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Research Article

Coefficient Bounds for Certain Subclasses of Bi-Univalent Function

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We introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Also consequences of the results are pointed out.

1. Introduction

Denote by $\mathscr{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, denote by $\mathscr{S}$ the class of all functions in $\mathscr{A}$ which are univalent and normalized by $f(0) = 0 = f'(0)$ in $\mathbb{U}$. The well-investigated subclasses of the univalent function class $\mathscr{S}$ are the class of starlike functions of order α $(0 \le \alpha < 1)$, denoted by $\mathscr{S}^*(\alpha)$ and the class of convex functions of order α denoted by $\mathscr{K}(\alpha)$ in $\mathbb{U}$. It is well known that every function $f \in \mathscr{S}$ has an inverse f^{-1}, defined by

$$\begin{gathered} f^{-1}(f(z)) = z, \quad z \in \mathbb{U}, \\ f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \ge \frac{1}{4}, \end{gathered} \tag{2}$$

where

$$\begin{aligned} f^{-1}(w) = w - a_2 w^2 &+ \left(2a_2^2 - a_3\right) w^3 \\ &- \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \end{aligned} \tag{3}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Analogous to the function class $\mathscr{S}$, the bi-univalent function class Σ include, for example, the class $\mathscr{S}_{\Sigma}^*(\alpha)$ of bi-starlike functions of order α $(0 \le \alpha < 1)$, the class $\mathscr{K}_{\Sigma}(\alpha)$ of biconvex functions of order α $(0 \le \alpha < 1)$, and the class $\mathscr{S}_{\Sigma}^{\alpha}$ of strongly bi-starlike functions of order α $(0 < \alpha \le 1)$. For some intriguing examples of functions and characterization of the class Σ, one could refer to Srivastava et al. [1] and the references stated therein (see also [2]). Recently there has been triggering, interest to study the bi-univalent functions class Σ (see [2–5]) and obtain nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$ is presumably still an open problem.

Motivated by the earlier works of Srivastava et al. [1] and Frasin and Aouf [3] in the present paper we introduce the following two new subclasses of the function class Σ.

Definition 1. A function $f(z)$ given by (1) is said to be in the class $\mathscr{G}_{\Sigma}(\alpha, \lambda)$ if the following conditions are satisfied:

$$\begin{gathered} f \in \Sigma, \quad \left| \arg \left(\frac{z f'(z)}{(1-\lambda) f(z) + \lambda z f'(z)} \right) \right| < \frac{\alpha \pi}{2}, \\ 0 < \alpha \le 1, \ 0 \le \lambda < 1, \ z \in \mathbb{U}, \end{gathered}$$

$$\left|\arg\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right)\right|<\frac{\alpha\pi}{2},$$
$$0<\alpha\leq 1,\ 0\leq\lambda<1,\ w\in\mathbb{U}, \tag{4}$$

where the function g is given by

$$g(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots. \tag{5}$$

Definition 2. A function $f(z)$ given by (1) is said to be in the class $\mathscr{M}_\Sigma(\beta,\lambda)$ if the following conditions are satisfied:

$$f\in\Sigma,\quad \Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right)>\beta,$$
$$0\leq\beta<1,\ 0\leq\lambda<1,\ z\in\mathbb{U},$$
$$\Re\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right)>\beta,$$
$$0\leq\beta<1,\ 0\leq\lambda<1,\ w\in\mathbb{U}, \tag{6}$$

where the function g is given by (5).

It is of interest to note that, for $\lambda=0$, the class $\mathscr{G}_\Sigma(\alpha,\lambda)$ reduces to $\mathscr{S}_\Sigma^\alpha$ of strongly bi-starlike functions of order α and the class $\mathscr{M}_\Sigma(\beta,\lambda)$ leads to $\mathscr{S}_\Sigma^*(\beta)$ bi-starlike functions of order β.

The object of the present paper is to find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathscr{G}_\Sigma(\alpha,\lambda)$ and $\mathscr{M}_\Sigma(\beta,\lambda)$ of the function class Σ by employing the techniques used earlier by Srivastava et al. [1].

In order to derive our main results, we recall the following lemma.

Lemma 3 (see [6]). *If $h\in\mathscr{P}$, then $|c_k|\leq 2$ for each k, where $\mathscr{P}$, is the family of all functions h analytic in $\mathbb{U}$ for which $\Re\{h(z)\}>0$, where $h(z)=1+c_1z+c_2z^2+\cdots$ for $z\in\mathbb{U}$.*

2. Coefficient Bounds for the Function Class $\mathscr{G}_\Sigma(\alpha,\lambda)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $f\in\mathscr{G}_\Sigma(\alpha,\lambda)$.

Theorem 4. *Let $f(z)$ given by* (1) *be in the class $\mathscr{G}_\Sigma(\alpha,\lambda)$, $0<\alpha\leq 1$, and $0\leq\lambda<1$. Then*

$$|a_2|\leq\frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}}, \tag{7}$$

$$|a_3|\leq\frac{4\alpha^2}{(1-\lambda)^2}+\frac{\alpha}{1-\lambda}. \tag{8}$$

Proof. It follows from (4) that

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}=[p(z)]^\alpha,$$
$$\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}=[q(w)]^\alpha, \tag{9}$$

where $p(z)$ and $q(w)$ in $\mathscr{P}$ have the forms

$$p(z)=1+p_1z+p_2z^2+\cdots, \tag{10}$$

$$q(z)=1+q_1w+q_2w^2+\cdots. \tag{11}$$

Now, equating the coefficients in (9), we get

$$(1-\lambda)a_2=\alpha p_1, \tag{12}$$

$$\left(\lambda^2-1\right)a_2^2+2(1-\lambda)a_3=\frac{1}{2}\left[\alpha(\alpha-1)p_1^2+2\alpha p_2\right], \tag{13}$$

$$-(1-\lambda)a_2=\alpha q_1, \tag{14}$$

$$\left(\lambda^2-4\lambda+3\right)a_2^2-2(1-\lambda)a_3=\frac{1}{2}\left[\alpha(\alpha-1)q_1^2+2\alpha q_2\right]. \tag{15}$$

From (12) and (14), we get

$$p_1=-q_1, \tag{16}$$

$$2(1-\lambda)^2a_2^2=\alpha^2\left(p_1^2+q_1^2\right). \tag{17}$$

From (13), (15), and (17), we obtain

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{(\alpha+1)(1-\lambda)^2}. \tag{18}$$

Applying Lemma 3 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\leq\frac{2\alpha}{(1-\lambda)\sqrt{(1+\alpha)}}. \tag{19}$$

This gives the bound on $|a_2|$ as asserted in (7).

Next, in order to find the bound on $|a_3|$, by subtracting (15) from (13), we get

$$4(1-\lambda)a_3-4(1-\lambda)a_2^2=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right). \tag{20}$$

It follows from (16), (17), and (20) that

$$a_3=\frac{\alpha(p_2-q_2)}{4(1-\lambda)}+\frac{\alpha^2\left(p_1^2+q_1^2\right)}{2(1-\lambda)^2}. \tag{21}$$

Applying Lemma 3 once again for the coefficients p_1, p_2, q_1, and q_2, we readily get

$$|a_3| \leq \frac{4\alpha^2}{(1-\lambda)^2} + \frac{\alpha}{1-\lambda}. \tag{22}$$

This completes the proof of Theorem 4. □

In the following section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathscr{M}_\Sigma(\beta, \lambda)$.

3. Coefficient Bounds for the Function Class $\mathscr{M}_\Sigma(\beta,\lambda)$

Theorem 5. *Let $f(z)$ given by* (1) *be in the class $\mathscr{M}_\Sigma(\beta, \lambda)$, $0 \leq \beta < 1$, and $0 \leq \lambda < 1$. Then*

$$\begin{gathered} |a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\lambda}, \\ |a_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2} + \frac{(1-\beta)}{1-\lambda}. \end{gathered} \tag{23}$$

Proof. It follows from (6) that there exists $p, q \in \mathscr{P}$ such that

$$\begin{gathered} \frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)} = \beta + (1-\beta) p(z), \\ \frac{wg'(w)}{(1-\lambda) g(w) + \lambda w g'(w)} = \beta + (1-\beta) q(w), \end{gathered} \tag{24}$$

where $p(z)$ and $q(w)$ have the forms of (10) and (11), respectively. Equating coefficients in (24) we get

$$\begin{gathered} (1-\lambda) a_2 = (1-\beta) p_1, \\ (\lambda^2 - 1) a_2^2 + 2(1-\lambda) a_3 = (1-\beta) p_2, \\ -(1-\lambda) a_2 = (1-\beta) q_1, \\ (\lambda^2 - 4\lambda + 3) a_2^2 - 2(1-\lambda) a_3 = (1-\beta) q_2. \end{gathered} \tag{25}$$

The proof follows, by employing the techniques used in the proof of Theorem 4. □

Taking $\lambda = 0$ in Theorems 4 and 5 one can get the following corollaries.

Corollary 6. *Let $f(z)$ given by* (1) *be in the class $\mathscr{S}_\Sigma^\alpha$ and $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}}, \qquad |a_3| \leq 4\alpha^2 + \alpha. \tag{26}$$

Corollary 7. *Let $f(z)$ given by* (1) *be in the class $\mathscr{S}_\Sigma^*(\beta)$ and $0 \leq \beta < 1$. Then*

$$|a_2| \leq \sqrt{2-2\beta}, \qquad |a_3| \leq 4(1-\beta)^2 + (1-\beta). \tag{27}$$

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