $\theta\ll c$, there exists $\varepsilon>0$ such that $c+B(0,\varepsilon)\subset P$. For this ε, there exists n_0 such that

$$D(x_n,x)=\inf\left\{\|u\| : u\in P,\ u\geq\frac{1}{s}d(x_n,x)\right\}\leq\frac{\varepsilon}{4}. \tag{20}$$

Then, there exist $v\in P$ and $d(x_n,x)\leq v$ such that $\|v\|\leq\varepsilon/2$. So $-v\in B(0,\varepsilon)$, and we have $c-v\in\operatorname{int}P$. Therefore, $d(x_n,x)\leq v\ll c$ for all $n\geq n_0$. By Lemma 9(1), we get $d(x_n,x)\ll c$ for all $n\geq n_0$. This proves that $\lim_{n\to\infty}x_n=x$ in the cone b-metric space (X,d).

(3) *Necessity.* Let $\{x_n\}$ be a Cauchy sequence in the cone b-metric space (X,d). For each $\varepsilon>0$, by Lemma 9(6), if $\theta\ll c$, then $\theta\ll s\cdot\varepsilon\cdot(c/\|c\|)$. Then for each $c\in E$ with $\theta\ll c$, there exists n_0 such that $d(x_n,x_m)\ll s\cdot\varepsilon\cdot(c/\|c\|)$ for all $n,m\geq n_0$. Using Lemma 9(6) again, we get $(1/s)d(x_n,x_m)\ll\varepsilon\cdot(c/\|c\|)$. It implies that

$$\begin{aligned}D(x_n,x_m)&=\inf\left\{\|u\| : u\in P,\ u\geq\frac{1}{s}d(x_n,x_m)\right\}\\ &\leq\varepsilon\cdot\left\|\frac{c}{\|c\|}\right\|=\varepsilon,\end{aligned} \tag{21}$$

for all $n,m\geq n_0$. This proves that $\{x_n\}$ is a Cauchy sequence in the b-metric space (X,D).

Sufficiency. Let $\{x_n\}$ be a Cauchy sequence in the b-metric space (X,D). Then $\lim_{n,m\to\infty}D(x_n,x_m)=0$. For each $\theta\ll c$, there exists $\varepsilon>0$ such that $c+B(0,\varepsilon)\subset P$. For this ε, there exists n_0 such that

$$D(x_n,x_m)=\inf\left\{\|u\| : u\in P,\ u\geq\frac{1}{s}d(x_n,x_m)\right\}\leq\frac{\varepsilon}{4}, \tag{22}$$

for all $n,m\geq n_0$. Then, there exists $v\in P$, $d(x_n,x_m)\leq v$ such that $\|v\|\leq\varepsilon/2$. So $-v\in B(0,\varepsilon)$, and we have $c-v\in\operatorname{int}P$. Therefore, $d(x_n,x_m)\leq v\ll c$ for all $n,m\geq n_0$. By Lemma 9(1), we get $d(x_n,x_m)\ll c$ for all $n,m\geq n_0$. This proves that $\{x_n\}$ is a Cauchy sequence in the cone b-metric space (X,d).

(4) It is a direct consequence of (2) and (3). □

By choosing $s=1$ in Theorem 14, we get the following results.

Corollary 15 (see [13, Lemma 2.1]). *Let (X,d) be a cone metric space. Then*

$$D(x,y)=\inf\left\{\|u\| : u\in P,\ u\geq d(x,y)\right\}, \tag{23}$$

for all $x,y\in X$ is a metric on X.

Corollary 16 (see [10, Theorem 2.2]). *Let (X,d) be a cone metric space and*

$$D(x,y)=\inf\left\{\|u\| : u\in P,\ u\geq d(x,y)\right\}, \tag{24}$$

for all $x,y\in X$. Then the metric space (X,D) is complete if and only if the cone metric space (X,d) is complete.

The following examples show that Corollaries 15 and 16 are not applicable to cone b-metric spaces in general.

Example 17. Let (X, d) be a cone b-metric space as in Example 11. We have

$$\begin{aligned} D(x,y) &= \inf\{\|u\| : u \in P,\ u \geq d(x,y)\} \\ &= \|d(x,y)\| = \sup\left\{|x-y|^2 e^t : t \in [0,1]\right\} \\ &= e|x-y|^2. \end{aligned} \tag{25}$$

It implies that

$$D(0,2) = 4e > D(0,1) + D(1,2) = e + e = 2e. \tag{26}$$

Then D is not a metric on X. This proves that Corollaries 15 and 16 are not applicable to given cone b-metric space (X, d).

Example 18. Let (X, d) be a cone b-metric space as in Example 12. We have

$$\begin{aligned} D(u,v) &= \inf\{\|z\| : z \in P,\ z \geq d(u,v)\} \\ &= \|d(u,v)\| \\ &= \sup\left\{e^t \int_0^1 |u(s) - v(s)|^2 ds : t \in [0,1]\right\} \\ &= e \int_0^1 |u(s) - v(s)|^2 ds. \end{aligned} \tag{27}$$

For $u(s) = 0$, $v(s) = 1$, and $w(s) = 2$ for all $s \in [0, 1]$, we have

$$D(u,w) = 4e > D(u,v) + D(v,w) = e + e = 2e. \tag{28}$$

Then D is not a metric on X. This proves that Corollaries 15 and 16 are not applicable to given cone b-metric space (X, d).

Next, by using Theorem 14, we show that some contraction conditions on cone b-metric spaces can be obtained from certain contraction conditions on b-metric spaces.

Corollary 19. *Let (X, d) be a cone b-metric space with coefficient s, let $T : X \to X$ be a map, and let D be defined as in Theorem 14. Then the following statements hold.*

(1) *If $d(Tx, Ty) \leq kd(x, y)$ for some $k \in [0, 1)$ and all $x, y \in X$, then*

$$D(Tx,Ty) \leq kD(x,y), \tag{29}$$

for all $x, y \in X$.

(2) *If $d(Tx,Ty) \leq \lambda_1 d(x,Tx) + \lambda_2 d(y,Ty) + \lambda_3 d(x,Ty) + \lambda_4 d(y,Tx)$ for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ with $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, 2/s\}$ and all $x, y \in X$, then*

$$\begin{aligned} D(Tx,Ty) \leq{}& \lambda_1 D(x,Tx) + \lambda_2 D(y,Ty) \\ &+ \lambda_3 D(x,Ty) + \lambda_4 D(y,Tx), \end{aligned} \tag{30}$$

for all $x, y \in X$.

Proof. (1) For each $x, y \in X$ and $v \in P$ with $v \geq (1/s)d(x, y)$, it follows from Lemma 9(8) that

$$kv \geq k\frac{1}{s}d(x,y) \geq \frac{1}{s}d(Tx,Ty). \tag{31}$$

Thus, $\{kv : v \in P,\ v \geq (1/s)d(x, y)\} \subset \{u : u \in P,\ u \geq (1/s)d(Tx, Ty)\}$. Then we have

$$\begin{aligned} D(Tx,Ty) &= \inf\left\{\|u\| : u \in P,\ u \geq \frac{1}{s}d(Tx,Ty)\right\} \\ &\leq \inf\left\{\|kv\| : v \in P,\ v \geq \frac{1}{s}d(x,y)\right\} \\ &= k\inf\left\{\|v\| : v \in P,\ v \geq \frac{1}{s}d(x,y)\right\} \\ &= kD(x,y). \end{aligned} \tag{32}$$

It implies that $D(Tx, Ty) \leq kD(x, y)$.

(2) Let $x, y \in X$ and $v_1, v_2, v_3, v_4 \in P$ satisfy

$$\begin{aligned} v_1 &\geq \frac{1}{s}d(x,Tx), & v_2 &\geq \frac{1}{s}d(y,Ty), \\ v_3 &\geq \frac{1}{s}d(x,Ty), & v_4 &\geq \frac{1}{s}d(y,Tx). \end{aligned} \tag{33}$$

From Lemma 9(8), we have

$$\begin{aligned} &\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 \\ &\quad\geq \frac{1}{s}\left[\lambda_1 d(x,Tx) + \lambda_2 d(y,Ty) + \lambda_3 d(x,Ty) + \lambda_4 d(y,Tx)\right] \\ &\quad\geq \frac{1}{s}d(Tx,Ty). \end{aligned} \tag{34}$$

It implies that

$$\begin{aligned} &\left\{v : v \in P,\ v \geq \frac{1}{s}d(Tx,Ty)\right\} \\ &\supset \Big\{\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 : \\ &\qquad v_1, v_2, v_3, v_4 \in P,\ v_1 \geq \frac{1}{s}d(x,Tx), \\ &\qquad v_2 \geq \frac{1}{s}d(y,Ty),\ v_3 \geq \frac{1}{s}d(x,Ty), \\ &\qquad v_4 \geq \frac{1}{s}d(y,Tx)\Big\}. \end{aligned} \tag{35}$$

Then we have

$$\begin{aligned} &D(Tx,Ty) \\ &\quad= \inf\left\{\|v\| : v \in P, v \geq \frac{1}{s}d(Tx,Ty)\right\} \\ &\quad\leq \inf\Big\{\|\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4\| : \\ &\qquad\quad v_1, v_2, v_3, v_4 \in P,\ v_1 \geq \frac{1}{s}d(x,Tx), \\ &\qquad\quad v_2 \geq \frac{1}{s}d(y,Ty),\ v_3 \geq \frac{1}{s}d(x,Ty), \\ &\qquad\quad v_4 \geq \frac{1}{s}d(y,Tx)\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \inf \Big\{\lambda_1 \|v_1\| + \lambda_2 \|v_2\| \\
&\qquad + \lambda_3 \|v_3\| + \lambda_4 \|v_4\| : \\
&\qquad v_1, v_2, v_3, v_4 \in P, \\
&\qquad v_1 \ge \frac{1}{s} d(x, Tx),\ v_2 \ge \frac{1}{s} d(y, Ty), \\
&\qquad v_3 \ge \frac{1}{s} d(x, Ty),\ v_4 \ge \frac{1}{s} d(y, Tx)\Big\} \\
&= \inf \Big\{\lambda_1 \|v_1\| : v_1 \in P,\ v_1 \ge \frac{1}{s} d(x, Tx)\Big\} \\
&\quad + \inf \Big\{\lambda_2 \|v_2\| : v_2 \in P,\ v_2 \ge \frac{1}{s} d(y, Ty)\Big\} \\
&\quad + \inf \Big\{\lambda_3 \|v_3\| : v_3 \in P,\ v_3 \ge \frac{1}{s} d(x, Ty)\Big\} \\
&\quad + \inf \Big\{\lambda_4 \|v_4\| : v_4 \in P,\ v_4 \ge \frac{1}{s} d(y, Tx)\Big\} \\
&= \lambda_1 \inf \Big\{\|v_1\| : v_1 \in P,\ v_1 \ge \frac{1}{s} d(x, Tx)\Big\} \\
&\quad + \lambda_2 \inf \Big\{\|v_2\| : v_2 \in P,\ v_2 \ge \frac{1}{s} d(y, Ty)\Big\} \\
&\quad + \lambda_3 \cdot \inf \Big\{\|v_3\| : v_3 \in P,\ v_3 \ge \frac{1}{s} d(x, Ty)\Big\} \\
&\quad + \lambda_4 \cdot \inf \Big\{\|v_4\| : v_4 \in P,\ v_4 \ge \frac{1}{s} d(y, Tx)\Big\} \\
&= \lambda_1 D(x, Tx) + \lambda_2 D(y, Ty) \\
&\quad + \lambda_3 D(x, Ty) + \lambda_4 D(y, Tx).
\end{aligned}
\tag{36}
$$

This proves that $D(Tx, Ty) \le \lambda_1 D(x, Tx) + \lambda_2 D(y, Ty) + \lambda_3 D(x, Ty) + \lambda_4 D(y, Tx)$. □

Now, we show that main results in [9] are consequences of preceding results on b-metric spaces.

Corollary 20. *Let (X, d) be a complete cone b-metric space with coefficient s, and let $T : X \to X$ be a map. Then the following statements hold.*

(1) *(see [9, Theorem 2.1]) If $d(Tx, Ty) \le kd(x, y)$ for all $x, y \in X$, then T has a unique fixed point.*

(2) *(see [9, Theorem 2.3]) If $d(Tx, Ty) \le \lambda_1 d(x, Tx) + \lambda_2 d(y, Ty) + \lambda_3 d(x, Ty) + \lambda_4 d(y, Tx)$ for some $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ with $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, 2/s\}$ and all $x, y \in X$, then T has a unique fixed point.*

Proof. Let D be defined as in Theorem 14. It follows from Theorem 14(4) that (X, D) is a complete b-metric space.

(1) By Corollary 19(1), we see that T satisfies all assumptions of [5, Theorem 2]. Then T has a unique fixed point.

(2) By Corollary 19(2), we see that T satisfies all assumptions in [6, Theorem 3.7], where $K = s$, $f = T$, g is the identity, and $a_1 = 0$, $a_2 = \lambda_1$, $a_3 = \lambda_2$, and $a_4 = \lambda_3$, $a_5 = \lambda_4$. Note that condition (3.10) in [6, Theorem 3.7] was used to prove (3.16) and $K(a_2+a_3+a_4+a_5) < 2$ at line 3, page 7 in the proof of [6, Theorem 3.7]. These claims also hold if $a_1 = 0$ and $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, 2/s\}$. Then T has a unique fixed point.

□

Remark 21. By similar arguments as in Corollaries 19 and 20, we may get fixed point theorems on cone b-metric spaces in [8, 10] from preceding ones on b-metric spaces in [3, 5].

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