Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 539736, 7 pages
http://dx.doi.org/10.1155/2013/539736

Research Article

Robust Density of Periodic Orbits for Skew Products with High Dimensional Fiber

Fatemeh Helen Ghane,[1] Mahboubeh Nazari,[1] Mohsen Saleh,[2] and Zahra Shabani[1]

[1] *Department of Mathematics, Ferdowsi University of Mashhad, Mashhad 91775-1159, Iran*
[2] *Department of Mathematics, University of Neyshabur, Neyshabur 93137 66835, Iran*

Correspondence should be addressed to Zahra Shabani; zahrashabani88@yahoo.com

Received 2 June 2013; Revised 13 August 2013; Accepted 9 September 2013

Academic Editor: Ondřej Došlý

We consider step and soft skew products over the Bernoulli shift which have an m-dimensional closed manifold as a fiber. It is assumed that the fiber maps Hölder continuously depend on a point in the base. We prove that, in the space of skew product maps with this property, there exists an open domain such that maps from this open domain have dense sets of periodic points that are attracting and repelling along the fiber. Moreover, robust properties of invariant sets of diffeomorphisms, including the coexistence of dense sets of periodic points with different indices, are obtained.

1. Introduction

In [1], Gorodetski and Ilyashenko studied certain properties of skew product maps over the Bernoulli shift and the Smale-Williams solenoid, with a fiber S^1. They provided an open set in the space of these skew products such that each mapping from this open set has a dense set of periodic orbits that are attracting and repelling along the fiber.

In this paper, we improve their results to skew product maps which have an m-dimensional closed manifold M as a fiber. Moreover, we prove that small perturbations of these skew products in the space of all diffeomorphisms have partially hyperbolic invariant sets. Also, they admit dense subsets of periodic points with different indices.

To be more precise, let us describe skew product maps which apply here in detail.

From now on, the ambient fiber space M will be an m-dimensional closed manifold and its metric is geodesic distance and the measure is the Riemannian volume.

Consider diffeomorphisms f_i, $i = 1, \ldots, k$, defined on M. The *iterated function system* $\mathscr{F}(M; f_1, \ldots, f_k)$ is the semigroup generated by $f_1, \ldots, f_k$, that is, the set of all maps $f_{t_j} \circ \cdots \circ f_{t_1}$, where $t_j, \ldots, t_1 \in \{1, \ldots, k\}$.

The $\mathscr{F}$*-orbit* of $x \in M$ is the set of points $f_{t_i} \circ \cdots \circ f_{t_1}(x)$, $t_j \geq 0$.

An iterated function system $F(M; f_1, \ldots, f_k)$ is called *minimal* if each closed subset A with $f_i(A) \subset A$, for all i, is empty or coincides with M. This means that $\mathscr{F}$-orbit of each $x \in M$ is dense in M.

Let f_i, $i = 0, 1$, be diffeomorphisms of M. A *step skew product* over the Bernoulli shift $\sigma : \Sigma^2 \to \Sigma^2$ is defined by

$$F : \Sigma^2 \times M \longrightarrow \Sigma^2 \times M; \qquad (\omega, x) \longrightarrow \left(\sigma\omega, f_{w_0}(x)\right), \tag{1}$$

where Σ^2 is the space of two-sided sequences of 2 symbols $\{0, 1\}$. Consider the following standard metric on Σ^2:

$$d\left(\omega, \omega'\right) = 2^{-n}, \tag{2}$$

where $n = \min\{|k|; \omega_k \neq \omega'_k\}$ and $\omega, \omega' \in \Sigma^2$.

Let us note that an iterated function system can be embedded in a single dynamical system, the skew product F of the form (1), such that the action orbits of the iterated function system $\mathscr{F}$ with generators f_i coincide with the projections of positive semitrajectories of the skew product F onto the fiber along the base.

A *soft skew product* over the Bernoulli shift is a map

$$G : \Sigma^2 \times M \longrightarrow \Sigma^2 \times M; \qquad (\omega, x) \longrightarrow (\sigma\omega, g_\omega(x)), \tag{3}$$

where the fiber maps g_ω are diffeomorphisms of the fiber into itself.

We would like to mention that in contrast to step skew products, the fiber maps of soft skew products depend on the whole sequence ω.

Skew products play an important role in the theory of dynamical systems. Many properties observed for these products appear to persist as properties of diffeomorphisms [1, 2].

Let w be a finite segment on the alphabets $\{0,1\}$. We denote by $\{\cdots \mid w\cdots\}$ an arbitrary infinite sequence ω in which w occurs starting from the zeroth position. In a similar way, we introduce the notation $\{\cdots w \mid \cdots\}$ and $\{\cdots w \mid w'\cdots\}$. We also denote by $|w|$ the length of w.

We recall that a map F is called *topologically mixing* if for each nonempty open sets $U, V \in \Sigma^2 \times M$, $F^n(U)$ intersects with V for all large enough $n \in \mathbb{N}$.

For a diffeomorphism f of M, a compact f-invariant set Λ has a *dominated splitting* if

$$T_\Lambda M = E_1 \oplus \cdots \oplus E_k, \tag{4}$$

where each E_i is nontrivial and Df-invariant for $1 \le i \le k$ and there exists an $m \in \mathbb{N}$ such that

$$\left\| Df^n|_{E_i(x)} \right\| \left\| \left(Df^n|_{E_j(x)} \right)^{-1} \right\| \le \frac{1}{2}, \tag{5}$$

for every $n \ge m$, $i > j$ and $x \in \Lambda$.

The set Λ is *partially hyperbolic* if it has a dominated splitting

$$T_\Lambda M = E_1 \oplus \cdots \oplus E_k \tag{6}$$

and there exists some $n \in \mathbb{N}$ such that Df^n either uniformly contracts E_1 or uniformly expands E_k.

We are now ready to state our main results. The first result describes the robust density of attracting and repelling periodic orbits along the fiber.

Theorem 1. *There exist C^1 diffeomorphisms $f_i : M \to M$, $i = 0, 1$, and C^1-neighborhoods $U_0(f_0)$, $U_1(f_1) \subset Diff^1(M)$ such that for any $g_0 \in U_0$ and $g_1 \in U_1$, the periodic orbits of the step skew product F of the form* (1) *with the fiber maps $g_i, i = 0, 1$, which are attracting (or repelling) along M, are dense in $\Sigma^2 \times M$.*

By applying the Hölder property, one can translate the properties of step skew products to the case of soft skew products.

Theorem 2. *There exist diffeomorphisms f_0 and f_1 on any m-dimensional closed manifold M, and C^2 neighborhoods $U_0(f_0)$, $U_1(f_1) \subset Diff^2(M)$ such that, for each $C > 1$ and $\alpha > 0$, if a soft skew product map G of the form* (3) *satisfies the following conditions:*

(1) $g_\omega \in U_{\omega_0}$, *for any* $\omega \in \Sigma^2$,

(2) $d_{C^1}(g_\omega, g_{\omega'}) \le C(d_{\Sigma^2}(\omega, \omega'))^\alpha$, *for* $\omega, \omega' \in \Sigma^2$,

(3) $L \cdot 2^{-\alpha} < 1$,

then the periodic orbits of G which are attracting (or repelling) along the fiber are dense in $\Sigma^2 \times M$.

Now by using the smooth realizations of step skew products, we prove that the above properties are preserved under small perturbations of these products in the space of C^2 diffeomorphisms.

Theorem 3. *Let n and m be positive integers with $n \ge m + 3$, $n \ge 5$, and $m \ge 1$. Suppose that N is an n-dimensional closed manifold. Then there exists an open set $\mathcal{U} \subset Diff^2(N)$ such that, for any $f \in \mathcal{U}$, there is a partially hyperbolic locally maximal invariant set $\Delta \subset N$ and two numbers l_1 and $l_2 = l_1 + m$, such that the hyperbolic periodic orbits with stable manifolds of dimension l_i are dense in Δ.*

2. Step Skew Products

This section is devoted to prove Theorem 1. We will show that there exists an open set $\mathcal{U}$ in the space of step skew product maps of the form (1) such that, for any map $F \in \mathcal{U}$, the periodic orbits of F which are attracting along M are dense in $\Sigma^2 \times M$. The same property holds for periodic orbits which are repelling along M.

First, let us recall some notations and definitions. We consider the iterations of step skew product map F. Clearly, for $n > 0$

$$\begin{aligned} F^n(\omega, x) &= \left(\sigma^n \omega, \overline{f}_n[w](x)\right), \\ F^{-n}(\omega, x) &= \left(\sigma^{-n}\omega, \overline{f}_{-n}[w](x)\right), \end{aligned} \tag{7}$$

where $\overline{f}_n[\omega] = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$, $\overline{f}_{-n}[\omega] = f^{-1}_{\omega_{-n}} \circ \cdots \circ f^{-1}_{\omega_{-1}}$, $\overline{f}_0[\omega] = \mathrm{id}$. A periodic orbit of a step skew product map F is determined by its initial point (ω, x), where $x \in M$ and $\omega \in \Sigma^2$ is a periodic sequence

$$\omega = \cdots www \cdots = (w), \tag{8}$$

with a finite zero-one segment $w = (w_0 \cdots w_{n-1})$. We say that a periodic orbit $((w), x)$ is *attracting along* M if $\|D\overline{f}_{|w|}[w](x)\| < 1$ and is *repelling along* M if $\|D\overline{f}_{|w|}[w](x)\| > 1$.

From now on, the ambient M is a compact connected m-dimensional manifold without boundary. Also, let $U, W \subset M$ be two disjoint open neighborhoods which are the domains of two local charts (W, φ), (U, ψ) of M. Take two gradient Morse-Smale vector fields on M, each of which possesses a unique hyperbolic repelling equilibrium q_i and a unique hyperbolic attracting equilibrium p_i, $i = 0, 1$, and finitely many saddle points r^i_j, $i = 0, 1$, $j = 1, \ldots, l$, contained in open domains $V_j \subset M \setminus (U \cup W)$.

Assume that the fixed points p_0 and q_1 are distinct points contained in U and p_1 and q_0 are also distinct points that are contained in W. Let f_0 and f_1 be their time-1 maps. Suppose that the mappings f_i, $i = 0, 1$, have no saddle connection. Also, we can choose the coordinate functions φ and ψ satisfying the following conditions.

(i) If we take $\widehat{f_i} := \psi \circ f_i \circ \psi^{-1}$, then $\widehat{f_i}$ are affine maps which are defined by

$$\widehat{f_0}(x_1, \ldots, x_m) = (\pm r x_m + s, r x_1, \ldots, r x_{m-1}),$$
$$\widehat{f_1}(x_1, \ldots, x_m) = \left(-a x_1, a x_2, \ldots, a x_{m-1}, -a x_m - 2\frac{s}{r}\right), \tag{9}$$

for constants $0 < r < 1$, $0 < s < a - 1$, $a > 1$ and $ar < 1$. We consider a minus sign for even m and a plus sign for odd m. By construction,

$$\widehat{f_0} \circ \widehat{f_1}(x_1, \ldots, x_m) = (\pm a r x_m - s, -a r x_1, a r x_2, \ldots, a r x_{m-1}), \tag{10}$$

is a contracting map.

(ii) If we take $\widetilde{f_i} := \varphi \circ f_i \circ \varphi^{-1}$, $i = 0, 1$, then $\widetilde{f_0} = \widehat{f_0}^{-1}$ and $\widetilde{f_1} = \widehat{f_0} \circ \widehat{f_1}^{-1} \circ \widehat{f_0}^{-1}$. So $\widetilde{f_0}^{-1} = \widehat{f_0}$ and $(\widetilde{f_0} \circ \widetilde{f_1})^{-1} = \widehat{f_0} \circ \widehat{f_1}$. Moreover, $\widetilde{f_1}$ is an affine contracting map.

Note that there is a compact invariant set $\Delta = \Delta_{\mathcal{F}} \subset U$ with nonempty interior which contains the fixed points p_0 and q_1, such that the acting of the iterated function system generated by $\{f_0, f_0 \circ f_1\}$ on Δ is minimal. Moreover, the iterated function system $\mathcal{F}(M; f_0, f_1)$ is C^1-robustly minimal (see [3] for more detail).

Put $h_0 := f_0$ and $h_1 := f_0 \circ f_1$. Let us define $\mathcal{L}(\Delta) = h_0(\Delta) \cup h_1(\Delta)$. Suppose that $\Delta_{\text{in}} \subset \Delta \subset \Delta_{\text{out}}$ are two open sets close to Δ on which h_0 and h_1 are contracting. Then

$$\Delta_{\text{in}} \subset \mathcal{L}(\Delta_{\text{in}}) \subset \Delta \subset \mathcal{L}(\Delta_{\text{out}}) \subset \Delta_{\text{out}}, \tag{11}$$

and $\mathcal{L}^i(\Delta_{\text{in}})$, $\mathcal{L}^i(\Delta_{\text{out}})$ converge to Δ in the Hausdorff topology, as $i \to \infty$, provided that the fiber maps f_i are sufficiently close to the identity map. This requires that the constants a and r are sufficiently close to 1.

Moreover, our construction shows that the iterated function system $\mathcal{F}(M; f_0^{-1}, f_1^{-1})$ is also minimal. Also, there exists a compact invariant set $\Delta' = \Delta'_{\mathcal{F}} \subset W$ that contains the fixed points q_0 and p_1 in its interior such that the iterated function system $\mathcal{F}(\Delta'; f_0^{-1}, (f_0 \circ f_1)^{-1})$ is minimal. In particular, there exist open sets $\Delta'_{\text{in}} \subset \Delta' \subset \Delta'_{\text{out}}$ satisfying the inclusion relations (11).

In the rest of this section, we fix the mappings f_i, $i = 0, 1$, satisfying all the properties mentioned above and we consider the skew product map

$$F : \Sigma^2 \times M \longrightarrow \Sigma^2 \times M, \qquad (\omega, x) \longmapsto \left(\sigma\omega, f_{\omega_0}(x)\right), \tag{12}$$

with the fiber maps f_i, $i = 0, 1$.

In [3], the authors proved that F is C^1-robustly topologically mixing on $\Sigma^2_{11} \times \Delta$, where $\Sigma^2_{11} \subset \Sigma^2$ is the set of all sequences from Σ^2 in which the segment “11” is not encountered to the right of any element.

Since f_i, $i = 0, 1$, are Morse-Smale diffeomorphisms with a unique attracting fixed point p_i and unique repelling fixed point q_i and they have not any saddle connection, so the stable and unstable sets $W^s(p_0, f_0)$ and $W^u(q_1, f_1)$ are open and dense subsets of M.

Lemma 4. *Consider the iterated function system $\mathcal{F}(M; f_0, f_1)$ as aforementioned. For every nonempty open set $U \subset M$, there exist $k \le k_0 \in \mathbb{N}$ and $\rho = \rho(U) > 0$ such that, for every ball $B \subset M$ of radius less than ρ, there exists a finite word $w = t_1 \cdots t_k$ on the alphabets $\{0, 1\}$ and with the length $k \le k_0$ such that $\overline{f}_k[w](B) \subset U$.*

Proof. Let $U \subset M$ be an open subset. Since the acting of $\mathcal{F}$ on M is minimal, for each $x \in M$ there exists a word $w(x)$ on the alphabets $\{0, 1\}$ such that $\overline{f}_{|w(x)|}[w(x)](x) \in U$. By continuity, there is a neighborhood V_x of x such that $\overline{f}_{|w(x)|}[w(x)](V_x) \subset U$.

Since M is compact, we can cover M by finitely many open sets V_{x_i}, $i = 1, \ldots, n$. We take k_0 as the maximum of the lengths of the words $w(x_i)$, $i = 1, \ldots, n$, and $\rho > 0$ the Lebesgue number of this covering. Then every ball $B \subset M$ of radius less than ρ is contained in some V_{x_i}. So there exists a word $w = t_1 \cdots t_k$ on the alphabets $\{0, 1\}$ of the length $k \le k_0$ such that $\overline{f}_k[w](B) \subset U$. □

Remark 5. Since the iterated function system $\mathcal{F}(M; f_0^{-1}, f_1^{-1})$ is minimal, we can apply the argument used in the proof of Lemma 4 to prove the following statement: for every nonempty open set $U \subset M$, there exists $l \le l_0 \in \mathbb{N}$ and $\varrho = \varrho(U) > 0$ such that, for every ball $B \subset M$ of radius less than ϱ, there exists a finite word $w = s_1 \cdots s_l$ on the alphabets $\{0, 1\}$ of the length $l \le l_0$ such that $f_{s_l}^{-1} \circ \cdots \circ f_{s_1}^{-1}(B) \subset U$.

In the following, we will use the notation

$$C_{\overline{\alpha}} = \left\{\omega \in \Sigma^2 \mid \omega_j = \alpha_j,\ -n \le j \le n - 1\right\}, \tag{13}$$

where $\overline{\alpha} = \alpha_{-n} \cdots \alpha_0 \cdots \alpha_{n-1}$ is a segment of the symbols $\{0, 1\}$.

The rest of this section is devoted to prove Theorem 1.

Proof. First, we will prove that the statement of Theorem 1 holds for the step skew product map F with generators f_0, f_1 which are introduced in the aforementioned. Note that the open sets $C_{\overline{\alpha}} \times U \subset \Sigma^2 \times M$, form a base of the topology of the space $\Sigma^2 \times M$ where $\overline{\alpha} = \alpha_{-n} \cdots \alpha_0 \cdots \alpha_{n-1}$ is a segment of $\{0, 1\}$, $C_{\overline{\alpha}}$ is the cylinder set corresponding to the segment $\overline{\alpha}$, and U is an open set of M.

Suppose that the segment $\overline{\alpha} = \alpha_{-n} \cdots \alpha_0 \cdots \alpha_{n-1}$ and open subset $U \subset M$ are given. We seek a periodic point $((\overline{\beta}), x) \in C_{\overline{\alpha}} \times U$ of the skew product map F which is attracting along M. From now on, we fix the open subset $C_{\overline{\alpha}} \times U \subset \Sigma^2 \times M$.

Let U_0 be an open ball which is contained in the basin of the attracting fixed point p_0 of f_0 such that $\|Df_0|_{U_0}\| \le \lambda < 1$, for some $0 < \lambda < 1$. By Lemma 4, there exist $\rho_0 := \rho_0(U_0)$ and $k_0 := k_0(U_0) \in \mathbb{N}$ such that, for every open neighborhood V of diameter less than ρ_0, there exists a word $w = w(V, U_0)$ on the alphabets $\{0, 1\}$ and with the length at most k, such that $\overline{f}_{|w|}[w](V) \subset U_0$.

Now the following statements hold.

(a) Consider an open ball $W \subset U$ of radius less than ρ_0/L^n. Take $W_{\alpha^+} := \overline{f}_n[\overline{\alpha}](W)$; then $\operatorname{diam}(W_{\alpha^+}) < \rho_0$. By Lemma 4, there exists a finite word $w = t_1 \cdots t_{l_1}$ on the alphabets $\{0,1\}$ of the length at most k_0, such that $\overline{f}_{l_1}[w](W_{\alpha^+}) \subset U_0$.

(b) Take $W_{\alpha^-} := \overline{f}_{-n}[\overline{\alpha}](W)$. So there exist $\rho_2 := \rho_2(W_{\alpha^-})$ and $k_2 := k_2(W_{\alpha^-}) \in \mathbb{N}$ satisfying the statement of Lemma 4.

Since U_0 is contained in the basin of attracting fixed point p_0 of f_0, so there exists a positive integer l_2 such that

$$\operatorname{diam}\left(f_0^{l_2}\left(\overline{f}_{l_1}[w](W_{\alpha^+})\right)\right) < \rho_2, \qquad L^{k_1+k_2}\lambda^{l_2} < 1. \tag{14}$$

By statement (b), there exists a word $w' = s_1 \cdots s_{l_3}$ on the alphabets $\{0,1\}$ and with the length $l_3 \le k_2$ such that $\overline{f}_{l_3}[w'](f_0^{l_2}(\overline{f}_{l_1}[w](W_{\alpha^+}))) \subset W_{\alpha^-}$.

We set $\overline{\beta} = \beta_{-m} \cdots \beta_{-1}\beta_0 \cdots \beta_{m-1}$, where

$$\begin{aligned}\beta_{-m} \cdots \beta_{-1} &= \beta_0 \cdots \beta_{m-1} \\ &= \alpha_0 \cdots \alpha_{n-1} t_1 \ldots t_{l_1} \underbrace{0 \cdots 0}_{l_2 \text{ times}} s_1 \cdots s_{l_3} \alpha_{-n} \cdots \alpha_{-1}\end{aligned} \tag{15}$$

and $m = l_1 + l_2 + l_3 + 2n$, which implies that $\overline{f}_{2m}[\overline{\beta}](W) \subset W$. Moreover, the choice of l_2 shows that $\|D\overline{f}_{2m}[\overline{\beta}]|_W\| < 1$.

According to these facts, there exists an attracting fixed point x for the mapping $\overline{f}_{2m}[\overline{\beta}]$ which is contained in $W \subset U$. So the periodic point $((\overline{\beta}), x)$ which is attracting along the fiber lies in $C_{\overline{\alpha}} \times U$.

Density of periodic orbits which are repelling along M can be established similarly.

Indeed, by applying Remark 5 and since the mapping f_1^{-1} is contracting on Δ', there exist an open set $W \subset U$ and a finite word $w'' = r_1 \cdots r_k$ on the alphabets $\{0,1\}$, such that

$$\begin{gathered} f_{r_k}^{-1} \circ \cdots \circ f_{r_1}^{-1} \circ \overline{f}_{-n}[\overline{\alpha}](W) \subset \overline{f}_n[\overline{\alpha}](W), \\ \left\| \left(\overline{f}_n[\overline{\alpha}]\right)^{-1} \circ f_{r_k}^{-1} \circ \cdots \circ f_{r_1}^{-1} \circ \overline{f}_{-n}[\overline{\alpha}]|_W \right\| < 1. \end{gathered} \tag{16}$$

So there exists an attracting fixed point y for the map

$$\begin{aligned} &\left(f_{\alpha_{-1}} \circ \cdots \circ f_{\alpha_{-n}} \circ \overline{f}_k[w''] \circ \overline{f}_n[\overline{\alpha}]\right)^{-1} \\ &\quad = \left(\overline{f}_n[\overline{\alpha}]\right)^{-1} \circ f_{r_k}^{-1} \circ \cdots \circ f_{r_1}^{-1} \circ \overline{f}_{-n}[\overline{\alpha}] \end{aligned} \tag{17}$$

which is contained in W.

Now, we take $\overline{\gamma} = \gamma_{-l} \cdots \gamma_{-1}\gamma_0 \cdots \gamma_{l-1}$, where

$$\gamma_{-l} \cdots \gamma_{-1} = \gamma_0 \cdots \gamma_{l-1} = \alpha_0 \cdots \alpha_{n-1} r_1 \cdots r_k \alpha_{-n} \cdots \alpha_{-1} \tag{18}$$

and $l = k + 2n$. Then $((\overline{\gamma}), y)$ is a periodic point for the skew product map F which is repelling along M and lies in $C_{\overline{\alpha}} \times U$.

Now, let us prove that the statement holds for small perturbations of F, that is, step skew product maps generated by small perturbations of f_0 and f_1. Choose $g_0 \in U_0$ and $g_1 \in U_1$, sufficiently close to f_0 and f_1 and consider the step skew product map G given by (1) and with the fiber maps g_i, $i = 0,1$. Therefore, g_i, $i = 0,1$, possesses a unique hyperbolic repelling fixed point close to q_i, $i = 0,1$, a unique hyperbolic attracting fixed point close to p_i, $i = 0,1$, and finitely many saddle points which are close to r_j^i, $i = 0,1$, $j = 1,\ldots,l$. Moreover, the iterated function system $\mathcal{G}(M; g_0, g_1)$ is minimal and admits an invariant set $\Delta = \Delta_{\mathcal{G}}$ with nonempty interior which contains the attracting fixed point of g_0 and the repelling fixed of g_1, such that $\mathcal{G}(\Delta; g_0, g_0 \circ g_1)$ is minimal. Moreover, the iterated function system $\mathcal{G}(M; g_0^{-1}, g_1^{-1})$ is also minimal. So similar reasoning implies the existence of an attracting (repelling) periodic orbit for the map G which is contained in $C_{\overline{\alpha}} \times U$. This terminates the proof of Theorem 1. □

3. Soft Skew Products

In this section, we prove Theorem 2. In fact, we describe the properties of soft skew product maps which have an m-dimensional closed manifold M as a fiber. To translate the properties of step skew product maps to the case of soft skew product maps, we need a Hölder property.

In the following, we provide an open set in the space of soft systems (3) with the Hölder property that has the same properties of step systems.

To be more precise, let us describe them in details.

First, note that if G is a soft skew product of the form (3), then it is obvious that, for $n \in \mathbb{N}$,

$$\begin{gathered} G^n(\omega, x) = \left(\sigma^n \omega, \overline{g}_n[w](x)\right), \\ G^{-n}(\omega, x) = \left(\sigma^{-n}\omega, \overline{g}_{-n}[w](x)\right), \end{gathered} \tag{19}$$

where

$$\begin{gathered} \overline{g}_n[\omega] = g_{\sigma^{n-1}\omega} \circ \cdots \circ g_{\sigma\omega} \circ g_{\omega}, \\ \overline{g}_{-n}[\omega] = g_{\sigma^{-n}\omega}^{-1} \circ \cdots \circ g_{\sigma^{-1}\omega}^{-1}, \qquad \overline{g}_0[\omega] = \mathrm{id}. \end{gathered} \tag{20}$$

Let f_0 and f_1 be two diffeomorphisms on M generating a robustly minimal iterated function system as in the previous section. Write $h_0 := f_0$, $h_1 := f_0 \circ f_1$ and let $\mathcal{F}$ be the iterated function system generated by h_0 and h_1. Recall that the iterated function system $\mathcal{F}$ acts minimally on a compact invariant set Δ. Also, there are open sets $\Delta_{\text{in}} \subset \Delta \subset \Delta_{\text{out}}$ on which

$$\Delta_{\text{in}} \subset \mathcal{F}(\Delta_{\text{in}}) \subset \Delta \subset \mathcal{F}(\Delta_{\text{out}}) \subset \Delta_{\text{out}}, \tag{21}$$

and h_0 and h_1 are contractions on Δ_{out}.

Moreover, our construction in Section 2 shows that the iterated function system $\mathcal{F}(M; f_0^{-1}, f_1^{-1})$ is also minimal. Also, there exists a compact invariant set Δ' which contains the attracting fixed point of f_1 and repelling fixed point of f_0 in its interior such that the iterated function system $\mathcal{F}(\Delta'; f_0^{-1}, (f_0 \circ f_1)^{-1})$ is minimal. In particular, there exist open sets $\Delta'_{\text{in}} \subset \Delta' \subset \Delta'_{\text{out}}$ satisfying the inclusion relations (21) corresponding to $\mathcal{F}(\Delta'; f_0^{-1}, (f_0 \circ f_1)^{-1})$.

Let F on $\Sigma^2 \times M$ be defined by

$$F(\omega, x) = \left(\sigma(\omega), h_{\omega_0}(x)\right), \tag{22}$$

where $(\sigma(\omega))_k = \omega_{k+1}$ is the left shift operator. Suppose that G is a soft skew product map of the form (3) such that g_ω depends continuously on ω and is uniformly close to h_{ω_0}, by a uniform bound $\delta > 0$. Then the inclusions (21) get replaced by

$$\Sigma^2 \times \Delta_{\text{in}} \subset G\left(\Sigma^2 \times \Delta_{\text{in}}\right), \qquad \left(\Sigma^2 \times \Delta_{\text{out}}\right) \subset \Sigma^2 \times \Delta_{\text{out}}, \tag{23}$$

for sufficiently small δ. Moreover, the choice of Δ_{in} can be independent of skew product map G. This means that if G is any soft skew product of the form (3) with the fiber maps g_ω, with $d_{C^1}(g_\omega, h_{\omega_0}) < \delta$, for any $\omega \in \Sigma^2$, then the inclusions (23) hold for G. By the argument used in the proof of [3, Proposition 5.1], the next lemma follows; see also [4, Proposition 5.1].

Lemma 6. *Let F be the step skew product map as in the aforementioned and by fiber maps h_i, $i = 0, 1$. Then any soft skew product map G of the form* (3) *which is sufficiently close to F possesses a maximal invariant set $\Lambda_G \subset \Sigma^2 \times \Delta_{out}$ on which the acting G is topologically mixing. Moreover, there is an open set Δ_{in} such that for any soft system G, $\Delta_{in} \subset \pi(\Lambda_G)$, where $\pi : \Sigma^2 \times M \to M$ is the natural projection.*

Since the diffeomorphisms f_i, $i = 0, 1$, are Morse-Smale and the set of all Morse-Smale diffeomorphisms is open subset of $\text{Diff}^2(M)$, so we can choose two neighborhoods $U_0(f_0)$, $U_1(f_1) \subset \text{Diff}^2(M)$ sufficiently small such that the following statements hold.

If G is a soft skew product of the form (3) with fiber maps $g_\omega \in U_{\omega_0}(f_{\omega_0})$, $\omega \in \Sigma^2$, then

(i) the mapping g_ω has one hyperbolic attracting fixed point $p(\omega)$, one hyperbolic repelling fixed point $q(\omega)$, and finitely many saddle points $r_i(\omega)$, $i = 1, \dots, l$;

(ii) all attracting fixed points of the mappings g_ω, with $\omega_0 = 0$, and all repelling fixed points of the mappings g_ω, with $\omega_0 = 1$, lie strictly inside Δ_{in};

(iii) all attracting fixed points of the mappings g_ω, with $\omega_0 = 1$, and all repelling fixed points of the mappings g_ω, with $\omega_0 = 0$, lie strictly inside Δ'_{in};

(iv) stable sets $W^s(p_\omega, g_\omega)$ are open and dense subsets of M, for any $\omega \in \Sigma^2$ with $\omega_0 = 0$;

(v) unstable sets $W^u(q_\omega, g_\omega)$ are open and dense subsets of M, for any $\omega \in \Sigma^2$ with $\omega_0 = 1$.

We say that the soft skew product map G is *controllable* if its fiber maps g_ω, $\omega \in \Sigma^2$, satisfying the assumptions of Theorem 2 and all of the properties mentioned above.

In the following, we establish the density of periodic points of a controllable soft skew product map G which are attracting along the fiber M.

Indeed, we will find a periodic point in any open set of the form $C_{\overline{\alpha}} \times U \subset \Sigma^2 \times M$, where $\overline{\alpha} = \alpha_{-n} \cdots \alpha_0 \cdots \alpha_{n-1}$ is a finite segment of the alphabets $\{0, 1\}$, $C_{\overline{\alpha}}$ is the cylinder set corresponding to it, and U is an open subset of M.

First, we need the following lemma which controls the error in the coordinate along the fiber. It is obtained by an argument used in [1, Lemma 3.1].

Lemma 7. *Let G be a controllable soft skew product map. Then there exists $K > 0$, with $K = K(L, C, \alpha)$ and being independent of $\delta > 0$, such that, for any $m \in \mathbb{N}$, the inequality $d_{\Sigma^2}(\omega, \omega') \leq 2^{-m}$ implies*

$$d_{C^0}\left(\overline{g}_{\pm m}[\omega], \overline{g}_{\pm m}[\omega']\right) \leq \gamma := K\delta^\beta, \tag{24}$$

where $\beta = 1 - \ln L / \ln 2^\alpha$.

According to Lemma 7, for each controllable soft skew product G with the fiber maps g_ω,

$$\text{diam}\left\{\overline{g}_{\pm m}[\omega](x) \mid \omega = \{\cdots w^\star \cdots\}\right\} \leq \gamma, \tag{25}$$

for any $x \in M$, any $m \in \mathbb{N}$, and any finite word $w^\star = w_{-m} \cdots w_{-1} \cdot w_0 \cdots w_m$.

Let us note that if $\delta > 0$ is sufficiently small, then $\gamma > 0$ is also small enough. By Lemma 6, the controllable soft skew product G is topologically mixing on $\Sigma^2_{11} \times \Delta$, where $\Sigma^2_{11} \subset \Sigma^2$ is the set of all sequences from Σ^2 in which the segment "11" is not encountered to the right of any element.

We now begin the proof of Theorem 2.

Proof. Suppose that the segment $\overline{\alpha} = \alpha_{-n} \cdots \alpha_0 \cdots \alpha_{n-1}$ and open neighborhood $U \subset M$ are given. Our aim is to find a periodic point in $C_{\overline{\alpha}} \times U$, where $C_{\overline{\alpha}}$ is the cylinder set corresponding to $\overline{\alpha}$.

We recall that the stable sets $W^s(p_\omega, g_\omega)$ are open and dense subsets of manifold M, for any $\omega \in \Sigma^2$ with $\omega_0 = 0$, so

$$\overline{g}_m[\omega](U) \cap W^s\left(p_{\sigma^m\omega}, g_{\sigma^m\omega}\right) \neq \emptyset, \tag{26}$$

for any $m \in \mathbb{N}$. This implies that there exists a neighborhood $U^1_\omega \subset U$, such that $\overline{g}_n[\omega](U^1_\omega) \subset W^s(p_{\sigma^n\omega}, g_{\sigma^n\omega})$, for any sequence $\omega = \{\cdots \mid \alpha_0 \cdots \alpha_{n-1} 0 \cdots\}$.

Similarly, $\overline{g}_{n+1}[\omega](U^1_\omega) \cap W^s(p_{\sigma^{n+1}\omega}, g_{\sigma^{n+1}\omega}) \neq \emptyset$, which implies that there is a neighborhood $U^2_\omega \subset U^1_\omega$, such that $\overline{g}_{n+1}[\omega](U^2_\omega)$ is contained in $W^s(p_{\sigma^{n+1}\omega}, g_{\sigma^{n+1}\omega})$, for any sequence $\omega = \{\cdots \mid \alpha_0 \cdots \alpha_{n-1} 00 \cdots\}$.

By continuing the above procedure, we obtain neighborhoods

$$U^k_\omega \subset U^{k-1}_\omega \subset \cdots U^1_\omega \subset U \tag{27}$$

such that

$$\overline{g}_{n+k-1}[\omega]\left(U^k_\omega\right) \subset W^s\left(p_{\sigma^{n+k-1}\omega}, g_{\sigma^{n+k-1}\omega}\right), \tag{28}$$

for any sequence

$$\omega = \Big\{\cdots \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k} \cdots\Big\}. \tag{29}$$

Since attracting fixed points of mappings g_ω, for any $\omega \in \Sigma^2$, are contained in Δ_{in}, so by increasing k, the subset $\overline{g}_{n+k}[\omega](U_\omega^k)$ intersects with Δ_{in}. Therefore, there exists a positive integer k_0 such that $\overline{g}_{n+k_0}[\omega](U_\omega^{k_0}) \cap \Delta_{\text{in}} \neq \emptyset$. Also, there is an open set $\widetilde{U}_\omega \subset U_\omega^{k_0}$ such that $\overline{g}_{n+k_0}[\omega](\widetilde{U}_\omega) \subset \Delta_{\text{in}}$, for any sequence $\omega = \{\cdots \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} \cdots\}$.

By shrinking $\overline{g}_{n+k_0}[\omega](\widetilde{U}_\omega)$, we can control the error in the coordinate along the fiber. To do this, we note that the map g_ω, with $\omega_0 = 0$, and the map $\overline{g}_2[\omega]$, with $\omega_0 = 1, \omega_1 = 0$, are contracting on Δ_{in}, so there exists a finite word $T = t_1 \cdots t_{l_1}$ such that $\overline{g}_{n+k_0+l_1}[\omega](\widetilde{U}_\omega)$ is contained in an open ball U_ω^+ of Δ_{in} with diameter 2γ, for any $\omega = \{\cdots \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\}$.

Analogously, since the unstable subsets $W^u(p_\omega, g_\omega)$ are open and dense subsets of manifold M, for any $\omega \in \Sigma^2$ with $\omega_0 = 1$, so

$$\overline{g}_{-m}[\omega](U) \cap W^s\left(q_{\sigma^{-m-1}\omega}, g^{-1}_{\sigma^{-m-1}\omega}\right) \neq \emptyset, \tag{30}$$

for any $m \in \mathbb{N}$ and $\omega \in \Sigma^2$ with $\omega_{-m-1} = 1$. This implies that there exists a neighborhood $W_\omega^1 \subset \widetilde{U}_\omega$, such that $\overline{g}_{-n}[\omega](W_\omega^1) \subset W^s(q_{\sigma^{-n-1}\omega}, g^{-1}_{\sigma^{-n-1}\omega})$, for any sequence $\omega = \{\cdots 1\alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\}$.

Similarly, $\overline{g}_{-n-1}[\omega](W_\omega^1) \cap W^s(q_{\sigma^{-n-2}\omega}, g^{-1}_{\sigma^{-n-2}\omega}) \neq \emptyset$, so there exists a neighborhood $W_\omega^2 \subset W_\omega^1$, such that $\overline{g}_{-n-1}[\omega](W_\omega^2) \subset W^s(q_{\sigma^{-n-2}\omega}, g^{-1}_{\sigma^{-n-2}\omega})$, for any $\omega = \{\cdots 11\alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\}$.

By induction, we obtain neighborhoods

$$W_\omega^m \subset W_\omega^{m-1} \subset \cdots W_\omega^1 \subset \widetilde{U}_\omega \tag{31}$$

such that

$$\overline{g}_{-n-m+1}[\omega](W_\omega^m) \subset W^s\left(q_{\sigma^{-n-m}\omega}, g^{-1}_{\sigma^{-n-m}\omega}\right), \tag{32}$$

for any sequence of the form

$$\omega = \left\{\cdots \underbrace{1 \cdots 1}_{m} \alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\right\}. \tag{33}$$

Since repelling fixed points of mappings g_ω, for any $\omega \in \Sigma^2$ with $\omega_0 = 1$, are contained in Δ_{in}, so by increasing m, $\overline{g}_{-n-m}[\omega](W_\omega^m)$ intersects with Δ_{in}; therefore, there exist a positive integer m_0 such that $\overline{g}_{-n-m_0}[\omega](W_\omega^{m_0}) \cap \Delta_{\text{in}} \neq \emptyset$ and an open set $\widetilde{W}_\omega \subset W_\omega^{m_0}$ such that $\overline{g}_{-n-m_0}[\omega](\widetilde{W}_\omega) \subset \Delta_{\text{in}}$, for any sequence

$$\omega = \left\{\cdots \underbrace{1 \cdots 1}_{m_0} \alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\right\}. \tag{34}$$

The construction shows that the mapping $\overline{g}_{-1}[\omega]$, with $\omega_{-1} = 0$, and the mapping $\overline{g}_{-2}[\omega]$, with $\omega_{-1} = 0$ and $\omega_{-2} = 1$, are expanding on $\Delta_{\text{in}} \subset \Delta$, so there exists a finite word $S = s_{l_0} \cdots s_1$ such that, for any sequence ω of the form

$$\omega = \left\{\cdots S \underbrace{1 \cdots 1}_{m_0} \alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\right\}, \tag{35}$$

$\overline{g}_{-(n+m_0+l_0)}[\omega](\widetilde{W}_\omega)$ contains an open ball W_ω^- of diameter 6γ.

Note that by shrinking the C^2-neighborhoods $U_0(g_0)$, $U_1(g_1) \subset \text{Diff}^2(M)$, if it is necessary, we may assume that $6\gamma < \text{diam}(\Delta_{\text{in}})$.

Since $\widetilde{W}_\omega \subset \widetilde{U}_\omega$, the subset $\overline{g}_{n+k_0+l_1}[\omega](\widetilde{W}_\omega)$ is contained in an open ball U_ω^+ of Δ_{in} with diameter 2γ, for any sequence of the form

$$\omega = \left\{\cdots S \underbrace{1 \cdots 1}_{m_0} \alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} T \cdots\right\}. \tag{36}$$

We recall that the acting of G is topologically mixing on $\Sigma_{11}^2 \times \Delta$, so there exists a finite word $R_\omega = r_1 \cdots r_{k_\omega} \in \Sigma_{11}^2$, $k_\omega > k_0$, such that, for any sequence $\omega = \{\cdots \mid R_\omega \cdots\}$, $B_\gamma(\overline{g}_{k_\omega}[\omega](U_\omega^+)) \subset W_\omega^-$.

Take the segment

$$w := S \underbrace{1 \cdots 1}_{m_0} \alpha_{-n} \cdots \alpha_{-1} \mid \alpha_0 \cdots \alpha_{n-1} \underbrace{0 \cdots 0}_{k_0} TR \tag{37}$$

and the periodic sequence $\widetilde{\beta} := (w)$.

Now the constructions show that $\overline{g}_{n+k_0+l_1}[\widetilde{\beta}](\widetilde{W}_{\widetilde{\beta}})$ is contained in an open ball $U^+_{\widetilde{\beta}}$ in Δ_{in} of diam 2γ, and $\overline{g}_{-(n+m_0+l_0)}[\widetilde{\beta}](\widetilde{W}_{\widetilde{\beta}})$ contains an open ball $W^-_{\widetilde{\beta}}$ of diam 6γ. So

$$\begin{aligned}&\left(\overline{g}_{-(n+m_0+l_0)}[\widetilde{\beta}]\right)^{-1}\left(W^-_{\widetilde{\beta}}\right)\\ &\quad = \overline{g}_{n+m_0+l_0}\left[\sigma^{-(n+m_0+l_0)}\widetilde{\beta}\right]\left(W^-_{\widetilde{\beta}}\right) \subset \widetilde{W}_{\widetilde{\beta}}.\end{aligned} \tag{38}$$

Let $m = 2n + l_0 + l_1 + m_0 + K_0 + k$. According to Lemma 7 and the fact $\overline{g}_{k_{\widetilde{\beta}}}[\sigma^{n+k+l_1}\widetilde{\beta}](U^+_{\widetilde{\beta}}) \subset W^-_{\widetilde{\beta}}$, we conclude that

$$\overline{g}_m[\widetilde{\beta}](\widetilde{W}_{\widetilde{\beta}}) \subset \widetilde{W}_{\widetilde{\beta}}. \tag{39}$$

Note that the acting of g_ω, with $\omega_0 = 0$, and $g_{\omega'}$, with $\omega'_0 = 1$ and $\omega'_1 = 0$, are contracting on Δ_{in}, so we can choose $k_{\widetilde{\beta}}$ sufficiently large such that $\|D\overline{g}_m[\widetilde{\beta}]\| < 1$ on $\widetilde{W}_{\widetilde{\beta}}$.

Hence, $\overline{g}_m[\widetilde{\beta}]$ has an attracting fixed point $\widetilde{y} \in \widetilde{W}_{\widetilde{\beta}}$. So $\widetilde{Y} = ((\widetilde{\beta}), \widetilde{y})$ is a periodic point in $C_{\overline{\alpha}} \times U$ which is attracting along the fiber. By a similar argument, we conclude the existence of a periodic point in $C_{\overline{\alpha}} \times U$ which is repelling along the fiber. This completes the proof of Theorem 2. □

4. Perturbations

Let n and m be positive integers with $n \geq m + 3$, $n \geq 5$, and $m \geq 1$. Suppose that N is an n-dimensional closed manifold.

In this section, we will construct an open set $\mathcal{U}$ of $\mathrm{Diff}^2(N)$ that satisfies the following property: each diffeomorphism of $\mathcal{U}$ possesses a partially hyperbolic locally maximal invariant set with a dense subset of periodic points with different indices.

In fact, we will find diffeomorphisms such that the restriction of them to their locally maximal invariant sets is conjugated to step random dynamical systems of the form (1).

As we have mentioned before, many properties observed for these products appear to persist as properties of diffeomorphisms [1, 2].

In the following, first we need to introduce skew products over the horseshoe which can be considered as smooth realizations of skew products over the Bernoulli shift of the forms (1) and (3).

Indeed, suppose that $h : S^2 \to S^2$ is a diffeomorphism with a horseshoe type hyperbolic set Λ, which has a Markov partition with two rectangles D_0, D_1 such that $D_0 \cap D_1 = \emptyset$, with the rate of contraction $k \in (0, 1)$ which is small enough (see [1, Theorem 2]). Put $D := D_0 \cup D_1$ and $h(D) := D'$. It is well known that the hyperbolic invariant set Λ is homeomorphic to Σ^2 with restriction of h to Λ being conjugate to the Bernoulli shift σ on Σ^2.

Now we define a skew product over the horseshoe map $h : \Lambda \to \Lambda$ with the fiber map M as follows:

$$\mathcal{F} : D \times M \longrightarrow D' \times M, \qquad \mathcal{F}|_{D_i \times M} = h \times f_i, \quad i = 0, 1, \tag{40}$$

where the diffeomorphism $f_i : M \to M$, $i = 0, 1$, are the generators of a skew products F of the form (1). The skew product $\mathcal{F}$ is called a *smooth realization* of the skew product F. It is easy to see that $\Lambda \times M$ is partially hyperbolic for $\mathcal{F}$ and $\mathcal{F}|_{\Lambda\times M}$ is conjugate to step skew product F. This fact implies that the properties found during the investigation of a semigroup generated by the diffeomorphisms $f_i : M \to M$ are realized by smooth mapping $\mathcal{F}$.

Suppose that $\mathcal{G}$ is a C^2 skew product which is C^1-close to $\mathcal{F}$. Then $\mathcal{G}$ has an invariant set $\mathcal{Y}_{\mathcal{G}}$ homeomorphic to $\Sigma^2 \times M$ by a homeomorphism K (see [2]). Let $\pi : \Sigma^2 \times M \to M$ be the projection to the fiber along the base. The homeomorphism $K : \Sigma^2 \times M \to \mathcal{Y}_{\mathcal{G}}$, $\mathcal{Y}_{\mathcal{G}} \subset D \times M$, can be taken so that the coordinate x is preserved, and hence the restriction of K to a single fiber is a C^2-diffeomorphism. One can consider the induced mapping

$$G = K^{-1} \circ \mathcal{G} \circ K : \Sigma^2 \times M \longrightarrow \Sigma^2 \times M. \tag{41}$$

Let us denote the mapping $\pi \circ K^{-1} \circ \mathcal{G} \circ K(\omega, \cdot) : M \to M$ by g_ω which depends on ω. Then g_ω is C^2 and the mapping G has the following form:

$$G : \Sigma^2 \times M \longrightarrow \Sigma^2 \times M, \qquad (\omega, x) \longrightarrow (\sigma\omega, g_\omega(x)), \tag{42}$$

which is a soft skew product (see [2] for more detail). We say that G is a soft skew product corresponding to $\mathcal{G}$ or $\mathcal{G}$ is a k-realization of G. Moreover, the bundle map g_ω is C^1-close to f_{ω_0} for each $\omega \in \Sigma^2$.

Here, we take $M = S^m$, the m-dimensional sphere. Let f_0 and f_1 be two diffeomorphisms on S^m generating a robustly minimal iterated function system as in Sections 2 and 3. Also, let F be the step skew product map of the form (1) with the fiber maps f_0 and f_1, and let $\mathcal{F}$ be its smooth realization. Let us take neighborhoods U_0, U_1 as in Theorem 1.

Now, let $\mathcal{G}$ be C^1-close to $\mathcal{F}$. Then $\mathcal{G}$ is conjugate to a controllable soft skew product map G, with fiber maps g_ω which is C^1-close to f_{ω_0}; see Section 3 for more detail.

Let $\mathcal{H}$ be a C^2 diffeomorphism which is C^1-close to $\mathcal{G}$. Then, $\mathcal{H}$ has an invariant set $\mathcal{Y}_{\mathcal{H}}$ homeomorphic to $\Sigma^2 \times S^m$ such that the projection $(\mathcal{Y}_{\mathcal{H}}, \mathcal{H}) \mapsto (\Sigma^2, \sigma)$ is semiconjugacy and so the dynamics of $\mathcal{H}$ restricted to $\mathcal{Y}_{\mathcal{G}}$ resembles the dynamics of $\mathcal{F}|_{\Lambda\times S^m}$. Also, $\mathcal{H}$ restricted to $\mathcal{Y}_{\mathcal{H}}$ is conjugate to skew product H on $\Sigma^2 \times S^m$ (see [2]). In particular, the fiber maps h_ω are C^1-close to g_ω and therefore it is C^1-close to f_{ω_0}, for each $\omega \in \Sigma^2$.

Now, we can apply Theorem 2 to conclude that the periodic orbits of the skew product H which are attracting (repelling) along S^m are dense in $\Sigma^2 \times S^m$. Therefore, $\mathcal{H}$ restricted to $\mathcal{Y}_{\mathcal{H}}$ has a dense subset of periodic orbits of indices (dimension of their stable manifolds) $l_1 = 1$ and $l_2 = m + 1$.

Finally, one can see that $\mathcal{H}$ restricted to $\mathcal{Y}_{\mathcal{H}}$ can be extended to a diffeomorphism on the closed manifold N.

Indeed, one can embed the m-sphere S^m in $\mathbb{R}^{m+1}$ and a two-dimensional rectangle B in $\mathbb{R}^{n-m-1}$, where $D \subset B$, $D = D_0 \cup D_1$. So $B \times S^m$ can be embedded in the closed manifold N, by a local chart of N (see [2] for more detail). This completes the proof of Theorem 3.

Acknowledgment

The authors are very grateful to the referee for fruitful comments and valuable suggestions.

References

[1] A. Gorodetski and Y. S. Ilyashenko, "Some properties of skew products over a horseshoe and solenoid," *Proceedings of the Steklov Institute of Mathematics*, vol. 231, pp. 96–118, 2000.

[2] A. S. Gorodetski and Y. S. Ilyashenko, "Some new robust properties of invariant sets and attractors of dynamical systems," *Funktsional'nyi Analiz i Ego Prilozheniya*, vol. 33, no. 2, pp. 16–30, 1999.

[3] A. J. Homburg and M. Nassiri, "Robust minimality of iterated function systems with two generators," *Ergodic Theory and Dynamical Systems*, pp. 1–6, 2013.

[4] F. H. Ghane, M. Nazari, M. Saleh, and Z. Shabani, "Attractors and their invisible parts for skew products with high dimensional fiber," *International Journal of Bifurcation and Chaos*, vol. 22, no. 8, Article ID 1250182, 16 pages, 2012.