Research Article

Two Efficient Generalized Laguerre Spectral Algorithms for Fractional Initial Value Problems

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We present a direct solution technique for approximating linear multiterm fractional differential equations (FDEs) on semi-infinite interval, using generalized Laguerre polynomials. We derive the operational matrix of Caputo fractional derivative of the generalized Laguerre polynomials which is applied together with generalized Laguerre tau approximation for implementing a spectral solution of linear multiterm FDEs on semi-infinite interval subject to initial conditions. The generalized Laguerre pseudo-spectral approximation based on the generalized Laguerre operational matrix is investigated to reduce the nonlinear multiterm FDEs and its initial conditions to nonlinear algebraic system, thus greatly simplifying the problem. Through several numerical examples, we confirm the accuracy and performance of the proposed spectral algorithms. Indeed, the methods yield accurate results, and the exact solutions are achieved for some tested problems.

1. Introduction

Fractional calculus has been used to develop accurate models of many phenomena of science, engineering, economics, and applied mathematics. These models are found to be best described by FDEs [1–4].

One of the best methods, in terms of the accuracy, for investigating the numerical solution of various kinds of differential equations is spectral method (see, for instance, [5–8]). Because all types of spectral methods are global and numerical computational methods, they are very convenient for approximating linear and nonlinear FDEs [6, 7, 9]. We refer also to recent numerical and analytical methods for solving FEEs [10–16].

In the last few years, theory and numerical solution of FDEs by using spectral methods have received an increasing attention. In this direction, Doha et al. [17] proposed an effective way to approximate solutions of linear and nonlinear multiterm FDEs with constant and variable coefficients using Jacobi spectral approximation, in which they generalized the Chebyshev spectral methods [6] and quadrature Legendre tau method [9]; moreover, other very important cases can be obtained for that approach. Maleki et al. [18] proposed an efficient and accurate spectral collocation method based on shifted Legendre-Gauss quadrature nodes for solving fractional boundary value problems in finite interval. The authors of [19] used the spline functions methods for tackling the linear and nonlinear FDEs. Recently, Bhrawy et al. [20] investigated the fractional integrals of modified generalized Laguerre operational matrix to implement a numerical solution of the integrated form of the linear FDEs on semi-infinite interval. Furthermore, Yuzbasi [21] proposed a new collocation method based on Bessel functions to introduce an approximate solution of a class of FDEs. We refer also to the recent papers [22–26] where operational matrices of several orthogonal polynomials are developed for solving linear and nonlinear ODEs and FDEs.

In this paper, the Caputo fractional derivative of generalized Laguerre operational matrix (GLOM) is stated and proved. The main aim of this paper is to extend the application of generalized Laguerre spectral tau method based on GLOM to develop a direct solution technique for the numerical solution of linear multi-term FDEs on a semi-infinite interval. Moreover, we develop the generalized Laguerre pseudo-spectral approximation based on the GLOM for reducing the nonlinear multi-term FDEs subject to nonhomogeneous initial conditions to a system of nonlinear algebraic equations. Finally, the accuracy of the proposed algorithms is demonstrated by test problems. The numerical results are given to show that the proposed spectral algorithms based on generalized Laguerre operational matrix of Caputo fractional derivatives are very effective for linear and nonlinear FDEs.

The outline of the paper is as follows. In Section 2, we present some preliminaries. Section is devoted to drive the GLOM of Caputo fractional derivative. In Section 4, we extend the generalized Laguerre spectral tau and pseudo-spectral approximations based on the GLOM of fractional derivative for solving multiorder linear and nonlinear FDEs. Some numerical experiments are presented in Section 5. Finally, we conclude the paper with some remarks.

2. Some Basic Preliminaries

The two most commonly used definitions are the Riemann-Liouville operator and the Caputo operator. We give some definitions and properties of fractional derivatives and generalized Laguerre polynomials.

Definition 1. The fractional integral operator of Riemann-Liouville sense is defined as

$$J^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt, \quad \nu > 0, \ x > 0,$$

$$J^0 f(x) = f(x). \tag{1}$$

Definition 2. The Caputo fractional derivatives is given by

$$D^\nu f(x)$$

$$= J^{m-\nu} D^m f(x)$$

$$= \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} \frac{d^m}{dt^m} f(t)\, dt, \tag{2}$$

$$m-1 < \nu < m, \ x > 0,$$

where D^m is mth order differential operator.

The Caputo fractional derivative operator satisfies

$$D^\nu C = 0, \quad (C \text{ is a constant}), \tag{3}$$

$$D^\nu x^\beta$$

$$= \begin{cases} 0, & \text{for } \beta \in N_0, \ \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \text{for } \beta \in N_0, \ \beta \geq \lceil \nu \rceil \quad \text{or} \\ & \beta \notin N, \ \beta > \lfloor \nu \rfloor, \end{cases} \tag{4}$$

where $\lceil \nu \rceil$ and $\lfloor \nu \rfloor$ are the ceiling and floor functions, respectively, while $N = \{1, 2, \ldots\}$ and $N_0 = \{0, 1, 2, \ldots\}$.

The Caputo's derivative operator is a linear operation:

$$D^\nu (\lambda f(x) + \mu g(x)) = \lambda D^\nu f(x) + \mu D^\nu g(x), \tag{5}$$

where λ and μ are constants.

We recall below some relevant properties of the generalized Laguerre polynomials (Szego [27] and Funaro [28]). let $\Lambda = (0, \infty)$ and let $w^{(\alpha)}(x) = x^\alpha e^{-x}$ be a weight function on Λ in the usual sense. Define

$$L^2_{w^{(\alpha)}}(\Lambda)$$

$$= \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{w^{(\alpha)}} < \infty\}, \tag{6}$$

with the inner product and norm

$$(u, v)_{w^{(\alpha)}} = \int_\Lambda u(x) v(x) w^{(\alpha)}(x)\, dx,$$

$$\|v\|_{w^{(\alpha)}} = (v, v)^{1/2}_{w^{(\alpha)}}. \tag{7}$$

For $\alpha > -1$, the generalized Laguerre polynomials are given by

$$L_i^{(\alpha)}(x) = \frac{1}{i!} x^{-\alpha} e^x \partial_x^i \left(x^{i+\alpha} e^{-x} \right), \quad i = 1, 2, \ldots. \tag{8}$$

According to [29] for $\alpha > -1$, we get

$$\partial_x L_i^{(\alpha)}(x) = -L_{i-1}^{(\alpha+1)}(x),$$

$$L_{i+1}^{(\alpha)}(x)$$

$$= \frac{1}{i+1} \left[(2i + \alpha + 1 - x) L_i^{(\alpha)}(x) - (i+\alpha) L_{i-1}^{(\alpha)}(x) \right],$$

$$i = 1, 2, \ldots, \tag{9}$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = 1 + \alpha - x$.

The generalized Laguerre polynomials are the $L^2_{w^{(\alpha)}}(\Lambda)$-orthogonal system;

$$\int_0^\infty L_j^{(\alpha)}(x) L_k^{(\alpha)}(x) w^{(\alpha)}(x)\, dx = h_k \delta_{jk}, \tag{10}$$

where $h_k = \Gamma(k + \alpha + 1)/k!$.

The generalized Laguerre polynomials on Λ are obtained from

$$L_i^{(\alpha)}(x) = \sum_{k=0}^i (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)(i-k)!k!} x^k, \quad i = 0, 1, \ldots. \tag{11}$$

The special value

$$D^q L_i^{(\alpha)}(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)!(i-j-q)!} L_j^{(\alpha)}(0), \quad i \geq q, \tag{12}$$

where $L_j^{(\alpha)}(0) = \Gamma(j + \alpha + 1)/\Gamma(\alpha + 1)j!$, will be of important use later, for treating the initial conditions of the given FDEs.

Let $u(x) \in L_{w^{(\alpha)}}^2(\Lambda)$, then $u(x)$ may be expressed in terms of generalized Laguerre polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j L_j^{(\alpha)}(x), \tag{13}$$

$$a_j = \frac{1}{h_k} \int_0^{\infty} u(x) L_j^{(\alpha)}(x) w^{(\alpha)}(x) dx, \quad j = 0, 1, 2, \dots. \tag{14}$$

In particular applications, the generalized Laguerre polynomials up to degree $N + 1$ are considered. Then we have

$$u_N(x) = \sum_{j=0}^{N} a_j L_j^{(\alpha)}(x). \tag{15}$$

We will present the Laguerre-Gauss quadrature. Let $\{x_j^{(\alpha)}, \varpi_j^{(\alpha)}\}$ be the set of generalized Laguerre-Gauss quadrature nodes and weights:

$$\int_{\Lambda} \phi(x) w^{(\alpha)}(x) dx = \sum_{j=0}^{N} \phi\left(x_j^{(\alpha)}\right) \varpi_j^{(\alpha)}. \tag{16}$$

For the generalized Laguerre-Gauss quadrature, $\{x_j^{(\alpha)}\}$ are the zeros of $L_{i+1}^{(\alpha)}(x)$, and

$$\begin{aligned}
\varpi_j^{(\alpha)} &= -\frac{\Gamma(i + \alpha + 1)}{(i + 1)! L_i^{(\alpha)}\left(x_j^{(\alpha)}\right) \partial_x L_{i+1}^{(\alpha)}\left(x_j^{(\alpha)}\right)} \\
&= \frac{\Gamma(i + \alpha + 1) x_j^{(\alpha)}}{(i + \alpha + 1)(i + 1)! \left[L_i^{(\alpha)}\left(x_j^{(\alpha)}\right)\right]^2}, \quad 0 \leq j \leq i.
\end{aligned} \tag{17}$$

3. GLOM of Fractional Derivatives

Let $u(x) \in L_{w^{(\alpha)}}^2(\Lambda)$, and then $u(x)$ may be expanded using generalized Laguerre polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j L_j^{(\alpha)}(x),$$

$$a_j = \frac{1}{h_k} \int_0^{\infty} u(x) L_j^{(\alpha)}(x) w^{(\alpha)}(x) dx, \quad j = 0, 1, 2, \dots. \tag{18}$$

In particular applications, the generalized Laguerre polynomials up to degree $N + 1$ are considered. Then we have

$$u_N(x) = \sum_{j=0}^{N} a_j L_j^{(\alpha)}(x) = C^T \phi(x), \tag{19}$$

where the vector C and vector $\phi(x)$ are given by

$$C^T = [c_0, c_1, \dots, c_N],$$
$$\phi(x) = \left[L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), \dots, L_N^{(\alpha)}(x)\right]^T. \tag{20}$$

Then the derivative of the vector $\phi(x)$ can be expressed by

$$\frac{d\phi(x)}{dx} = \mathbf{D}^{(1)} \phi(x), \tag{21}$$

where $\mathbf{D}^{(1)}$ is the $(N + 1) \times (N + 1)$ operational matrix of the derivative given by

$$\mathbf{D}^{(1)} = -\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0
\end{pmatrix}. \tag{22}$$

By using (21), it is clear that

$$\frac{d^n \phi(x)}{dx^n} = \left(\mathbf{D}^{(1)}\right)^n \phi(x), \tag{23}$$

where $n \in N$ and the superscript in $\mathbf{D}^{(1)}$ denotes matrix powers. Thus

$$\mathbf{D}^{(n)} = \left(\mathbf{D}^{(1)}\right)^n, \quad n = 1, 2, \dots. \tag{24}$$

Lemma 3. *Let $L_i^{(\alpha)}(x)$ be a generalized Laguerre polynomial Then*

$$D^{\nu} L_i^{(\alpha)}(x) = 0, \quad i = 0, 1, \dots, \lceil \nu \rceil - 1, \; \nu > 0. \tag{25}$$

In the following theorem we prove the operational matrix of Caputo fractional derivative for the generalized Laguerre vector (20).

Theorem 4. *Suppose $\nu > 0$, the fractional derivative of order ν of $\phi(x)$ is given by*

$$D^{\nu} \phi(x) \simeq \mathbf{D}^{(\nu)} \phi(x), \tag{26}$$

where $\mathbf{D}^{(\nu)}$ is the $(N+1) \times (N+1)$ operational matrix of Caputo fractional derivative and is given by

$$\mathbf{D}^{(\nu)}$$

$$= \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
S_{\nu}(\lceil \nu \rceil, 0) & S_{\nu}(\lceil \nu \rceil, 1) & S_{\nu}(\lceil \nu \rceil, 2) & \cdots & S_{\nu}(\lceil \nu \rceil, N) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
S_{\nu}(i, 0) & S_{\nu}(i, 1) & S_{\nu}(i, 2) & \cdots & S_{\nu}(i, N) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
S_{\nu}(N, 0) & S_{\nu}(N, 1) & S_{\nu}(N, 2) & \cdots & S_{\nu}(N, N)
\end{pmatrix}, \tag{27}$$

where

$$S_\nu(i, j)$$

$$= \sum_{k=\lceil \nu \rceil}^{i} \sum_{\ell=0}^{j} \left((-1)^{k+\ell} j! \Gamma(i + \alpha + 1) \right.$$

$$\times \Gamma(k - \nu + \alpha + \ell + 1) \right)$$

$$\times \left((i - k)! (j - \ell)! \ell! \Gamma(k - \nu + 1) \right.$$

$$\left. \times \Gamma(k + \alpha + 1) \Gamma(\alpha + \ell + 1) \right)^{-1}. \tag{28}$$

Proof. Applying (4) to (11) gives

$$D^\nu L_i^{(\alpha)}(x)$$

$$= \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i + \alpha + 1)}{(i - k)! k! \Gamma(k + \alpha + 1)} D^\nu x^k$$

$$= \sum_{k=\lceil \nu \rceil}^{i} (-1)^k \frac{\Gamma(i + \alpha + 1)}{(i - k)! \Gamma(k - \nu + 1) \Gamma(k + \alpha + 1)} x^{k-\nu}, \tag{29}$$

$$i = \lceil \nu \rceil, \ldots, N.$$

Now, $x^{k-\nu}$ can be approximated by $N + 1$ terms of the generalized Laguerre polynomials to get

$$x^{k-\nu} = \sum_{j=0}^{N} b_{kj} L_j^{(\alpha)}(x), \tag{30}$$

where b_j is directly obtained from (18), and

$$b_j = \sum_{\ell=0}^{j} (-1)^\ell \frac{j! \Gamma(k - \nu + \alpha + \ell + 1)}{(j - \ell)! (\ell)! \Gamma(\ell + \alpha + 1)}. \tag{31}$$

Employing (29)–(31) yields

$$D^\nu L_i^{(\alpha)}(x) = \sum_{j=0}^{N} S_\nu(i, j) L_j^{(\alpha)}(x), \quad i = \lceil \nu \rceil, \ldots, N, \tag{32}$$

where

$$S_\nu(i, j) = \sum_{k=\lceil \nu \rceil}^{i} \sum_{\ell=0}^{j} \left((-1)^{k+\ell} j! \Gamma(i + \alpha + 1) \right.$$

$$\times \Gamma(k - \nu + \alpha + \ell + 1) \right)$$

$$\left((i - k)! (j - \ell)! \ell! \right.$$

$$\times \Gamma(k - \nu + 1) \Gamma(k + \alpha + 1)$$

$$\left. \times \Gamma(\alpha + \ell + 1) \right)^{-1}. \tag{33}$$

Accordingly, (32) can be written in a vector form as follows:

$$D^\nu L_i^{(\alpha)}(x)$$

$$\simeq [S_\nu(i, 0), S_\nu(i, 1), S_\nu(i, 2), \ldots, S_\nu(i, N)] \phi(x), \tag{34}$$

$$i = \lceil \nu \rceil, \ldots, N.$$

Also according to Lemma 3, we can write

$$D^\nu L_i^{(\alpha)}(x) \simeq [0, 0, 0, \ldots, 0] \phi(x), \quad i = 0, 1, \ldots, \lceil \nu \rceil - 1. \tag{35}$$

A combination of (34) and (35) leads to the desired result. $\square$

Remark 5. In the case of $\nu = n \in N$, Theorem 4 gives the same result as (23).

4. Applications of GLOM for Multiterm FDEs

In this section, we are interested in using GLOM in combination with two types of spectral methods for solving linear and nonlinear FDEs.

4.1. Linear Multiorder FDEs. Here, we propose a direct solution technique to approximate linear multi-term FDEs with constant coefficients using the generalized Laguerre tau method in combination with GLOM.

Consider the linear multi-order FDE

$$D^\nu u(x) = \sum_{j=1}^{k} \gamma_j D^{\beta_j} u(x) + \gamma_{k+1} u(x) + g(x), \quad \text{in } \Lambda, \tag{36}$$

with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0, \ldots, m - 1, \tag{37}$$

where γ_j $(j = 1, \ldots, k + 1)$ are constants and $m - 1 < \nu \le m$, $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Also $D^\nu u(x) \equiv u^{(\nu)}(x)$ is ν order of Caputo fractional derivative for $u(x)$, d_i $(i = 0, \ldots, m-1)$ are the initial values, and $g(x)$ is a source function.

To solve the fractional FDE (36)-(37), we approximate $u(x)$ and $g(x)$ by generalized Laguerre polynomials as

$$u(x) \simeq \sum_{i=0}^{N} c_i L_i^{(\alpha)}(x) = C^T \phi(x), \tag{38}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i L_i^{(\alpha)}(x) = G^T \phi(x), \tag{39}$$

where vector $G = [g_0, \ldots, g_N]^T$ is known.

By using Theorem 4 (relation equations (26), and (38)) we have

$$D^\nu u(x) \simeq C^T D^\nu \phi(x) = C^T D^{(\nu)} \phi(x),$$

$$D^{\beta_j} u(x) \simeq C^T D^{\beta_j} \phi(x) = C^T D^{(\beta_j)} \phi(x), \quad j = 1, \ldots, k. \tag{40}$$

Making use of (38)–(40), the residual $R_N(x)$ for (36) can be given from

$$R_N(x) = \left(C^T D^{(\nu)} - C^T \sum_{j=1}^{k} \gamma_j \mathbf{D}^{(\beta_j)} - \gamma_{k+1} C^T - G^T \right) \phi(x). \tag{41}$$

The use of generalized Laguerre tau approximation generates $(N - m + 1)$ system of linear equations

$$\left\langle R_N(x), L_j^{(\alpha)}(x) \right\rangle$$

$$= \int_0^\infty w(x) R_N(x) L_j^{(\alpha)}(x) \, dx = 0, \qquad (42)$$

$$j = 0, 1, \ldots, N - m.$$

Substituting (23) and (38) into (37) generates m set of linear equations

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)} \phi(0) = d_i, \quad i = 0, 1, \ldots, m - 1. \qquad (43)$$

The combination of (42) and (43) reduces the solution of (36)-(37) to a linear system of algebraic equations, which can be solved for unknown coefficients of the vector C by any direct solver technique to find the spectral solution $u(x)$.

4.2. Nonlinear Multiorder FDEs. In this section, we present the generalized Laguerre pseudo-spectral approximation in combination with GLOM of fractional derivative to find the approximate solution $u_N(x)$.

Let us consider the nonlinear multi-term FDE

$$D^\gamma u(x) = F\left(x, u(x), D^{\beta_1} u(x), \ldots, D^{\beta_k} u(x)\right), \quad \text{in } \Lambda, \qquad (44)$$

subject to the nonhomogeneous initial conditions (37), where F can be nonlinear in general. In [30], the authors studied the existence of solutions of a class of nonlinear FDEs.

Now, we will implement the generalized Laguerre operational matrix for treating this nonlinear problem. To do this, firstly, we approximate $u(x)$, $D^\gamma u(x)$, and $D^{\beta_j} u(x)$, for $j = 1, \ldots, k$ by using (38), (40), respectively, and then the operational matrices formulation of (44) can be expressed as

$$C^T \mathbf{D}^{(\gamma)} \phi(x)$$

$$\simeq F\left(x, C^T \phi(x), \ C^T \mathbf{D}^{(\beta_1)} \phi(x), \ldots, C^T \mathbf{D}^{(\beta_k)} \phi(x)\right). \qquad (45)$$

Also, making use of (38) and (23) in (44) yields

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)} \phi(0) = d_i, \quad i = 0, 1, \ldots, m - 1. \qquad (46)$$

Collocating the operational matrix equation (45) at $(N-m+1)$ nodes of the generalized Laguerre-Gauss quadrature on Λ,

$$C^T \mathbf{D}^{(\gamma)} \phi\left(x_j^{(\alpha)}\right)$$

$$\simeq F\left(x, C^T \phi\left(x_j^{(\alpha)}\right), C^T \mathbf{D}^{(\beta_1)} \phi\left(x_j^{(\alpha)}\right), \qquad (47)$$

$$\ldots, C^T \mathbf{D}^{(\beta_k)} \phi\left(x_j^{(\alpha)}\right)\right).$$

Combining $(N - m + 1)$ algebraic equations (47) with m initial conditions (46) generates a system of $(N + 1)$ nonlinear algebraic equations. This system may be evaluated by implementing Newton's iterative method to find the spectral solution $u(x)$.

5. Numerical Results

This section considers several numerical examples to demonstrate the accuracy and applicability of the proposed spectral algorithms based on operational matrix of fractional derivatives of generalized Laguerre polynomials. A comparison of the results obtained by adopting different choices of the generalized Laguerre parameter α reveals that the present algorithms are very convenient for all choices of α and produces accurate solutions to multi-term FDEs on semi-infinite interval.

Example 1. Consider the linear FDE

$$D^2 u(x) + D^{3/2} u(x) + u(x) = 1 + x,$$

$$u(0) = 1, \qquad u'(0) = 1, \qquad x \in \Lambda, \qquad (48)$$

where $u(x) = 1 + x$ is the exact solution.

If we apply the operational matrix formulation, the generalized Laguerre spectral tau method with $N = 2$, we get

$$u(x) = c_0 L_0^{(\alpha)}(x) + c_1 L_1^{(\alpha)}(x) + c_2 L_2^{(\alpha)}(x) = C^T \phi(x). \qquad (49)$$

Here, we have

$$\mathbf{D}^{(1)} = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(3/2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ S_{3/2}(2,0) & S_{3/2}(2,1) & S_{3/2}(2,2) \end{pmatrix}, \qquad (50)$$

$$G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix},$$

where g_j and $S_\nu(i, j)$ are defined in (18) and (27).

Applying variational formulation of the tau method of (48) yields

$$c_0 + c_2 + S_{3/2}(2, 0) = g_0. \qquad (51)$$

The treatment of initial conditions using (43) gives

$$c_0 + (\alpha + 1) c_1 + \frac{(\alpha + 1)(\alpha + 2)}{2} c_2 = 1,$$

$$-c_1 - (\alpha + 2) c_2 - 1 = 0. \qquad (52)$$

Solving the resulted system of algebraic equations (51)-(52) provides the unknown coefficients in terms of the parameter α:

$$c_0 = \alpha + 2, \qquad c_1 = -1, \qquad c_2 = 0. \qquad (53)$$

TABLE 1: c_0, c_1, and c_2 for different values of α for Example 1.

α	c_0	c_1	c_2
−0.5	1.5	−1	0
0	2	−1	0
0.5	2.5	−1	0
1	3	−1	0
2	4	−1	0
3	5	−1	0

TABLE 2: c_0, c_1, and c_2 for different values of α for Example 2.

α	c_0	c_1	c_2
−0.5	0.75	−3	2
0	2	−4	2
0.5	3.75	−5	2
1	6	−6	2
2	12	−8	2
3	20	−10	2

Accordingly, the approximate solution can be written as

$$u(x) = (c_0, c_1, c_2) \begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ L_2^{(\alpha)}(x) \end{pmatrix} = 1 + x, \qquad (54)$$

which is the exact solution.

Table 1 lists the values of c_0, c_1, and c_2 with different choices of (α). Indeed, we can achieve the exact solution of this problem with all choices of the generalized Laguerre parameters α.

Example 2. Consider linear FED

$$D^2 u(x) + D^{1/2} u(x) + u(x)$$
$$= x^2 + 2 + \frac{2.6666666667}{\Gamma(0.5)} x^{1.5}, \qquad (55)$$
$$u(0) = 0, \qquad u'(0) = 0, \qquad x \in \Lambda.$$

The analytical solution is $u(x) = x^2$.

The use of technique described in Section 4.1 with $N = 2$ enables one to approximate the solution as

$$u(x) = c_0 L_0^{(\alpha)}(x) + c_1 L_0^{(\alpha)}(x) + c_2 L_2^{(\alpha)}(x) = C^T \phi(x). \quad (56)$$

Here, we have

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(1/2)} = \begin{pmatrix} 0 & 0 & 0 \\ S_{1/2}(1,0) & S_{1/2}(1,1) & S_{1/2}(1,2) \\ S_{1/2}(2,0) & S_{1/2}(2,0) & S_{1/2}(2,0) \end{pmatrix}, \quad (57)$$

$$G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}.$$

Therefore using (42), we obtain

$$c_0 + S_{1/2}(1,0) c_1 + [S_{1/2}(2,0) + 1] c_2 = 6. \qquad (58)$$

Now, making use of (43) yields

$$c_0 + (\alpha + 1) c_1 + \frac{(\alpha + 1)(\alpha + 2)}{2} c_2 = 0.$$
$$-c_1 - (\alpha + 2) c_2 = 0. \qquad (59)$$

Finally by solving (58)-(59), then, we get

$$c_0 = \alpha^2 + 3\alpha + 2, \qquad c_1 = -2\alpha - 4, \qquad c_2 = 2. \quad (60)$$

Thus we can write

$$u(x) = (c_0, c_1, c_2) \begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ L_2^{(\alpha)}(x) \end{pmatrix} = x^2, \qquad (61)$$

which is the exact solution.

In Table 2, we exhibit the values of c_0, c_1 and c_2 with different choices of (α).

Example 3. Consider linear initial value problem of fractional order (see [31])

$$D^2 u(x) - 2Du(x) + D^{1/2} u(x) + u(x)$$
$$= x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}} x^{2.5}, \qquad (62)$$
$$u(0) = 0, \qquad u'(0) = 0, \qquad x \in \Lambda,$$

whose exact solution is given by $u(x) = x^3$.

If we apply the operational matrix formulation, the generalized Laguerre spectral tau method with $N = 2$, we get

$$u(x) = \sum_{i=0}^{3} c_i L_i^{(\alpha)}(x) = C^T \phi(x). \qquad (63)$$

Also,

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(1)} = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(1/2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ S_{1/2}(1,0) & S_{1/2}(1,1) & S_{1/2}(1,2) & S_{1/2}(1,3) \\ S_{1/2}(2,0) & S_{1/2}(2,1) & S_{1/2}(2,2) & S_{1/2}(2,3) \\ S_{1/2}(3,0) & S_{1/2}(3,1) & S_{1/2}(3,2) & S_{1/2}(3,3) \end{pmatrix},$$

$$G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

(64)

Therefore using (42), we obtain

$$c_0 + [2c_1 + S_{1/2}(1,0)] c_1$$
$$+ [3 + S_{1/2}(2,0)] c_2$$
$$+ [4 + S_{1/2}(3,0)] c_3 - g_0 = 0,$$

$$[1 + S_{1/2}(1,1)] c_1$$
$$+ [2 + S_{1/2}(2,1)] c_2$$
$$+ [3 + S_{1/2}(3,1)] c_3 - g_1 = 0,$$

$$C^T \phi(0) = c_0 + (\alpha + 1) c_1 + \frac{(\alpha+1)(\alpha+2)}{2} c_2$$
$$+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6} c_3 = 0,$$

$$C^T \mathbf{D}^{(1)} \phi(0) = -c_1 - (\alpha + 2) c_2 - \frac{(\alpha+3)(\alpha+2)}{2} c_3 = 0.$$

(65)

Accordingly, we get

$$c_0 = \alpha^3 + 6\alpha + 11\alpha + 6,$$
$$c_1 = -3\alpha^2 - 15\alpha - 18,$$
$$c_2 = 6\alpha + 18, \qquad c_3 = -6.$$

(66)

Thus we can write

$$u(x) = (c_0, \ c_1, \ c_2, \ c_3) \begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ L_2^{(\alpha)}(x) \\ L_3^{(\alpha)}(x) \end{pmatrix} = x^3,$$

(67)

which is the exact solution. The 4 unknown coefficients with various choices of α are listed in Table 3.

TABLE 3: $c_0, c_1, c_2,$ and c_3 for different values of α for Example 3.

α	c_0	c_1	c_2	c_3
−0.5	15/8	−45/4	15	−6
0	6	−18	18	−6
0.5	105/8	−105/4	21	−6
1	24	−36	24	−6
2	60	−60	30	−6
3	120	−90	36	−6

Example 4. Consider the following FDE:

$$D^{5/2} u(x) + D^2 u(x) - 2D^{1/2} u(x) + 4u(x) = g(x),$$
$$u(0) = 0, \qquad u'(0) = 0, \qquad x \in \Lambda,$$

(68)

where

$$g(x) = 4x^9 + 72x^7 - \frac{2\Gamma(10)}{\Gamma(19/2)} x^{17/2} + \frac{\Gamma(10)}{\Gamma(15/2)} x^{13/2}$$

(69)

and the exact solution is $u(x) = x^9$.

Now, if we use the spectral tau approximation based on with $N = 9$ and $x \in \Lambda$, then we obtain

$$u(x) = \sum_{i=0}^9 c_i L_i^{(\alpha)}(x) = x^9,$$

(70)

which is the exact solution.

Example 5. Consider the nonhomogeneous fractional initial value problem

$$D^2 u(x) + D^\nu u(x) + u(x) = g(x),$$
$$u(0) = 1, \qquad u'(0) = 0, \qquad x \in (0, 10),$$

(71)

where

$$g(x)$$
$$= \nu^6 \left(e^{((1/\nu^3)x)} + \sin\left(\frac{1}{\nu^3} x\right) \right) + e^{((1/\nu^3)x)} - \sin\left(\frac{1}{\nu^3} x\right)$$
$$+ \frac{1}{\Gamma(-\nu)} \int_0^x (x-t)^{-\nu-1} \left(e^{((1/\nu^3)x)} - \sin\left(\frac{1}{\nu^3} x\right) \right) dt$$

(72)

and the exact solution is given by $u(x) = e^{((1/\nu^3)x)} - \sin((1/\nu^3)x)$.

Table 4 introduces the maximum absolute errors, using the tau method based on GLOM of fractional derivative, at $\nu = 1.5$ and with different choices of the parameters α and N. The curves of exact solutions and approximate solutions obtained by the proposed method for $\alpha = 1$, $N = 10$, and $\nu = 1.5, 1.7, 1.9$ are shown in Figures 1 and 2. From these figures, the exact and approximate solutions are completely *conciet*.

TABLE 4: Maximum absolute error for various choices of α and N for Example 5.

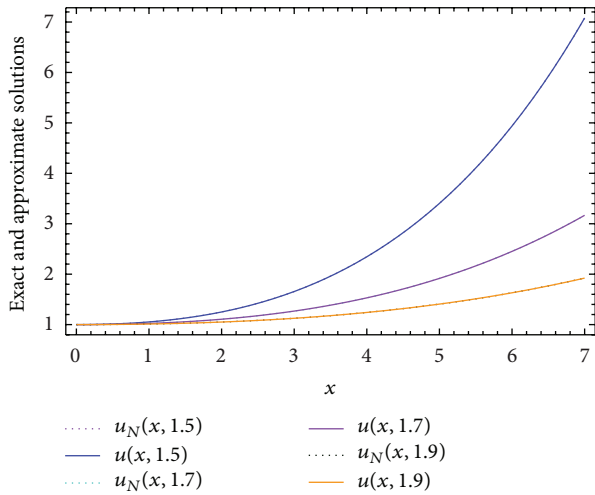| N	ν	α	GLOM	α	GLOM	α	GLOM
5	1.5	0	$1.69 \cdot 10^{-2}$	1	$4.41 \cdot 10^{-2}$	2	$1.07 \cdot 10^{-1}$
10			$1.08 \cdot 10^{-3}$		$4.17 \cdot 10^{-4}$		$6.52 \cdot 10^{-4}$
15			$4.99 \cdot 10^{-6}$		$1.46 \cdot 10^{-5}$		$1.33 \cdot 10^{-5}$
20			$3.57 \cdot 10^{-8}$		$1.22 \cdot 10^{-7}$		$2.41 \cdot 10^{-7}$
25			$3.50 \cdot 10^{-10}$		$1.30 \cdot 10^{-9}$		$4.07 \cdot 10^{-9}$
30			$4.03 \cdot 10^{-12}$		$1.55 \cdot 10^{-11}$		$3.47 \cdot 10^{-11}$
35			$5.10 \cdot 10^{-14}$		$2.08 \cdot 10^{-13}$		$6.50 \cdot 10^{-13}$
40			$6.87 \cdot 10^{-16}$		$2.64 \cdot 10^{-15}$		$5.29 \cdot 10^{-15}$

FIGURE 1: Comparing the exact solution and the approximate solutions at $N = 10$, $\alpha = 1$, and $\nu = 1.5, 1.7, 1.9$.

Example 6. Let us consider the nonlinear fractional initial value problem

$$D^2 u(x) + D^{1/2} u(x) + u^2(x) = g(x),$$
$$u(0) = 1, \qquad u'(0) = 0, \qquad x \in (0, 20), \tag{73}$$

where

$$g(x) = \cos^2(\gamma x) - \gamma^2 \cos(\gamma x)$$
$$+ \frac{1}{\Gamma(-1/2)} \int_0^x (x-t)^{-\gamma-1} \cos(\gamma t)\, dt \tag{74}$$

and the exact solution is given by $u(x) = \cos(\gamma x)$.

The solution of this problem is obtained by applying the generalized Laguerre-Gauss collocation method based on generalized Laguerre operational matrix. The absolute error between the exact and the approximate solution obtained by the proposed method $\gamma = 0.01$, $\alpha = 1$ and $N = 30$ is given in Figure 3.
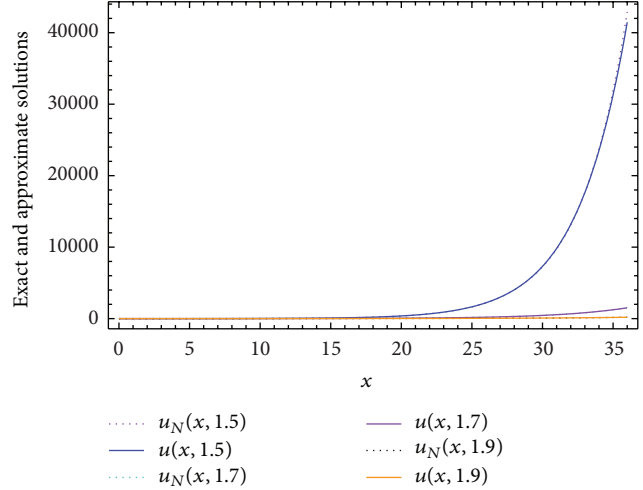

FIGURE 2: Comparing the exact solution and the approximate solutions at $N = 10$, $\alpha = 1$, and $\nu = 1.5, 1.7, 1.9$.
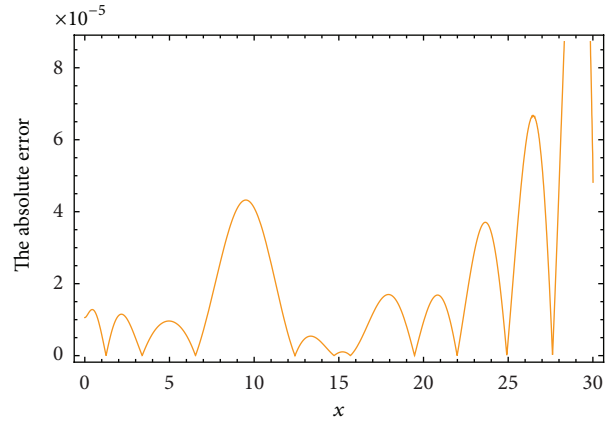

FIGURE 3: The absolute error for $\gamma = 1/100$, $\alpha = 1$ at $N = 30$.

Example 7. We consider the nonlinear fractional initial value problem

$$D^\xi u(x) + D^\eta u(x) D^\theta u(x) + u^2(x) = g(x),$$
$$u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = 0, \qquad x \in (0, 1),$$
$$\zeta \in (2, 3), \qquad \eta \in (1, 2), \qquad \theta \in (0, 1), \tag{75}$$

where

$$g(x) = x^6 + \frac{6x^{3-\xi}}{\Gamma(4-\xi)} + \frac{36 x^{6-\eta-\theta}}{\Gamma(4-\eta)\,\Gamma(4-\theta)} \tag{76}$$

and the exact solution is $u(x) = x^3$.

The solution of this problem is obtained by applying the generalized Laguerre-Gauss collocation method based on generalized Laguerre operational matrix for $\xi = 2.001$, $\eta = 1.001$, and $\theta = 0.001$. The exact solution and approximate
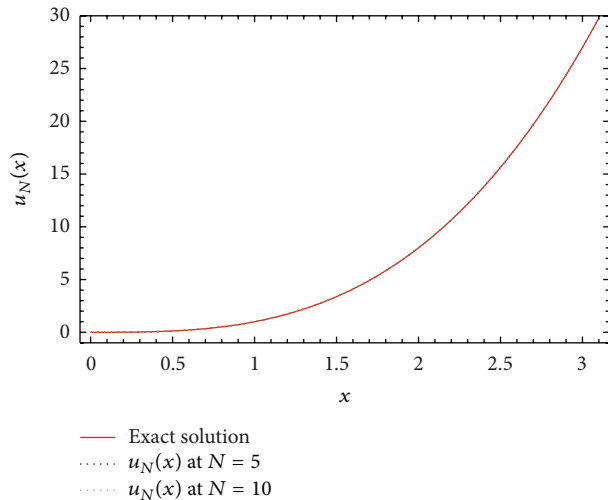
FIGURE 4: Comparing the exact solution and the approximate solutions at $N = 5, 10$ and $\alpha = 1$.

solutions obtained by the proposed method for $\alpha = 1$ and two choices of N are shown in Figure 4.

6. Conclusion

In this paper, the generalized Laguerre operational matrix of Caputo fractional derivative was derived. Furthermore, we have implemented the generalized Laguerre tau approximation in combination with the GLOM with the generalized Laguerre family to solve the linear FDEs. In addition, combining the pseudo-spectral approximation and the GLOM of fractional derivative was applied to develop an accurate approximate solution of nonlinear FDEs. The generalized Laguerre-Gauss quadrature points were used as a collocation nodes. This method reduces the nonlinear FDEs to a system of algebraic equation in the expansion coefficients which can be solved by any standard technique. The numerical results demonstrate that the proposed spectral algorithms are accurate and efficient.

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