Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 547682, 10 pages
http://dx.doi.org/10.1155/2013/547682

Research Article

Nonexistence of Homoclinic Orbits for a Class of Hamiltonian Systems

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Received 31 March 2013; Accepted 22 July 2013

Academic Editor: Marco Squassina

We give several sufficient conditions under which the first-order nonlinear Hamiltonian system $x'(t) = \alpha(t)x(t)+f(t, y(t))$, $y'(t) = -g(t, x(t))-\alpha(t)y(t)$ has no solution $(x(t), y(t))$ satisfying condition $0 < \int_{-\infty}^{+\infty}[|x(t)|^{\nu}+(1+\beta(t))|y(t)|^{\mu}]dt < +\infty$, where $\mu, \nu > 1$ and $(1/\mu) + (1/\nu) = 1$, $0 \le xf(t, x) \le \beta(t)|x|^{\mu}$, $xg(t, x) \le \gamma_0(t)|x|^{\nu}$, $\beta(t), \gamma_0(t) \ge 0$, and $\alpha(t)$ are locally Lebesgue integrable real-valued functions defined on $\mathbb{R}$.

1. Introduction

In 1897, Poincaré [1] studied the existence of homoclinic solutions for Hamiltonian systems and realized that homoclinic solutions play a very important role in the study of the behavior of dynamical systems. Since then many methods have been developed to this study ([2–6]). Recently, the critical point theory has been successfully applied to establish the existence and multiplicity of homoclinic solutions for Hamiltonian systems; see [1, 7–20] and references therein.

Among the above-mentioned literature, there are two classes of Hamiltonian systems that have been widely investigated: one is the second-order Hamiltonian system

$$\left(p(t)u'(t)\right)' - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1}$$

and the other is the first-order Hamiltonian system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} H_x(t, x(t), y(t)) \\ H_y(t, x(t), y(t)) \end{pmatrix}. \tag{2}$$

By means of variational methods, in order to seek the homoclinic solutions for system (1), one usually defines a functional $\varphi(u)$ on the Banach space

$$E = \left\{u \in W^{1,2}(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \left[a_1(t)\left|u'(t)\right|^2 + a_2(t)|u(t)|^2\right] dt < +\infty\right\}, \tag{3}$$

where $a_1, a_2 \in C(\mathbb{R}, (0, +\infty))$ associated with the coefficients $p(t)$ and $L(t)$ of system (1). And then one proves that φ possesses critical points on E which are homoclinic solutions of system (1). Thus, the nontrivial homoclinic solutions of system (1) which were studied in the existing work are actually a class of special solutions satisfying condition

$$0 < \int_{\mathbb{R}} \left[(1 + a_1(t))\left|u'(t)\right|^2 + (1 + a_2(t))|u(t)|^2\right] dt < +\infty. \tag{4}$$

Similarly, the non-trivial homoclinic solutions of system (2) which were studied in the literature are also a class of special solutions satisfying condition

$$0 < \int_{\mathbb{R}} \left[(1 + b_1(t))|x(t)|^2 + (1 + b_2(t))\left|y(t)\right|^2\right] dt < +\infty, \tag{5}$$

where $b_1, b_2 \in C(\mathbb{R}, (0, +\infty))$ associated with the potential H of system (2).

As mentioned earlier, the existence and multiplicity of homoclinic solutions for Hamiltonian systems have been studied extensively via critical point theory in recent years; various sufficient conditions for existence are established. However, as we know, there are no results on nonexistence of homoclinic solutions for Hamiltonian systems in the literature. For the simplest second-order Hamiltonian system,

$$x''(t) + q(t)x(t) = 0 \tag{6}$$

has no non-trivial homoclinic solutions as $q(t) \equiv$ constant, but when $q(t) \not\equiv$ constant, there seem to be no results on existence or non-existence of homoclinic solutions in the literature either.

In this paper, we consider the first-order nonlinear Hamiltonian system

$$\begin{aligned} x'(t) &= \alpha(t)x(t) + f(t, y(t)), \\ y'(t) &= -g(t, x(t)) - \alpha(t)y(t), \end{aligned} \tag{7}$$

where $\alpha(t)$ is locally Lebesgue integrable real-valued function defined on $\mathbb{R}$, $f, g : \mathbb{R}^2 \to \mathbb{R}$. For every $t \in \mathbb{R}$, $f(t,x)$ and $g(t,x)$ are continuous on x in $\mathbb{R}$, and for every $x \in \mathbb{R}$, $f(t,x)$ and $g(t,x)$ are locally Lebesgue integrable real-valued functions on t.

For the sake of convenience, we give the following assumptions on f and g.

(F) For any $c \neq 0$, $\text{meas}\{t \in \mathbb{R} : f(t, c\exp(-\int_0^t \alpha(s)ds)) \neq 0\} > 0$, and there exist a constant $\mu > 1$ and a locally Lebesgue integrable nonnegative function $\beta(t)$ defined on $\mathbb{R}$ such that

$$0 \le xf(t,x) \le \beta(t)|x|^{\mu}, \quad \forall (t,x) \in \mathbb{R}^2. \tag{8}$$

(G) $g(t,0) = 0$ for $t \in \mathbb{R}$, and there exists a locally Lebesgue integrable nonnegative function $\gamma_0(t)$ defined on $\mathbb{R}$ such that

$$xg(t,x) \le \gamma_0(t)|x|^{\nu}, \quad \forall (t,x) \in \mathbb{R}^2, \tag{9}$$

where $\nu > 1$ and $(1/\mu) + (1/\nu) = 1$.

Remark 1. In case $g(t,x) = \gamma(t)\varphi(x)$, where $\gamma(t)$ is locally Lebesgue integrable real-valued function defined on $\mathbb{R}$, $\varphi \in C(\mathbb{R}, \mathbb{R})$, and satisfies that

$$0 \le x\varphi(x) \le |x|^{\nu}, \quad \forall (t,x) \in \mathbb{R}^2, \tag{10}$$

then we can choose $\gamma_0(t) = \gamma^+(t) = \max\{\gamma(t), 0\}$.

Let $u(t) = (x(t), y(t))^{\top}$, $H(t,x,y) = \int_0^x g(t,s)ds + \alpha(t)xy + \int_0^y f(t,s)ds$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

Then we can rewrite (7) as a standard first-order Hamiltonian system

$$u'(t) = J\nabla H(t, u(t)). \tag{12}$$

There are two special forms of system (7) which have been dealt with extensively in the literature: one is the first-order linear Hamiltonian system

$$\begin{aligned} x'(t) &= \alpha(t)x(t) + \beta(t)y(t), \\ y'(t) &= -\gamma(t)x(t) - \alpha(t)y(t) \end{aligned} \tag{13}$$

and the other is the first-order quasilinear Hamiltonian system

$$\begin{aligned} x'(t) &= \alpha(t)x(t) + \beta(t)|y(t)|^{\mu-2}y(t), \\ y'(t) &= -\gamma(t)|x(t)|^{\nu-2}x(t) - \alpha(t)y(t), \end{aligned} \tag{14}$$

(see [21, 22] and the references therein), where $\mu, \nu > 1$ and $(1/\mu) + (1/\nu) = 1$, and $\beta(t)$ and $\gamma(t)$ are locally Lebesgue integrable real-valued functions defined on $\mathbb{R}$. In addition, the special forms of system (7) also contain many other well-known second-order differential equations such as the second-order linear differential equation

$$\left(p(t)x'(t)\right)' + q(t)x(t) = 0, \tag{15}$$

the second-order half-linear differential equation

$$\left[p(t)\left|x'(t)\right|^{r-2}x'(t)\right]' + q(t)|x(t)|^{r-2}x(t) = 0, \tag{16}$$

and the second-order nonlinear differential equation

$$\left[p(t)\phi\left(x'(t)\right)\right]' + h(t, x(t)) = 0, \tag{17}$$

where $r > 1$, $p(t)$ and $q(t)$ are locally Lebesgue integrable real-valued functions defined on $\mathbb{R}$ and $p(t) > 0$, $\phi \in C(\mathbb{R}, \mathbb{R})$, and $h \in C(\mathbb{R}^2, \mathbb{R})$. Indeed, we can rewrite the above-mentioned second-order differential equations as the form of system (7). For example, let

$$y(t) = p(t)\left|x'(t)\right|^{r-2}x'(t). \tag{18}$$

Then (16) can be written as the form of (13):

$$\begin{aligned} x'(t) &= [p(t)]^{1/(1-r)}|y(t)|^{(2-r)/(r-1)}y(t), \\ y'(t) &= -q(t)|x(t)|^{r-2}x(t), \end{aligned} \tag{19}$$

where $\mu = r/(r-1)$, $\nu = r$ and $\alpha(t) = 0$, $\beta(t) = [p(t)]^{1/(1-r)}$, and $\gamma(t) = q(t)$. If ϕ has an inverse ϕ^{-1}, then let

$$y(t) = p(t)\phi\left(x'(t)\right). \tag{20}$$

Hence, (17) can be written as the form of (7):

$$x'(t) = \phi^{-1}\left(\frac{y(t)}{p(t)}\right),$$
$$y'(t) = -h(t, x(t)), \tag{21}$$

where $\alpha(t) = 0$, $f(t, x) = \phi^{-1}(x/p(t))$, and $g(t, x) = h(t, x)$.

In Sections 2 and 3, we will give some necessary conditions for existence of homoclinic solutions of systems (7) and (13), which satisfy conditions

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^{\nu} + (1 + \beta(t))|y(t)|^{\mu}\right] dt < +\infty,$$
$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^{2} + (1 + \beta(t))|y(t)|^{2}\right] dt < +\infty, \tag{22}$$

respectively. These necessary conditions are actually Lyapunov-type inequalities, which generalize the classical Lyapunov inequality for system (6); see [21–25]. Taking advantage of these necessary conditions, we are able to establish some criteria for non-existence of homoclinic solutions of systems (7) and (13) in Section 4.

2. Lyapunov-Type Inequalities for System (7)

In this section, we will establish some Lyapunov-type inequalities for system (7). For the sake of convenience, we list some assumptions on $\alpha(t)$ and $\beta(t)$ as follows:

(A0) $\liminf_{|t|\to+\infty} \int_0^t \alpha(s)ds > -\infty$,

(A1) $\int_{-\infty}^{+\infty} |\alpha(s)|ds < +\infty$,

(B0) $\beta(t) \geq (\not\equiv)\, 0$, $\forall t \in \mathbb{R}$,

(B1) $\beta(t) > 0$, $\forall t \in \mathbb{R}$,

(B2) $\int_{-\infty}^{+\infty} \beta(\tau)\exp(-\mu \int_0^{\tau} \alpha(s)ds)d\tau < +\infty$.

Denote

$$\zeta(t) := \left[\int_{-\infty}^{t} \beta(\tau)\exp\left(\mu\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right]^{\nu/\mu}, \tag{23}$$

$$\eta(t) := \left[\int_{t}^{+\infty} \beta(\tau)\exp\left(-\mu\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right]^{\nu/\mu}. \tag{24}$$

Theorem 2. *Suppose that hypotheses (F), (G), (A0), (B0), and (B2) are satisfied. If system (7) has a solution $(x(t), y(t))$ satisfying*

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^{\nu} + (1 + \beta(t))|y(t)|^{\mu}\right] dt < +\infty, \tag{25}$$

then one has the following inequality:

$$\int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\,dt \geq 1. \tag{26}$$

Proof. Hypothesis (B2) implies that functions $\zeta(t)$ and $\eta(t)$ are well defined on $\mathbb{R}$. Without loss of generality, we can assume that

$$\int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\,dt < +\infty. \tag{27}$$

It follows from (F), (25), and (B0) that

$$\liminf_{t\to-\infty} |x(t)| = \liminf_{t\to+\infty} |x(t)| = 0, \tag{28}$$

$$\liminf_{t\to-\infty} |y(t)| = \liminf_{t\to+\infty} |y(t)| = 0, \tag{29}$$

$$\int_{-\infty}^{+\infty} f(\tau, y(\tau))\,y(\tau)\,d\tau \leq \int_{-\infty}^{+\infty} \beta(\tau)|y(\tau)|^{\mu}d\tau < +\infty. \tag{30}$$

Set $A(t) = \{\tau \in (-\infty, t] : \beta(\tau) > 0\}$ for $t \in \mathbb{R}$, and then it follows from (F) that

$$\begin{aligned}
&[\beta(\tau)]^{-\nu/\mu}|f(\tau,z)|^{\nu} \\
&\quad= |z|^{-\nu}[\beta(\tau)]^{-\nu/\mu}|zf(\tau,z)|^{\nu} \\
&\quad\leq |z|^{-\nu}[\beta(\tau)]^{-\nu/\mu}\left|\beta(\tau)|z|^{\mu}\right|^{\nu-1}zf(\tau,z) \\
&\quad= zf(\tau,z), \quad \tau \in A(t),\ t \in \mathbb{R},\ z \neq 0.
\end{aligned} \tag{31}$$

Since $f(\tau, 0) = 0$ for $\tau \in \mathbb{R}$, it follows that

$$[\beta(\tau)]^{-\nu/\mu}|f(\tau,z)|^{\nu} \leq zf(\tau,z), \quad \tau \in A(t),\ t \in \mathbb{R}. \tag{32}$$

Hence, from (F), (23), (24), (30), (32), and the Hölder inequality, one has

$$\begin{aligned}
&\int_{-\infty}^{t} |f(\tau, y(\tau))|\exp\left(\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau \\
&\quad= \int_{A(t)} |f(\tau, y(\tau))|\exp\left(\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau \\
&\quad\leq \left[\int_{A(t)} \beta(\tau)\exp\left(\mu\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right]^{1/\mu} \\
&\qquad\times\left[\int_{A(t)} [\beta(\tau)]^{-\nu/\mu}|f(\tau, y(\tau))|^{\nu}d\tau\right]^{1/\nu} \\
&\quad\leq \left[\int_{A(t)} \beta(\tau)\exp\left(\mu\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right]^{1/\mu}
\end{aligned}$$

$$\times \left[\int_{A(t)} f(\tau, y(\tau)) y(\tau) d\tau \right]^{1/\nu}$$
$$\le \left[\int_{-\infty}^{t} \beta(\tau) \exp\left(\mu \int_{\tau}^{t} \alpha(s) ds \right) d\tau \right]^{1/\mu}$$
$$\times \left[\int_{-\infty}^{t} f(\tau, y(\tau)) y(\tau) d\tau \right]^{1/\nu}$$
$$= [\zeta(t)]^{1/\nu} \left[\int_{-\infty}^{t} f(\tau, y(\tau)) y(\tau) d\tau \right]^{1/\nu}$$
$$< +\infty, \quad \forall t \in \mathbb{R}, \tag{33}$$

$$\int_{t}^{+\infty} |f(\tau, y(\tau))| \exp\left(- \int_{t}^{\tau} \alpha(s) ds \right) d\tau$$
$$\le \left[\int_{t}^{+\infty} \beta(\tau) \exp\left(-\mu \int_{t}^{\tau} \alpha(s) ds \right) d\tau \right]^{1/\mu}$$
$$\times \left[\int_{t}^{+\infty} f(\tau, y(\tau)) y(\tau) d\tau \right]^{1/\nu} \tag{34}$$
$$= [\eta(t)]^{1/\nu} \left[\int_{t}^{+\infty} f(\tau, y(\tau)) y(\tau) d\tau \right]^{1/\nu}$$
$$< +\infty, \quad \forall t \in \mathbb{R}.$$

From (A0), (28), (33), (34), and the first equation of system (7), we have

$$x(-\infty) := \lim_{t \to -\infty} x(t) = 0 = \lim_{t \to +\infty} x(t) := x(+\infty), \tag{35}$$

$$x(t) = \int_{-\infty}^{t} f(\tau, y(\tau)) \exp\left(\int_{\tau}^{t} \alpha(s) ds \right) d\tau, \quad \forall t \in \mathbb{R}, \tag{36}$$

$$x(t) = -\int_{t}^{+\infty} f(\tau, y(\tau)) \exp\left(- \int_{t}^{\tau} \alpha(s) ds \right) d\tau, \quad \forall t \in \mathbb{R}. \tag{37}$$

Combining (33) with (36), one has

$$|x(t)|^{\nu} = \left| \int_{-\infty}^{t} f(\tau, y(\tau)) \exp\left(\int_{\tau}^{t} \alpha(s) ds \right) d\tau \right|^{\nu}$$
$$\le \zeta(t) \int_{-\infty}^{t} f(\tau, y(\tau)) y(\tau) d\tau, \quad \forall t \in \mathbb{R}. \tag{38}$$

Similarly, it follows from (34) and (37) that

$$|x(t)|^{\nu} = \left| \int_{t}^{+\infty} f(\tau, y(\tau)) \exp\left(- \int_{\tau}^{t} \alpha(s) ds \right) d\tau \right|^{\nu}$$
$$\le \eta(t) \int_{t}^{+\infty} f(\tau, y(\tau)) y(\tau) d\tau, \quad \forall t \in \mathbb{R}. \tag{39}$$

Hence, from (38) and (39), one has

$$|x(t)|^{\nu} \le \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \int_{-\infty}^{+\infty} f(\tau, y(\tau)) y(\tau) d\tau, \quad \forall t \in \mathbb{R}. \tag{40}$$

Now, it follows from (27), (30), and (40) that

$$\int_{-\infty}^{+\infty} \gamma_0(t) |x(t)|^{\nu} dt$$
$$\le \int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \gamma_0(t) dt \int_{-\infty}^{+\infty} f(t, y(t)) y(t) dt$$
$$< +\infty. \tag{41}$$

By (29), we can choose two sequences $\{t_{-n}\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ such that

$$-\infty < \cdots < t_{-3} < t_{-2} < t_{-1} < t_1 < t_2 < t_3 < \cdots < +\infty, \tag{42}$$

$$\lim_{n \to \infty} t_{-n} = -\infty, \qquad \lim_{n \to \infty} t_n = +\infty,$$
$$\lim_{n \to \infty} y(t_{-n}) = \lim_{n \to \infty} y(t_n) = 0. \tag{43}$$

By (7), we obtain

$$(x(t) y(t))' = f(t, y(t)) y(t) - g(t, x(t)) x(t). \tag{44}$$

Integrating the above equation from t_{-n} to t_n, we have

$$\int_{t_{-n}}^{t_n} g(t, x(t)) x(t) dt$$
$$= \int_{t_{-n}}^{t_n} f(t, y(t)) y(t) dt + x(t_{-n}) y(t_{-n}) \tag{45}$$
$$- x(t_n) y(t_n), \quad n = 1, 2, 3, \ldots.$$

Let $n \to \infty$ in the above equation, and using (30), (35), and (43) we obtain

$$\lim_{n \to \infty} \int_{t_{-n}}^{t_n} g(t, x(t)) x(t) dt = \int_{-\infty}^{+\infty} f(t, y(t)) y(t) dt, \tag{46}$$

which, together with (41), implies that

$$\int_{-\infty}^{+\infty} \gamma_0(t) |x(t)|^{\nu} dt$$
$$\le \int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \gamma_0(t) dt \int_{-\infty}^{+\infty} f(t, y(t)) y(t) dt$$
$$= \left[\int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \gamma_0(t) dt \right]$$
$$\times \lim_{n \to \infty} \int_{t_{-n}}^{t_n} g(t, x(t)) x(t) dt$$
$$\le \left[\int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \gamma_0(t) dt \right]$$
$$\times \lim_{n \to \infty} \int_{t_{-n}}^{t_n} \gamma_0(t) |x(t)|^{\nu} dt$$
$$= \int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)} \gamma_0(t) dt \int_{-\infty}^{+\infty} \gamma_0(t) |x(t)|^{\nu} dt. \tag{47}$$

We claim that

$$\int_{-\infty}^{+\infty} \gamma_0(t)|x(t)|^{\nu} dt > 0. \tag{48}$$

If (48) is not true, then

$$\int_{-\infty}^{+\infty} \gamma_0(t)|x(t)|^{\nu} dt = 0. \tag{49}$$

From (F), (G), (46), and (49), we have

$$\begin{aligned} 0 &\le \int_{-\infty}^{+\infty} f(t, y(t))\, y(t)\, dt \\ &= \lim_{n\to\infty} \int_{t_{-n}}^{t_n} g(t, x(t))\, x(t)\, dt \\ &\le \int_{-\infty}^{+\infty} \gamma_0(t)|x(t)|^{\nu} dt = 0, \end{aligned} \tag{50}$$

which, together with (F), implies that

$$f(t, y(t))\, y(t) = 0, \quad \text{a.e. } t \in \mathbb{R}. \tag{51}$$

Combining (36) with (51), we obtain that

$$x(t) \equiv 0, \quad \forall t \in \mathbb{R}, \tag{52}$$

which, together with (G) and the second equation of system (7), implies that

$$y(t) = y(0)\exp\left(-\int_0^t \alpha(s)\, ds\right), \quad \forall t \in \mathbb{R}. \tag{53}$$

From (F), (51), and the above, one has

$$y(t) \equiv 0, \quad \forall t \in \mathbb{R}. \tag{54}$$

Both (52) and (54) contradict (25). Therefore, (48) holds. Hence, it follows from (47) and (48) that (26) holds. □

Corollary 3. *Suppose that hypotheses (F), (G), (A1), (B0), and (B2) are satisfied. If system* (7) *has a solution* $(x(t), y(t))$ *satisfying* (25), *then one has the following inequalities:*

$$\begin{aligned} &\int_{-\infty}^{+\infty} \tilde{\gamma}(t)\left(\int_{-\infty}^{t} \tilde{\beta}(\tau)\, d\tau \int_{t}^{+\infty} \tilde{\beta}(\tau)\, d\tau\right)^{\nu/2\mu} dt \\ &\quad \ge 2\exp\left(-\frac{\nu}{2}\int_{-\infty}^{+\infty}\left(|\alpha(s)| + \mu^{-1}|\omega(s)|\right) ds\right), \end{aligned} \tag{55}$$

$$\begin{aligned} &\left(\int_{-\infty}^{+\infty} \tilde{\beta}(t)\, dt\right)^{1/\mu}\left(\int_{-\infty}^{+\infty} \tilde{\gamma}^{+}(t)\, dt\right)^{1/\nu} \\ &\quad \ge 2\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}\left(|\alpha(s)| + \mu^{-1}|\omega(s)|\right) ds\right), \end{aligned} \tag{56}$$

$$\begin{aligned} &\int_{-\infty}^{+\infty} \gamma_0(t)\left(\int_{-\infty}^{t} \beta(\tau)\, d\tau \int_{t}^{+\infty} \beta(\tau)\, d\tau\right)^{\nu/2\mu} dt \\ &\quad \ge 2\exp\left(-\frac{\nu}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\, ds\right), \end{aligned} \tag{57}$$

$$\begin{aligned} &\left(\int_{-\infty}^{+\infty} \beta(t)\, dt\right)^{1/\mu}\left(\int_{-\infty}^{+\infty} \gamma_0(t)\, dt\right)^{1/\nu} \\ &\quad \ge 2\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\, ds\right), \end{aligned} \tag{58}$$

where $\omega \in L^1(\mathbb{R})$ *is an arbitrary function and*

$$\begin{aligned} \tilde{\beta}(t) &= \beta(t)\exp\left(\int_{t_0}^{t} \omega(s)\, ds\right), \\ \tilde{\gamma}(t) &= \gamma_0(t)\exp\left(-\frac{\nu}{\mu}\int_{t_0}^{t} \omega(s)\, ds\right) \end{aligned} \tag{59}$$

for some $t_0 \in \mathbb{R}$.

Proof. (A1), (B0), and (B2) imply that (A0) and $\int_{-\infty}^{+\infty} \beta(s)ds < +\infty$. Since

$$\zeta(t) + \eta(t) \ge 2[\zeta(t)\eta(t)]^{1/2}, \tag{60}$$

then it follows from (23), (24), (26), (56), and (57) that

$$\begin{aligned} 1 &\le \int_{-\infty}^{+\infty} \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\, dt \\ &\le \frac{1}{2}\int_{-\infty}^{+\infty} [\zeta(t)\eta(t)]^{1/2}\gamma_0(t)\, dt \end{aligned}$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\gamma_0(t)\left[\int_{-\infty}^{t}\beta(\tau)\exp\left(\mu\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau \times \int_{t}^{+\infty}\beta(\tau) \times \exp\left(-\mu\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right]^{\nu/2\mu}dt$$
$$\le \frac{1}{2}\left[\int_{-\infty}^{+\infty}\gamma_0(t)\left(\int_{-\infty}^{t}\beta(\tau)\,d\tau\int_{t}^{+\infty}\beta(\tau)\,d\tau\right)^{\nu/2\mu}dt\right] \times \exp\left(\frac{\nu}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right)$$
$$\le \frac{1}{2}\left[\int_{-\infty}^{+\infty}\tilde{\gamma}(t)\left(\int_{-\infty}^{t}\tilde{\beta}(\tau)d\tau\int_{t}^{+\infty}\tilde{\beta}(\tau)d\tau\right)^{\nu/2\mu}dt\right] \times \exp\left(\frac{\nu}{2}\int_{-\infty}^{+\infty}\left(|\alpha(s)|+\mu^{-1}|\omega(s)|\right)ds\right), \tag{61}$$

which implies that (55) holds. Note that

$$\int_{-\infty}^{t}\tilde{\beta}(\tau)\,d\tau\int_{t}^{+\infty}\tilde{\beta}(\tau)\,d\tau \le \frac{1}{4}\left(\int_{-\infty}^{+\infty}\tilde{\beta}(\tau)\,d\tau\right)^2, \tag{62}$$

which, together with (55), yields that (56) holds. It follows from (55) and (56) that (57) and (58) hold. □

In case hypothesis (B0) is replaced by (B1) in the proof of Theorem 2, then (40) is strict; that is,

$$|x(t)|^{\nu} < \frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\int_{-\infty}^{+\infty}f(\tau,y(\tau))\,y(\tau)\,d\tau, \quad \forall t\in\mathbb{R}. \tag{63}$$

In fact, if (63) is not true, then there exists a $t_*\in\mathbb{R}$ such that

$$|x(t_*)|^{\nu} = \frac{\zeta(t_*)\eta(t_*)}{\zeta(t_*)+\eta(t_*)}\int_{-\infty}^{+\infty}f(\tau,y(\tau))\,y(\tau)\,d\tau. \tag{64}$$

Hence, from (38), (39), and (64), we obtain

$$|x(t_*)|^{\nu} = \zeta(t_*)\int_{-\infty}^{t_*}f(\tau,y(\tau))\,y(\tau)\,d\tau, \tag{65}$$
$$|x(t_*)|^{\nu} = \eta(t_*)\int_{t_*}^{+\infty}f(\tau,y(\tau))\,y(\tau)\,d\tau. \tag{66}$$

It follows from (23), (38), and (65) that

$$\left|\int_{-\infty}^{t_*}f(\tau,y(\tau))\exp\left(\int_{\tau}^{t_*}\alpha(s)ds\right)d\tau\right|^{\nu} = \left[\int_{-\infty}^{t_*}\beta(\tau)\exp\left(\mu\int_{\tau}^{t_*}\alpha(s)\,ds\right)d\tau\right]^{\nu/\mu} \times \int_{-\infty}^{t_*}f(\tau,y(\tau))\,y(\tau)\,d\tau, \tag{67}$$

which, together with the Hölder inequality, implies that there exists a constant c_1 such that

$$f(\tau,y(\tau))\,y(\tau) = c_1\beta(\tau)\exp\left(\mu\int_{\tau}^{t_*}\alpha(s)\,ds\right), \quad -\infty<\tau\le t_*. \tag{68}$$

Similarly, it follows from (24), (39), (66), and the Hölder inequality that there exists a constant c_2 such that

$$f(\tau,y(\tau))\,y(\tau) = c_2\beta(\tau)\exp\left(-\mu\int_{t_*}^{\tau}\alpha(s)\,ds\right), \quad t_*\le\tau<+\infty. \tag{69}$$

From (F), (68), and (69), one has that $c_1, c_2\ge 0$. If $c_1=c_2=0$, then $f(\tau,y(\tau))y(\tau)=0$ for $\tau\in\mathbb{R}$; it follows from (36) that $x(t)=0$ for $t\in\mathbb{R}$. Similar to the proof of (54), one has $y(t)=0$ for $t\in\mathbb{R}$, which contradicts (25). If $c_1+c_2>0$, then $f(\tau,y(\tau))y(\tau)>0$ for $\tau\in(-\infty,t_*]$ or for $\tau\in[t_*,+\infty)$; it follows from (A0) and (36) that $x(+\infty)\neq 0$, which contradicts (35). The above two cases show that (63) holds. Hence, in view of the proof of Theorem 2, we have the following theorem.

Theorem 4. *Suppose that hypotheses (F), (G), (A0), (B1), and (B2) are satisfied. If system* (7) *has a solution* $(x(t),y(t))$ *satisfying* (25), *then one has the following inequality:*

$$\int_{-\infty}^{+\infty}\frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\,dt > 1, \tag{70}$$

where $\zeta(t)$ *and* $\eta(t)$ *are defined by* (23) *and* (24), *respectively.*

Similar to the proof of Corollary 3, we can drive the following corollary from Theorem 4.

Corollary 5. *Suppose that hypotheses (F), (G), (A1), (B1), and (B2) are satisfied. If system* (7) *has a solution* $(x(t),y(t))$ *satisfying* (25), *then*

$$\int_{-\infty}^{+\infty}\tilde{\gamma}(t)\left(\int_{-\infty}^{t}\tilde{\beta}(\tau)\,d\tau\int_{t}^{+\infty}\tilde{\beta}(\tau)\,d\tau\right)^{\nu/2\mu}dt > 2\exp\left(-\frac{\nu}{2}\int_{-\infty}^{+\infty}\left(|\alpha(s)|+\mu^{-1}|\omega(s)|\right)ds\right),$$
$$\left(\int_{-\infty}^{+\infty}\tilde{\beta}(t)\,dt\right)^{1/\mu}\left(\int_{-\infty}^{+\infty}\tilde{\gamma}(t)\,dt\right)^{1/\nu} > 2\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}\left(|\alpha(s)|+\mu^{-1}|\omega(s)|\right)ds\right),$$

$$\int_{-\infty}^{+\infty} \gamma_0(t) \left(\int_{-\infty}^{t} \beta(\tau)\, d\tau \int_{t}^{+\infty} \beta(\tau)\, d\tau \right)^{\nu/2\mu} dt > 2\exp\left(-\frac{\nu}{2} \int_{-\infty}^{+\infty} |\alpha(s)|\, ds \right),$$
$$\left(\int_{-\infty}^{+\infty} \beta(t)\, dt \right)^{1/\mu} \left(\int_{-\infty}^{+\infty} \gamma_0(t)\, dt \right)^{1/\nu} > 2\exp\left(-\frac{1}{2} \int_{-\infty}^{+\infty} |\alpha(s)|\, ds \right), \tag{71}$$

where $\widetilde{\beta}(t)$ *and* $\widetilde{\gamma}(t)$ *are defined by* (59).

Applying Theorem 4 and Corollary 5 to system (19) (i.e., (16)), we have immediately the following two corollaries.

Corollary 6. *Suppose that* $r > 1$, $p(t) > 0$ *for* $t \in \mathbb{R}$ *and*

$$\int_{-\infty}^{+\infty} \frac{1}{[p(\tau)]^{1/(r-1)}} d\tau < +\infty. \tag{72}$$

If (16) *has a solution* $x(t)$ *satisfying*

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^r + [p(t)]^{1/(r-1)} (1 + p(t)) \left| x'(t) \right|^r \right] dt < +\infty, \tag{73}$$

then

$$\int_{-\infty}^{+\infty} \frac{\left(\int_{-\infty}^{t} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1} \left(\int_{t}^{+\infty} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1}}{\left(\int_{-\infty}^{t} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1} + \left(\int_{t}^{+\infty} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1}} \times q^{+}(t)\, dt > 1,$$
$$\int_{-\infty}^{+\infty} q^{+}(t) \left(\int_{-\infty}^{t} [p(\tau)]^{-1/(r-1)} d\tau \times \int_{t}^{+\infty} [p(\tau)]^{-1/(r-1)} d\tau \right)^{(r-1)/2} dt > 2. \tag{74}$$

Applying Theorem 4 to the second-order nonlinear differential equation (17) (i.e., system (21)), where $\alpha(t) = 0$, $f(t,x) = \phi^{-1}(x/p(t))$, and $g(t,x) = h(t,x)$, we have the following corollary.

Corollary 7. *Suppose that* $r > 1$ *and* $p(t) > 0$ *for* $t \in \mathbb{R}$, *and that* (72) *and the following hypothesis are satisfied:*

(H1) *There exists a locally Lebesgue integrable nonnegative function* $\gamma_0(t)$ *defined on* $\mathbb{R}$ *such that*

$$0 \le x\phi^{-1}(x) \le |x|^{r/(r-1)},$$
$$xh(t,x) \le \gamma_0(t)\, |x|^r, \quad \forall (t,x) \in \mathbb{R}^2. \tag{75}$$

If (17) *has a solution* $x(t)$ *satisfying* (73), *then*

$$\int_{-\infty}^{+\infty} \frac{\left(\int_{-\infty}^{t} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1} \left(\int_{t}^{+\infty} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1}}{\left(\int_{-\infty}^{t} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1} + \left(\int_{t}^{+\infty} [p(\tau)]^{-1/(r-1)} d\tau \right)^{r-1}} \times \gamma_0(t)\, dt > 1. \tag{76}$$

3. Lyapunov-Type Inequalities for System (13)

When $\mu = \nu = 2$, assumption (B2) reduces to the following form:

(B2′) $\int_{-\infty}^{+\infty} \beta(\tau) \exp(-2 \int_0^{\tau} \alpha(s) ds) d\tau < +\infty$.

Applying some results obtained in the last section to the first-order linear Hamiltonian system (13), we have immediately the following corollaries.

Corollary 8. *Suppose that hypotheses (A0), (B0), and* (B2′) *are satisfied. If system* (13) *has a solution* $(x(t), y(t))$ *satisfying*

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^2 + (1 + \beta(t)) \left| y(t) \right|^2 \right] dt < +\infty, \tag{77}$$

then

$$\int_{-\infty}^{+\infty} \left(\left(\left[\int_{-\infty}^{t} \beta(\tau) \exp\left(2\int_{\tau}^{t} \alpha(s)\, ds \right) d\tau \right] \times \left[\int_{t}^{+\infty} \beta(\tau) \exp\left(-2\int_{t}^{\tau} \alpha(s)\, ds \right) d\tau \right] \right) \times \left(\int_{-\infty}^{t} \beta(\tau) \exp\left(2\int_{\tau}^{t} \alpha(s)\, ds \right) d\tau + \int_{t}^{+\infty} \beta(\tau) \exp\left(-2\int_{t}^{\tau} \alpha(s)\, ds \right) d\tau \right)^{-1} \right) \times \gamma^{+}(t)\, dt \ge 1,$$
$$\int_{-\infty}^{+\infty} \gamma^{+}(t) \left(\int_{-\infty}^{t} \beta(\tau)\, d\tau \int_{t}^{+\infty} \beta(\tau) d\tau \right)^{1/2} dt \ge 2\exp\left(-\int_{-\infty}^{+\infty} |\alpha(s)|\, ds \right). \tag{78}$$

Corollary 9. *Suppose that hypotheses (A0), (B1), and* $(B2')$ *are satisfied. If system* (13) *has a solution* $(x(t), y(t))$ *satisfying* (77), *then*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\left(\left[\int_{-\infty}^{t}\beta(\tau)\exp\left(2\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right]\right.\right.\\&\qquad\times\left.\left[\int_{t}^{+\infty}\beta(\tau)\exp\left(-2\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right]\right)\\&\qquad\times\left(\int_{-\infty}^{t}\beta(\tau)\exp\left(2\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right.\\&\qquad\quad\left.\left.+\int_{t}^{+\infty}\beta(\tau)\exp\left(-2\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right)^{-1}\right)\\&\quad\times\gamma^{+}(t)\,dt>1,\\&\int_{-\infty}^{+\infty}\gamma^{+}(t)\left(\int_{-\infty}^{t}\beta(\tau)\,d\tau\int_{t}^{+\infty}\beta(\tau)\,d\tau\right)^{1/2}dt\\&\qquad>2\exp\left(-\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right).\end{aligned}\tag{79}$$

Corollary 10. *Suppose that* $p(t)>0$ *for* $t\in\mathbb{R}$ *and that*

$$\int_{-\infty}^{+\infty}\frac{d\tau}{p(\tau)}<+\infty.\tag{80}$$

If (15) *has a solution* $x(t)$ *satisfying*

$$0<\int_{-\infty}^{+\infty}\left[|x(t)|^{2}+p(t)(1+p(t))\left|x'(t)\right|^{2}\right]dt<+\infty,\tag{81}$$

then

$$\int_{-\infty}^{+\infty}q^{+}(t)\left(\int_{-\infty}^{t}\frac{d\tau}{p(\tau)}\int_{t}^{+\infty}\frac{d\tau}{p(\tau)}\right)dt>\int_{-\infty}^{+\infty}\frac{d\tau}{p(\tau)}.\tag{82}$$

Corollary 11. *Suppose that* $p(t)>0$ *for* $t\in\mathbb{R}$ *and that* (80) *and the following hypothesis are satisfied:*

(H2) *There exists a locally Lebesgue integrable nonnegative function* $\gamma_0(t)$ *defined on* $\mathbb{R}$ *such that*

$$\begin{gathered}0\le x\phi^{-1}(x)\le|x|^{2},\\xh(t,x)\le\gamma_0(t)|x|^{2},\quad\forall(t,x)\in\mathbb{R}^{2}.\end{gathered}\tag{83}$$

If (17) *has a solution* $x(t)$ *satisfying* (81), *then*

$$\int_{-\infty}^{+\infty}\gamma_0(t)\left(\int_{-\infty}^{t}\frac{d\tau}{p(\tau)}\int_{t}^{+\infty}\frac{d\tau}{p(\tau)}\right)dt>\int_{-\infty}^{+\infty}\frac{d\tau}{p(\tau)}.\tag{84}$$

4. Nonexistence of Homoclinic Solutions

Applying the results obtained in Sections 2 and 3, we can drive the following criteria for non-existence of homoclinic solutions of systems (7) and (13) immediately.

Corollary 12. *Suppose that hypotheses (F), (G), (A0), (B0), and (B2) are satisfied. If one of the conditions*

$$\int_{-\infty}^{+\infty}\frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\,dt<1,\tag{85}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty}\gamma_0(t)\left(\int_{-\infty}^{t}\beta(\tau)\,d\tau\int_{t}^{+\infty}\beta(\tau)\,d\tau\right)^{\nu/2\mu}dt\\&\qquad<2\exp\left(-\frac{\nu}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right),\end{aligned}\tag{86}$$

$$\begin{aligned}&\left(\int_{-\infty}^{+\infty}\beta(t)\,dt\right)^{1/\mu}\left(\int_{-\infty}^{+\infty}\gamma_0(t)\,dt\right)^{1/\nu}\\&\qquad<2\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right)\end{aligned}\tag{87}$$

holds, then system (7) *has no solution* $(x(t), y(t))$ *satisfying*

$$0<\int_{-\infty}^{+\infty}\left[|x(t)|^{\nu}+(1+\beta(t))|y(t)|^{\mu}\right]dt<+\infty.\tag{88}$$

Corollary 13. *Suppose that hypotheses (F), (G), (A0), (B1), and (B2) are satisfied. If one of the conditions*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{\zeta(t)\eta(t)}{\zeta(t)+\eta(t)}\gamma_0(t)\,dt\le1,\\&\int_{-\infty}^{+\infty}\gamma_0(t)\left(\int_{-\infty}^{t}\beta(\tau)\,d\tau\int_{t}^{+\infty}\beta(\tau)\,d\tau\right)^{\nu/2\mu}dt\\&\qquad\le2\exp\left(-\frac{\nu}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right),\\&\left(\int_{-\infty}^{+\infty}\beta(t)\,dt\right)^{1/\mu}\left(\int_{-\infty}^{+\infty}\gamma_0(t)\,dt\right)^{1/\nu}\\&\qquad\le2\exp\left(-\frac{1}{2}\int_{-\infty}^{+\infty}|\alpha(s)|\,ds\right)\end{aligned}\tag{89}$$

holds, then system (7) *has no solution* $(x(t), y(t))$ *satisfying* (86).

Corollary 14. *Suppose that hypotheses (A0), (B0), and* $(B2')$ *are satisfied. If*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\left(\left[\int_{-\infty}^{t}\beta(\tau)\exp\left(2\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right]\right.\right.\\&\qquad\times\left.\left[\int_{t}^{+\infty}\beta(\tau)\exp\left(-2\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right]\right)\\&\qquad\times\left(\int_{-\infty}^{t}\beta(\tau)\exp\left(2\int_{\tau}^{t}\alpha(s)\,ds\right)d\tau\right.\\&\qquad\quad\left.\left.+\int_{t}^{+\infty}\beta(\tau)\exp\left(-2\int_{t}^{\tau}\alpha(s)\,ds\right)d\tau\right)^{-1}\right)\\&\quad\times\gamma_0(t)\,dt<1\end{aligned}\tag{90}$$

or

$$\int_{-\infty}^{+\infty} \gamma^{+}(t)\left(\int_{-\infty}^{t} \beta(\tau)\,d\tau \int_{t}^{+\infty} \beta(\tau)\,d\tau\right)^{1/2} dt < 2\exp\left(-\int_{-\infty}^{+\infty} |\alpha(s)|\,ds\right) \quad (91)$$

holds, then system (13) *has no solution* $(x(t), y(t))$ *satisfying*

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^2 + (1+\beta(t))\left|y(t)\right|^2\right] dt < +\infty. \quad (92)$$

Corollary 15. *Suppose that hypotheses (A0), (B1), and (B2′) are satisfied. If*

$$\int_{-\infty}^{+\infty} \left(\left(\left[\int_{-\infty}^{t} \beta(\tau)\exp\left(2\int_{\tau}^{t} \alpha(s)\,ds\right) d\tau\right] \times \left[\int_{t}^{+\infty} \beta(\tau)\exp\left(-2\int_{t}^{\tau} \alpha(s)\,ds\right) d\tau\right]\right) \times \left(\int_{-\infty}^{t} \beta(\tau)\exp\left(2\int_{\tau}^{t} \alpha(s)\,ds\right) d\tau + \int_{t}^{+\infty} \beta(\tau)\exp\left(-2\int_{t}^{\tau} \alpha(s)\,ds\right) d\tau\right)^{-1}\right) \times \gamma^{+}(t)\,dt \le 1 \quad (93)$$

or

$$\int_{-\infty}^{+\infty} \gamma^{+}(t)\left(\int_{-\infty}^{t} \beta(\tau)\,d\tau \int_{t}^{+\infty} \beta(\tau)\,d\tau\right)^{1/2} dt \le 2\exp\left(-\int_{-\infty}^{+\infty} |\alpha(s)|\,ds\right) \quad (94)$$

holds, then system (13) *has no solution* $(x(t), y(t))$ *satisfying* (92).

Corollary 16. *Suppose that* $p(t) > 0$ *for* $t \in \mathbb{R}$ *and that* (80) *holds. If*

$$\int_{-\infty}^{+\infty} q^{+}(t)\left(\int_{-\infty}^{t} \frac{d\tau}{p(\tau)} \int_{t}^{+\infty} \frac{d\tau}{p(\tau)}\right) dt \le \int_{-\infty}^{+\infty} \frac{d\tau}{p(\tau)}, \quad (95)$$

then (15) *has no solution* $x(t)$ *satisfying* (81).

Corollary 17. *Suppose that* $p(t) > 0$ *for* $t \in \mathbb{R}$ *and that* (80) *and (H2) are satisfied. If*

$$\int_{-\infty}^{+\infty} \gamma_0(t)\left(\int_{-\infty}^{t} \frac{d\tau}{p(\tau)} \int_{t}^{+\infty} \frac{d\tau}{p(\tau)}\right) dt \le \int_{-\infty}^{+\infty} \frac{d\tau}{p(\tau)}, \quad (96)$$

then (17) *has no solution* $x(t)$ *satisfying* (81).

Example 18. Consider the second-order nonlinear differential equation

$$\left[\left(1+t^2\right)x'(t)\right]' + q(t)\,x(t)\left[1+\sin^2 x(t)\right] = 0, \quad (97)$$

where $q(t)$ is locally Lebesgue integrable real-valued function defined on $\mathbb{R}$. In view of Corollary 16, if

$$\int_{-\infty}^{+\infty} \left[\frac{\pi^2}{4} - (\arctan t)^2\right] q^{+}(t)\,dt \le \frac{\pi}{2}, \quad (98)$$

then (97) has no solution $x(t)$ satisfying

$$0 < \int_{-\infty}^{+\infty} \left[|x(t)|^2 + \left(1+t^2\right)^2 \left|x'(t)\right|^2\right] dt < +\infty. \quad (99)$$

Acknowledgments

This work is partially supported by the NNSF (no. 11201138) of China Hunan Provincial Natural Science Foundation (no. 11JJ2005), and the Scientific Research Fund of Hunan Provincial Education Department (12B034).

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