Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 539061, 11 pages
http://dx.doi.org/10.1155/2013/539061

Research Article

Strong Convergence of a General Iterative Method for a Countable Family of Nonexpansive Mappings in Banach Spaces

Chin-Tzong Pang[1] and Eskandar Naraghirad[2,3]

[1] *Department of Information Management, Yuan Ze University, Chungli 32003, Taiwan*
[2] *Department of Mathematics, Yasouj University, Yasouj 75918, Iran*
[3] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 804, Taiwan*

Correspondence should be addressed to Eskandar Naraghirad; eskandarrad@gmail.com

Received 22 August 2013; Accepted 7 September 2013

Academic Editor: Chi-Ming Chen

We introduce a general algorithm to approximate common fixed points for a countable family of nonexpansive mappings in a real Banach space. We prove strong convergence theorems for the sequences produced by the methods and approximate a common fixed point of a countable family of nonexpansive mappings which solves uniquely the corresponding variational inequality. Furthermore, we apply our results for finding a zero of an accretive operator. It is important to state clearly that the contribution of this paper in relation with the previous works (Marino and Xu, 2006) is a technical method to prove strong convergence theorems of a general iterative algorithm for an infinite family of nonexpansive mappings in Banach spaces. Our results improve and generalize many known results in the current literature.

1. Introduction

Viscosity approximation method for finding the fixed points of nonexpansive mappings was first proposed by Moudafi [1]. He proved the convergence of the sequence generated by the proposed method. In 2004, Xu [2] proved the strong convergence of the sequence generated by the viscosity approximation method to a unique solution of a certain variational inequality problem defined on the set of fixed points of a nonexpansive map (see also [3]). Marino and Xu [4] considered a general iterative method and proved that the sequence generated by the method converges strongly to a unique solution of a certain variational inequality problem which is the optimality condition for a particular minimization problem. Liu [5] and Qin et al. [6] also studied some applications of the iterative method considered in [4]. Yamada [7] introduced the so-called hybrid steepest-descent method for solving the variational inequality problem and also studied the convergence of the sequence generated by the proposed method. Recently, Tian [8] combined the iterative methods of [4, 7] in order to propose implicit and explicit schemes for constructing a fixed point of a nonexpansive mapping T defined on a real Hilbert space. He also proved the strong convergence of these two schemes to a fixed point of T under appropriate conditions. Related iterative methods for solving fixed point problems, variational inequalities, and optimization problems can be found in [9–14] and the references therein. By virtue of the projection, the authors in [13, 15] extended the implicit and explicit iterative schemes proposed in [8]. The approximation methods for common fixed points of a countable family of nonexpansive mappings have been recently studied by several authors; see, for example, [16, 17].

The purpose of this paper is to introduce a general algorithm to approximate common fixed points for a countable family of nonexpansive mappings in a Banach space. We prove strong convergence theorems for the sequences produced by the methods for a common fixed point of a countable family of nonexpansive mappings which solves uniquely the corresponding variational inequality. Furthermore, we apply our results for finding a zero of an accretive operator. Our results improve and generalize many known results in the current literature; see, for example, [4, 7, 8, 13–15, 18–20].

2. Preliminaries

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. For any sequence $\{x_n^*\}$ in E^*, we denote the strong convergence of $\{x_n^*\}$ to $x^* \in E^*$ by $x_n^* \to x^*$, the weak convergence by $x_n^* \rightharpoonup x^*$, and the weak-star convergence by $x_n^* \rightharpoonup^* x^*$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2, \|x\| = \|f\| \right\}, \quad \forall x \in E. \tag{1}$$

The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\} \tag{2}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists. In this case, E is called *smooth*. If the limit (3) is attained uniformly in $x, y \in S$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S$ and $x \ne y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [21]. Now, we define a mapping $\rho : [0, \infty) \to [0, \infty)$, the modulus of smoothness of E, as follows:

$$\begin{aligned} \rho(t) = \sup \Big\{ & \frac{1}{2} \left(\|x+y\| + \|x-y\| \right) - 1 : \\ & x, y \in E, \|x\| = 1, \|y\| = t \Big\}. \end{aligned} \tag{4}$$

It is well known that E is uniformly smooth if and only if $\lim_{t \to 0} (\rho(t)/t) = 0$. Let $q \in \mathbb{R}$ be such that $1 < q \le 2$. Then a Banach space E is said to be q-uniformly smooth if there exists a constant $c_q > 0$ such that $\rho(t) \le c_q t^q$ for all $t > 0$. If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak star topology, then J is single valued and also E is smooth; for more details, see [22]. In this case, the normalized duality mapping J is said to be weakly sequentially continuous; that is, if $\{x_n\} \subset E$ is a sequence with $x_n \rightharpoonup x \in E$, then $J(x_n) \rightharpoonup^* J(x)$ [22]. A Banach space E is said to satisfy the *Opial property* [23] if for any weakly convergent sequence $\{x_n\}$ in E with weak limit x,

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \tag{5}$$

for all $y \in E$ with $y \ne x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces, and the Banach spaces l^p $(1 \le p < \infty)$ satisfy the Opial property; for example, see [22, 23]. It is also known that if E admits a weakly sequentially continuous duality mapping, then E is smooth and enjoys the Opial property; see for more details [22].

Let E be a real Banach space and C a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 1. Let C be a nonempty, closed, and convex subset of a real Banach space E. An operator $A : C \to E$ is said to be

(i) *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge 0, \quad \forall x, y \in C; \tag{6}$$

(ii) *η-strongly accretive* if, for some $\eta > 0$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge \eta \|x-y\|^2, \quad \forall x, y \in C; \tag{7}$$

(iii) *l-Lipschitzian* if, for some $l > 0$,

$$\|Ax - Ay\| \le l \|x-y\|, \quad \forall x, y \in C; \tag{8}$$

in particular, if $l \in [0, 1)$, then A is called a contraction;

(iv) nonexpansive if

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x, y \in C. \tag{9}$$

A linear bounded operator $A : E \to E^*$ is said to be strongly positive if there exists $\overline{\gamma} > 0$ such that

$$\langle x, Ax \rangle \ge \overline{\gamma} \|x\|^2, \quad \forall x \in E. \tag{10}$$

Remark 2. Let C be a nonempty, closed, and convex subset of a real Banach space E and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is an accretive operator, where I is the identity mapping. Indeed, for any $x, y \in C$ we have

$$\begin{aligned} & \langle (I-T)x - (I-T)y, j(x-y) \rangle \\ & = \langle x-y, j(x-y) \rangle - \langle Tx - Ty, j(x-y) \rangle \\ & \ge \|x-y\|^2 - \|Tx-Ty\| \|x-y\| \\ & \ge \|x-y\|^2 - \|x-y\|^2 = 0, \end{aligned} \tag{11}$$

which means that $I - T$ is accretive.

The following result has been proved in [24].

Lemma 3. *Let E be a real 2-uniformly smooth Banach space. Then there exists a best uniformly smooth constant $\rho > 0$ such that*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + 2\rho^2\|y\|^2, \tag{12}$$

for all $x, y \in E$.

Let C and D be nonempty subsets of real Banach space E with $D \subset C$. A mapping $Q_D : C \to D$ is said to be sunny if

$$Q_D(Q_D x + t(x - Q_D x)) = Q_D x \tag{13}$$

for each $x \in E$ and $t \ge 0$. A mapping $Q_D : C \to D$ is said to be a retraction if $Q_D x = x$ for each $x \in C$.

The following result has been proved in [25].

Lemma 4. *Let C and D be nonempty subsets of a real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D(z), j(y - Q_D(z))\rangle \le 0 \tag{14}$$

for all $z \in C$ and $y \in D$.

Lemma 5 (demiclosedness principle [26]). *Let C be a closed and convex subset of a real 2-uniformly smooth Banach space E and let the normalized duality mapping $J : E \to E^*$ be weakly sequentially continuous at zero. Suppose that $T : C \to E$ is a nonexpansive mapping with $F(T) \ne \varnothing$. If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in F(T)$.*

Lemma 6 (see [27]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \le (1 - \gamma_n)s_n + \gamma_n\delta_n, \quad \forall n \ge 0, \tag{15}$$

where $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the conditions

(i) *$\{\gamma_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty}\gamma_n = \infty$, or equivalently, $\prod_{n=0}^{\infty}(1 - \gamma_n) = 0$;*

(ii) *$\limsup_{n\to\infty}\delta_n \le 0$, or*

(ii)$'$ *$\sum_{n=0}^{\infty}\gamma_n\delta_n < \infty$.*

Then, $\lim_{n\to\infty}s_n = 0$.

Lemma 7 (see [28]). *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space E such that*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n, \quad n \ge 1, \tag{16}$$

where $\{\beta_n\}$ satisfies the following conditions: $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. If $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$, then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Let C be a subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ a family of mappings of C such that $\cap_{n=1}^{\infty}F(T_n) \ne \varnothing$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the *AKTT*-condition [29] if for each bounded subset K of C,

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_n z\| : z \in K\} < \infty. \tag{17}$$

Lemma 8 (see [29]). *Let C be a subset of a real Banach space E and $\{T_n\}_{n=1}^{\infty}$ a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}_{n=1}^{\infty}$ converges strongly to a point in C. Moreover, let the mapping T be defined by*

$$Tx = \lim_{n\to\infty}T_n x, \quad \forall x \in C. \tag{18}$$

Then for each bounded subset K of C,

$$\limsup_{n\to\infty}\{\|T_n z - Tz\| : z \in K\} = 0. \tag{19}$$

In the sequel, one will write that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKKT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKKT-condition and T is defined by Lemma 8 with $F(T) = \cap_{n=1}^{\infty}F(T_n)$.

We end this section with the following simple examples of mappings satisfying the *AKTT*-condition (see also Lemma 19).

Example 9. (i) Let E be a Banach space. For any $n \in \mathbb{N}$, let a mapping $T_n : E \to E$ be defined by

$$T_n(x) = \frac{x}{n}, \quad \forall x \in E. \tag{20}$$

Then, T_n is a nonexpansive mapping for each $n \in \mathbb{N}$. It could easily be seen that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the *AKKT*-condition, where $T(x) = 0$ for all $x \in E$.

(ii) Let E be a smooth Banach space and let $x_0 \ne 0$ be any element of E. For any $j \in \mathbb{N}$, we define a mapping $T_j : E \to E$ by

$$T_j(x) = \begin{cases} \left(\dfrac{1}{2} + \dfrac{1}{2^{n+1}}\right)x_0, & \text{if } x = \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right)x_0; \\ \dfrac{-x}{j}, & \text{if } x \ne \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right)x_0, \end{cases} \tag{21}$$

for all $n \ge 0$. We define also a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \left(\dfrac{1}{2} + \dfrac{1}{2^{n+1}}\right)x_0, & \text{if } x = \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right)x_0; \\ 0, & \text{if } x \ne \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right)x_0, \end{cases} \tag{22}$$

for all $n \ge 0$. It is easy to verify that $(\{T_j\}_{j=1}^{\infty}, T)$ satisfies the *AKKT*-condition.

(iii) Let $E = l^2$, where

$$\begin{gathered} l^2 = \left\{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots) : \sum_{n=1}^{\infty}\|\sigma_n\|^2 < \infty\right\}, \\ \|\sigma\| = \left(\sum_{n=1}^{\infty}\|\sigma_n\|^2\right)^{1/2}, \quad \forall \sigma \in l^2, \\ \langle \sigma, \eta\rangle = \sum_{n=1}^{\infty}\sigma_n\eta_n, \\ \forall \delta = (\sigma_1, \sigma_2, \dots, \sigma_n, \dots), \\ \eta = (\eta_1, \eta_2, \dots, \eta_n, \dots) \in l^2. \end{gathered} \tag{23}$$

Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1,0,0,0,\ldots) \\ x_1 &= (1,1,0,0,0,\ldots) \\ x_2 &= (1,0,1,0,0,0,\ldots) \\ x_3 &= (1,0,0,1,0,0,0,\ldots) \\ &\vdots \\ x_n &= (\sigma_{n,1},\sigma_{n,2},\ldots,\sigma_{n,k},\ldots) \\ &\vdots, \end{aligned} \tag{24}$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases} \tag{25}$$

for all $n \in \mathbb{N}$. It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Indeed, for any $\Lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n,\ldots) \in l^2 = (l^2)^*$, we have

$$\Lambda(x_n - x_0) = \langle x_n - x_0, \Lambda\rangle = \sum_{k=2}^{\infty} \lambda_k \sigma_{n,k} \longrightarrow 0 \tag{26}$$

as $n \to \infty$. It is also obvious that $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$ with n, m sufficiently large. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. We define a countable family of mappings $T_j : E \to E$ by

$$T_j(x) = \begin{cases} \dfrac{n}{n+1} x, & \text{if } x = x_n; \\ \dfrac{-j}{j+1} x, & \text{if } x \neq x_n, \end{cases} \tag{27}$$

for all $j \geq 1$ and $n \geq 0$. It is clear that $F(T_j) = \{0\}$ for all $j \geq 1$. It is obvious that T_j is a quasi-nonexpansive mapping for each $j \in \mathbb{N}$. Thus $\{T_j\}_{j\in\mathbb{N}}$ is a countable family of quasi-nonexpansive mappings.

Let $Tx = \lim_{j\to\infty} T_j x$ for all $x \in E$. It is easy to see that

$$T(x) = \begin{cases} \dfrac{n}{n+1} x, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases} \tag{28}$$

Then, we obtain that T is a quasi-nonexpansive mapping with $F(T) = \{0\} = \widetilde{F}(T)$. Let D be a bounded subset of E. Then there exists $r > 0$ such that $D \subset B_r = \{z \in E : \|z\| < r\}$. On the other hand, for any $j \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{j=1}^{\infty} \sup\left\{\left\|T_{j+1}z - T_j z\right\| : z \in D\right\} \\ &\quad = \sum_{j=1}^{\infty} \sup\left\{\left\|\frac{-j-1}{j+2} z - \frac{-j}{j+1} z\right\| : z \in D\right\} \\ &\quad = \sum_{j=1}^{\infty} \frac{1}{(j+2)(j+1)} \sup\{\|z\| : z \in D\} < \infty. \end{aligned} \tag{29}$$

Furthermore, we have

$$\limsup_{j\to\infty} \left\{\left\|T_j z - Tz\right\| : z \in D\right\} = 0. \tag{30}$$

Therefore, $(\{T_j\}_{j=1}^{\infty}, T)$ satisfies the $AKKT$-condition.

3. Fixed Point and Convergence Theorems

Let E be a 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and let C be a closed and convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, let $B : C \to E$ be an l-Lipschitzian mapping with constant $l \geq 0$, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \varnothing$. Suppose that $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$. Define a mapping $f : [0,1] \to \mathbb{R}$ by

$$f(t) = \begin{cases} \dfrac{1 - \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)}}{t} & \text{if } t \in (0,1], \\ \mu\eta & \text{if } t = 0. \end{cases} \tag{31}$$

From the definition of f we deduce that

$$f(t) \leq \mu\eta, \quad \forall t \in [0,1]. \tag{32}$$

Indeed, for any $t \in (0,1]$, in view of (31) we obtain

$$\begin{aligned} f(t) < \mu\eta &\Longleftrightarrow \frac{1 - \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)}}{t} < \mu\eta \\ &\Longleftrightarrow 1 - \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)} < \mu\eta t \\ &\Longleftrightarrow 1 - \mu\eta t < \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)} \\ &\Longleftrightarrow 1 + \mu^2\eta^2 t^2 - 2\mu\eta t < 1 - 2t\mu\left(\eta - t\mu k^2\rho^2\right) \\ &\Longleftrightarrow 0 < \eta < \sqrt{2}k\rho. \end{aligned} \tag{33}$$

On the other hand, it is easy to see that f is continuous on compact interval $[0,1]$. In fact, employing L'Hôpital's Rule, we conclude that $\lim_{t\to 0} f(t) = \mu\eta$. Thus,

$$\exists t_0 \in [0,1] \text{ such that } f(t_0) = \min\{f(t) : t \in [0,1]\}. \tag{34}$$

Set $\tau_0 := \tau_{t_0} = f(t_0)$ and $\tau_t := f(t)$ if $t \in [0,1]$. Then we have

$$0 < \tau_0 \leq \tau_t \leq \mu\eta. \tag{35}$$

Assume now that γ satisfies $0 \leq \gamma l < \tau_0$. Then we get

$$0 < \frac{1}{\mu\eta - \gamma l} \leq \frac{1}{\tau_t - \gamma l} \leq \frac{1}{\tau_0 - \gamma l} < \infty \quad \forall t \in [0,1]. \tag{36}$$

In this section, we introduce the following implicit scheme that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = Q_C\left[t\gamma B x_t + (I - t\mu A) T x_t\right]. \tag{37}$$

We prove the strong convergence of $\{x_t\}$ to a fixed point $\tilde{x}$ of T which solves the variational inequality

$$\langle (\mu A - \gamma B)\tilde{x}, j(\tilde{x} - z)\rangle \le 0, \quad \forall z \in F(T). \tag{38}$$

We first prove the following extension of Lemma 3.1 in [7] in a 2-uniformly smooth Banach space.

Lemma 10. *Let E be a 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and let C be a closed and convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with $0 < \eta < \sqrt{2}k\rho, 0 < \mu < \eta/k^2\rho^2$, and $t \in (0,1)$. In association with a nonexpansive mapping $T : C \to C$, define the mapping $S_t : C \to E$ by*

$$S_t x := Tx - t\mu A(Tx), \quad \forall x \in C. \tag{39}$$

Then, S_t is a contraction with contraction constant $\tau_t = 1 - c_t$, where $c_t = \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)}$.

Proof. In view of Lemma 3, we conclude that

$$\begin{aligned}
\|S_t x - S_t y\|^2 &= \|(T - t\mu AT)x - (T - t\mu AT)y\|^2 \\
&= \|(Tx - Ty) - t\mu(ATx - ATy)\|^2 \\
&\le \|Tx - Ty\|^2 - 2t\mu\langle ATx - ATy, j(Tx - Ty)\rangle \\
&\quad + 2t^2\mu^2\rho^2\|ATx - ATy\|^2 \\
&\le \|Tx - Ty\|^2 - 2t\mu\eta\|Tx - Ty\|^2 \\
&\quad + 2t^2\mu^2k^2\rho^2\|Tx - Ty\|^2 \\
&= \left(1 - 2t\mu\left(\eta - t\mu k^2\rho^2\right)\right)\|Tx - Ty\|^2 \\
&\le \left(1 - 2t\mu\left(\eta - t\mu k^2\rho^2\right)\right)\|x - y\|^2,
\end{aligned} \tag{40}$$

for all $x, y \in C$. Put $c_t = \sqrt{1 - 2t\mu(\eta - t\mu k^2\rho^2)} \in (0,1)$. Then by the assumptions $t \in (0,1)$ and $0 < \eta < \sqrt{2}k\rho$, we infer that

$$\|S_t x - S_t y\| \le c_t\|x - y\|. \tag{41}$$

Let $\tau_t = (1 - c_t) \in (0,1)$. Then we have

$$\|S_t x - S_t y\| \le (1 - \tau_t)\|x - y\|. \tag{42}$$

Therefore, S_t is a contraction with contraction constant $1 - \tau_t$, which completes the proof. □

Remark 11. Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and C a closed convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Assume $T : C \to C$ is a nonexpansive mapping with $F(T) \ne \emptyset$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \le \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). For any $t \in (0,1)$, let the mapping $R_t : C \to E$ be defined by

$$R_t x := Q_C[t\gamma Bx + (I - t\mu A)Tx], \quad \forall x \in C. \tag{43}$$

Using Remark 11, it could easily be seen that

$$\|R_t x - R_t y\| \le (1 - t(\tau_0 - \gamma l))\|x - y\|, \quad \forall x, y \in C. \tag{44}$$

Thus in view of Banach contraction principle, the contraction mapping $R_t : C \to E$ has a unique fixed point x_t in C, which uniquely solves the fixed point equation (37).

Remark 12. Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and C a closed convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to E$ be an l-Lipschitzian mapping with constant $l \ge 0$. Assume $T : C \to C$ is a nonexpansive mapping with $F(T) \ne \emptyset$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \le \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). Then

$$\begin{aligned}
&\langle (\mu A - \gamma B)x - (\mu A - \gamma B)y, j(x - y)\rangle \\
&\quad \ge (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in C.
\end{aligned} \tag{45}$$

That is, $\mu A - \gamma B$ is strongly accretive with coefficient $\mu\eta - \gamma l$.

In the following result, we drive some important properties of the net $\{x_t\}_{t\in(0,1)}$ which will be used in the sequel.

Proposition 13. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and let C be a closed and convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Assume $T : C \to C$ is a nonexpansive mapping with $F(T) \ne \emptyset$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \le \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). For each $t \in (0,1)$, let x_t denote a unique solution of the fixed point equation (37). Then, the following properties hold for the net $\{x_t\}_{t\in(0,1)}$:*

(1) *$\{x_t\}_{t\in(0,1)}$ is bounded;*

(2) *$\lim_{t\to 0}\| x_t - Tx_t \| = 0$;*

(3) *x_t defines a continuous curve from $(0,1)$ into C.*

Proof. (1) Let $p \in F(T)$ be taken arbitrarily. Then, in view of Lemma 10 we obtain

$$\begin{aligned}
\|x_t - p\| &= \|Q_C[t\gamma Bx_t + (I - t\mu A)Tx_t] - Q_C p\| \\
&\le \|t\gamma Bx_t + (I - t\mu A)Tx_t - p\|
\end{aligned}$$

$$\begin{aligned}
&= \|(I - t\mu A)Tx_t + (I - t\mu A)p \\
&\quad + t(\gamma Bx_t - \mu A(p))\| \\
&\le (1 - t\tau_t)\|x_t - p\| \\
&\quad + t(\gamma l\|x_t - p\| + \|\gamma Bp - \mu Ap\|) \\
&= (1 - t(\tau_t - \gamma l))\|x_t - p\| + t\|(\gamma B - \mu A)p\| \\
&\le (1 - t(\tau_0 - \gamma l))\|x_t - p\| + t\|(\gamma B - \mu A)p\|.
\end{aligned} \tag{46}$$

This implies that

$$\|x_t - p\| \le \frac{\|(\gamma B - \mu A)p\|}{\tau_0 - \gamma l}. \tag{47}$$

This shows that $\{x_t\}$ is bounded.

(2) Since $\{x_t\}$ is bounded, we have that $\{Bx_t\}$ and $\{ATx_t\}$ are bounded too. In view of the definition of $\{x_t\}$ we conclude that

$$\begin{aligned}
\|x_t - Tx_t\| &= \|Q_C[t\gamma Bx_t + (I - t\mu A)Tx_t] - Q_C[Tx_t]\| \\
&\le \|t\gamma Bx_t + (I - t\mu A)Tx_t - Tx_t\| \\
&= t\|\gamma Bx_t - \mu ATx_t\| \longrightarrow 0,
\end{aligned} \tag{48}$$

as $t \to 0$.

(3) Take $t_1, t_2 \in (0,1)$ arbitrarily. Then, we have

$$\begin{aligned}
\|x_{t_1} - x_{t_2}\| &= \|Q_C[t_1\gamma Bx_{t_1} + (I - t_1\mu A)Tx_{t_1}] \\
&\quad - Q_C[t_2\gamma Bx_{t_2} + (I - t_2\mu A)Tx_{t_2}]\| \\
&\le \|t_1\gamma Bx_{t_1} + (I - t_1\mu A)Tx_{t_1} \\
&\quad - [t_2\gamma Bx_{t_2} + (I - t_2\mu A)Tx_{t_2}]\| \\
&= \|(t_2 - t_1)\gamma Bx_{t_2} + t_1\gamma(Bx_{t_2} - Bx_{t_1}) \\
&\quad + (t_1 - t_2)\mu ATx_{t_2} \\
&\quad + (I - t_1\mu A)Tx_{t_2} - (I - t_1\mu A)Tx_{t_1}\| \\
&\le (\gamma\|Bx_{t_2}\| + \mu\|ATx_{t_2}\|)|t_1 - t_2| \\
&\quad + (1 - t_1(\tau_{t_1} - \gamma l))\|x_{t_1} - x_{t_2}\| \\
&\le (\gamma\|Bx_{t_2}\| + \mu\|ATx_{t_2}\|)|t_1 - t_2| \\
&\quad + (1 - t_1(\tau_0 - \gamma l))\|x_{t_1} - x_{t_2}\|.
\end{aligned} \tag{49}$$

This implies that

$$\|x_{t_2} - x_{t_1}\| \le \frac{\gamma\|Bx_{t_2}\| + \mu\|ATx_{t_2}\|}{t_1(\tau_0 - \gamma l)}|t_2 - t_1|. \tag{50}$$

The boundedness of $\{x_t\}$ implies that x_t defines a continuous curve from $(0,1)$ into C. □

Theorem 14. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and let C be a closed and convex subset of E. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Assume $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \le \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). For each $t \in (0,1)$, let $\{x_t\}$ denote a unique solution of the fixed point equation (37). Then the net $\{x_t\}$ converges strongly, as $t \to 0$, to a fixed point $\tilde{x}$ of T which solves the variational inequality (38), or equivalently, $Q_{F(T)}(I - \mu A + \gamma B)\tilde{x} = \tilde{x}$.*

Proof. In view of Remark 11 the variational inequality (38) has a unique solution, say $\tilde{x} \in C$. We show that $x_t \to \tilde{x}$ as $t \to 0$. To this end, let $z \in F(T)$ be given arbitrary. Set

$$y_t = t\gamma Bx_t + (I - t\mu A)Tx_t, \quad \forall t \in (0,1). \tag{51}$$

Then we have $x_t = Q_C y_t$ and hence

$$\begin{aligned}
x_t - z &= Q_C y_t - y_t + y_t - z \\
&= Q_C y_t - y_t + t(\gamma Bx_t - \mu Az) \\
&\quad + (I - t\mu A)Tx_t - (I - t\mu A)Tz.
\end{aligned} \tag{52}$$

Since Q_C is a nonexpansive mapping from E onto C, in view of Lemma 4, we conclude that

$$\langle Q_C y_t - y_t, j(Q_C y_t - z)\rangle \le 0. \tag{53}$$

Exploiting Lemma 10, (37), and (52), we obtain

$$\begin{aligned}
\|x_t - z\|^2 &= \langle x_t - z, j(x_t - z)\rangle \\
&= \langle Q_C y_t - y_t, j(Q_C y_t - z)\rangle \\
&\quad + \langle (I - t\mu A)Tx_t - (I - t\mu A)z\rangle \\
&\quad + \langle t(\gamma Bx_t - \mu Az), j(x_t - z)\rangle \\
&\le \frac{1}{\tau_0}\left[\gamma l\|x_t - z\|^2 + \langle \gamma Bz - \mu Az, j(x_t - z)\rangle\right].
\end{aligned} \tag{54}$$

This implies that

$$\|x_t - z\|^2 \le \frac{1}{\tau_0 - \gamma l}\langle \gamma Bz - \mu Az, j(x_t - z)\rangle. \tag{55}$$

Let $\{t_n\} \subset (0,1)$ be such that $t_n \to 0^+$ as $n \to \infty$. Letting $x_n^* := x_{t_n}$, it follows from Proposition 13(2) that $\lim_{n\to\infty}\|x_n^* - Tx_n^*\| = 0$. The boundedness of $\{x_t\}$ implies that there exists $x^* \in C$ such that $x_n^* \rightharpoonup x^*$ as $n \to \infty$. In view of Lemma 5, we deduce that $x^* \in F$. Since $x_n^* \rightharpoonup x^*$ as $n \to \infty$, it follows from (55) that $\lim_{n\to\infty}\|x_n^* - x^*\| = 0$. Thus we have $\lim_{t\to 0^+} x_t = x^*$ well defined. Next, we show that x^* solves the variational inequality (38). We first notice that

$$x_t = Q_C y_t = Q_C y_t - y_t + t\gamma Bx_t + (I - t\mu A)Tx_t. \tag{56}$$

This, together with (52), implies that

$$(\mu A - \gamma B) x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I - T) x_t + \mu (A x_t - AT x_t). \tag{57}$$

Since T is nonexpansive, in view of Remark 2, we conclude that $I - T$ is accretive. This implies that

$$\begin{aligned}
&\langle (\gamma B - \mu A) x_t, j(x_t - z)\rangle \\
&= \frac{1}{t}\langle Q_C y_t - y_t, j(x_t - z)\rangle \\
&\quad - \frac{1}{t}\langle (I - T) x_t - (I - T) z, j(x_t - z)\rangle \\
&\quad + \mu \langle A x_t - AT x_t, j(x_t - z)\rangle \\
&\le \mu \langle A x_t - AT x_t, j(x_t - z)\rangle \\
&\le \mu l \|x_t - T x_t\| \|x_t - z\|.
\end{aligned} \tag{58}$$

Replacing t by t_n in (58), taking the limit $n \to \infty$, and noticing that $\{x_t - z\}_{t\in(0,1)}$ is bounded for $z \in F(T)$, we obtain

$$\langle (\mu A - \gamma B) x^*, j(x^* - z)\rangle \le 0. \tag{59}$$

Thus, we have $x^* \in F(T)$ a solution of the variational inequality (38). Consequently, $x^* = \tilde{x}$ by uniqueness. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. The variational inequality (38) can be written as

$$\langle (I - \mu A + \gamma B)\tilde{x} - \tilde{x}, j(\tilde{x} - z)\rangle \ge 0, \quad \forall z \in F(T). \tag{60}$$

Thus, in view of Lemma 4, it is equivalent to the following fixed point equation:

$$Q_{F(T)}(I - \mu A + \gamma B)\tilde{x} = \tilde{x}. \tag{61}$$

This completes the proof. □

Theorem 15. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and let C be a nonempty, closed and convex subset of E. Suppose that the normalized duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \le \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). Assume $\{T_n\}_{n=1}^{\infty}$ is a sequence of nonexpansive mappings from C into itself such that $\cap_{n=1}^{\infty}F(T_n) \ne \emptyset$. Suppose in addition that $T : C \to C$ is a nonexpansive mapping such that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition. For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{aligned}
y_n &= Q_C[\alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n], \\
x_{n+1} &= (1 - \beta_n) y_n + \beta_n T_n y_n, \quad n \in \mathbb{N},
\end{aligned} \tag{62}$$

where Q_C is the sunny nonexpansive retraction from E onto C and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$ satisfying the following control conditions:

$$\begin{gathered}
\text{(a)} : \lim_{n\to\infty}\alpha_n = 0, \\
\sum_{n=1}^{\infty}\alpha_n = \infty; \\
\text{(b)} : 0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1.
\end{gathered} \tag{63}$$

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \cap_{n=1}^{\infty}F(T_n)$ which solves the variational inequality

$$\langle (\mu A - \gamma B) x^*, j(x^* - z)\rangle \le 0, \quad z \in \bigcap_{n=1}^{\infty} F(T_n). \tag{64}$$

Proof. We divide the proof into several steps.

Step 1. We claim that the sequence $\{x_n\}$ is bounded. Let $p \in F$ be fixed. In view of (62)–(64) and Lemma 10, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|Q_C[\alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n] - Q_C p\| \\
&\le \|\alpha_n \gamma B x_n + (I - \alpha_n \mu A) x_n - p\| \\
&= \|\alpha_n (\gamma B x_n - \mu A p) \\
&\qquad + (I - \alpha_n \mu A) x_n - (I - \alpha_n \mu A) p\| \\
&= \|\alpha_n (\gamma B x_n - \gamma B p) + \alpha_n (\gamma B p - \mu A p) \\
&\qquad + (I - \alpha_n \mu A) x_n - (I - \alpha_n \mu A) p\| \\
&\le \alpha_n \gamma l \|x_n - p\| + \alpha_n \|(\gamma B - \mu A) p\| \\
&\qquad + (1 - \alpha_n \tau_0)\|x_n - p\| \\
&= (1 - \alpha_n(\tau_0 - \gamma l))\|x_n - p\| \\
&\qquad + \alpha_n \|(\gamma B - \mu A) p\| \\
&\le \max\left\{\|x_n - p\|, \frac{\|(\gamma B - \mu A) p\|}{\tau_0 - \gamma l}\right\}.
\end{aligned} \tag{65}$$

Since T_n is nonexpansive, for all $n \in \mathbb{N}$, it follows from (62) and (65) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \beta_n)(x_n - p) + \beta_n (T_n y_n - p)\| \\
&\le (1 - \beta_n)\|x_n - p\| + \beta_n \|T_n y_n - p\| \\
&\le (1 - \beta_n)\|x_n - p\| + \beta_n \|y_n - p\| \\
&\le (1 - \beta_n)\|x_n - p\| \\
&\qquad + \beta_n \max\left\{\|x_n - p\|, \frac{\|(\gamma B - \mu A) p\|}{\tau_0 - \gamma l}\right\} \\
&\le \max\left\{\|x_n - p\|, \frac{\|(\gamma B - \mu A) p\|}{\tau_0 - \gamma l}\right\}.
\end{aligned} \tag{66}$$

By induction, we conclude that $\{x_n\}$ is bounded. This implies that the sequences $\{Ax_n\}$, $\{Bx_n\}$, $\{y_n\}$, and $\{T_n y_n\}$ are bounded too. Let $M_1 = \sup\{\|x_n\|, \|Ax_n\|, \|Bx_n\|, \|y_n\|, \|T_n y_n\| : n \in \mathbb{N}\} < \infty$ and set $K = \{z \in E : \|z\| \leq M\}$. Then we have K a bounded subset of E and $\{x_n, Ax_n, Bx_n, y_n, T_n y_n\} \subset K$.

Step 2. We claim that $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$. For this purpose, we denote a sequence $\{z_n\}$ by $z_n = T_n y_n$. Then we have

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|T_{n+1}y_{n+1} - T_n y_n\| \\ &\leq \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_n y_n\| \\ &\leq \|y_{n+1} - y_n\| + \sup\{\|T_{n+1}z - T_n z\| : z \in K\}.\end{aligned} \tag{67}$$

This implies that

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|y_{n+1} - y_n\| \\ &\quad \leq \sup\{\|T_{n+1}z - T_n z\| : z \in K\}.\end{aligned} \tag{68}$$

In view of Lemma 8 and (63)(a) we conclude that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|y_{n+1} - y_n\|) \leq 0. \tag{69}$$

Utilizing Lemma 7, we deduce that

$$\lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{70}$$

It follows from (63)(b) and (70) that

$$\lim_{n\to\infty}\|y_{n+1} - y_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - y_n\| = 0. \tag{71}$$

Observe now that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|Q_C[\alpha_{n+1}\gamma Bx_{n+1} + (I - \alpha_{n+1}\mu A)T_{n+1}x_{n+1}] \\ &\qquad - Q_C[\alpha_n\gamma Bx_n + (I - \alpha_n\mu A)T_n x_n]\| \\ &\leq \|\alpha_{n+1}\gamma Bx_{n+1} + (I - \alpha_{n+1}\mu A)T_{n+1}x_{n+1} \\ &\qquad - [\alpha_n\gamma Bx_n + (I - \alpha_n\mu A)T_n x_n]\| \\ &= \|\alpha_{n+1}\gamma Bx_{n+1} + T_{n+1}x_{n+1} - \alpha_{n+1}\mu AT_{n+1}x_{n+1} \\ &\qquad - \alpha_n\gamma Bx_n - T_n x_n + \alpha_n\mu AT_n x_n\| \\ &\leq \alpha_{n+1}\gamma\|Bx_{n+1}\| + \alpha_n\gamma\|Bx_n\| \\ &\quad + \alpha_{n+1}\mu\|AT_{n+1}x_{n+1}\| + \alpha_n\mu\|AT_n x_n\| \\ &\quad + \|T_{n+1}x_{n+1} - T_n x_n\| \\ &\leq (\alpha_{n+1} + \alpha_n)(\gamma + \mu)M_1 \\ &\quad + \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| \\ &\leq (\alpha_{n+1} + \alpha_n)(\gamma + \mu)M_1 + \|x_{n+1} - x_n\| \\ &\quad + \sup\{\|T_{n+1}z - T_n z\| : z \in K\}.\end{aligned} \tag{72}$$

This implies that

$$\begin{aligned}&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &\quad \leq (\alpha_{n+1} + \alpha_n)(\gamma + \mu)M_1 \\ &\qquad + \sup\{\|T_{n+1}z - T_n z\| : z \in K\}.\end{aligned} \tag{73}$$

Utilizing Lemma 7 and taking into account $\alpha_n \to 0$, we deduce that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{74}$$

On the other hand, we have

$$\begin{aligned}\|y_n - Ty_n\| &\leq \|y_n - T_n y_n\| + \|T_n y_n - Ty_n\| \\ &\leq \|y_n - z_n\| + \sup\{\|T_n z - Tz\| : z \in K\}.\end{aligned} \tag{75}$$

Employing Lemma 8, we obtain

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{76}$$

Step 3. We prove that there exists $x^* \in F$ such that

$$\limsup_{n\to\infty}\langle(\mu A - \gamma B)x^*, j(x^* - y_n)\rangle \leq 0, \tag{77}$$

where x^* is as in Theorem 14. We first note that there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle\mu Ax^* - \gamma Bx^*, j(x^* - y_n)\rangle \\ &\quad = \lim_{i\to\infty}\left\langle\mu Ax^* - \gamma Bx^*, j\left(x^* - y_{n_i}\right)\right\rangle.\end{aligned} \tag{78}$$

Since $\{y_n\}$ is bounded, without loss of generality, we may assume that $y_{n_i} \rightharpoonup u \in C$ as $i \to \infty$. In view of Lemma 6 and Step 2, we conclude that $u \in F$. This, together with (78), implies that

$$\begin{aligned}&\limsup_{n\to\infty}\langle\mu Ax^* - \gamma Bx^*, j(x^* - y_n)\rangle \\ &\quad = \lim_{i\to\infty}\left\langle\mu Ax^* - \gamma Bx^*, j\left(x^* - y_{n_i}\right)\right\rangle \\ &\quad = \langle\mu Ax^* - \gamma Bx^*, j(x^* - u)\rangle \leq 0.\end{aligned} \tag{79}$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

For each $n \in \mathbb{N} \cup \{0\}$, by Lemma 10 and (36) we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &= \langle y_n - x^*, j(y_n - x^*)\rangle \\
&= \langle Q_C[\alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n] \\
&\quad -x^*, j(y_n - x^*)\rangle \\
&= \langle Q_C[\alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n] \\
&\quad - [\alpha_n \gamma B x_n + (I - \alpha_n \mu A) \\
&\quad \times T_n x_n - x^*], j(y_n - x^*)\rangle \\
&\quad + \langle \alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n \\
&\quad -x^*, j(y_n - x^*)\rangle \\
&\le \langle \alpha_n \gamma B x_n + (I - \alpha_n \mu A) T_n x_n \\
&\quad -x^*, j(y_n - x^*)\rangle \\
&= \alpha_n \langle \gamma B x_n - \mu A(x^*), j(y_n - x^*)\rangle \\
&\quad + \langle (I - \alpha_n \mu A) T x_n \\
&\quad - (I - \alpha_n \mu A) T x^*, j(y_n - x^*)\rangle \\
&= \alpha_n \gamma \langle B x_n - B x^*, j(y_n - x^*)\rangle \\
&\quad + \alpha_n \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle \\
&\quad + \langle (I - \alpha_n \mu A) T x_n \\
&\quad - (I - \alpha_n \mu A) T x^*, j(y_n - x^*)\rangle \\
&\le \alpha_n \gamma l \|x_n - x^*\| \|y_n - x^*\| \\
&\quad + \alpha_n \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle \\
&\quad + (1 - \alpha_n \tau_0) \|x_n - x^*\| \|y_n - x^*\| \\
&= (1 - \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\| \|y_n - x^*\| \\
&\quad + \alpha_n \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle \\
&\le (1 - \alpha_n(\tau_0 - \gamma l)) \frac{1}{2} \left(\|x_n - x^*\|^2 + \|y_n - x^*\|^2\right) \\
&\quad + \alpha_n \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle.
\end{aligned} \tag{80}$$

This implies that

$$\begin{aligned}
\|y_n - x^*\|^2 &\le \frac{(1 - \alpha_n(\tau_0 - \gamma l))}{(1 + \alpha_n(\tau_0 - \gamma l))} \|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n(\tau_0 - \gamma l)} \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle \\
&\le (1 - \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 + \alpha_n(\tau_0 - \gamma l)} \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle \\
&= (1 - \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\|^2 + \theta_n \xi_n,
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
\theta_n &= \alpha_n(\tau_0 - \gamma l), \\
\xi_n &= \frac{2}{(1 + \alpha_n(\tau_0 - \gamma l))(\tau_0 - \gamma l)} \\
&\quad \times \langle (\gamma B - \mu A) x^*, j(y_n - x^*)\rangle
\end{aligned} \tag{82}$$

In view of (81), we conclude that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n) x_n + \beta_n T_n y_n - x^*\|^2 \\
&\le (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|T_n y_n - x^*\|^2 \\
&\le (1 - \beta_n) \|x_n - x^*\|^2 + \beta_n \|y_n - x^*\|^2 \\
&\le (1 - \beta_n) \|x_n - x^*\|^2 \\
&\quad + \beta_n \left[(1 - \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\|^2 + \theta_n \xi_n\right] \\
&\le (1 - \beta_n \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\|^2 + \beta_n \theta_n \xi_n \\
&= (1 - \beta_n \alpha_n(\tau_0 - \gamma l)) \|x_n - x^*\|^2 \\
&\quad + \beta_n \alpha_n(\tau_0 - \gamma l) \xi_n \\
&= (1 - \gamma_n) \|x_n - x^*\|^2 + \gamma_n \xi_n,
\end{aligned} \tag{83}$$

where $\gamma_n = \beta_n \alpha_n(\tau_0 - \gamma l)$. It is easy to show that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=0}^{\infty} \gamma_n = \infty$, and $\limsup_{n\to\infty} \xi_n \le 0$. Hence, in view of Lemma 6 and (83), we conclude that the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$. This completes the proof. □

Remark 16. Theorem 15 improves and extends [19, Theorems 3.1 and 3.2] in the following aspects.

(i) The self-contractive mapping $f : C \to C$ in [19, Theorems 3.1 and 3.2] is extended to the case of a Lipschitzian (possibly nonself-) mapping $B : C \to E$ on a nonempty closed convex subset C of a Banach space E.

(ii) The identity mapping I is extended to the case of $I - A : C \to E$, where $A : C \to E$ is a k-Lipschitzian and η-strongly accretive (possibly nonself-) mapping.

(iii) The contractive coefficient $\alpha \in (0, 1)$ in [19, Theorems 3.1 and 3.2] is extended to the case where the Lipschitzian constant l lies in $[0, \infty)$.

(iv) In order to find a common fixed point of a countable family of nonexpansive self-mappings $T_n : C \to C$, the Mann type iterations in [19, Theorems 3.1 and 3.2] are extended to develop the new Mann type iteration (62).

(v) The new technique of argument is applied in deriving Theorem 14. For instance the characteristic properties (Lemma 4) of sunny nonexpansive retraction play an important role in proving the strong convergence of the net $\{x_t\}_{t\in(0,1)}$ in Theorem 14.

(vi) Whenever we have $C = E, B = f$ a contraction mapping with coefficient $\alpha \in (0,1)$, $A = I$ the identity mapping on C, and $l = \alpha$ with $0 < \gamma\alpha < \tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$, Theorem 14 reduces to [19, Theorems 3.1 and 3.2]. Thus, Theorem 14 covers [19, Theorems 3.1 and 3.2] as special cases.

Remark 17. Proposition 13 and Theorems 14 and 15 improve and generalize the corresponding results of [4] from Hilbert spaces to Banach spaces.

4. Applications

In this section, we apply Theorem 15 for finding a zero of an accretive operator. Let E be a real Banach space and let $S : E \to 2^E$ be a mapping. The effective domain of S is denoted by $\text{dom}(S)$; that is, $\text{dom}(S) = \{x \in E : Sx \neq \emptyset\}$. The range of S is denoted by $R(S)$. A multivalued mapping S is said to be accretive if for all $x, y \in E$ there exists $j \in J(x-y)$ such that $\langle x - y, j\rangle \geq 0$, where $J : E \to 2^{E^*}$ is the duality mapping. An accretive operator S is m-accretive if $R(I + rS) = E$ for each $r \geq 0$. Throughout this section, we assume that $S : E \to 2^E$ is m-accretive and has a zero. For an accretive operator S on E and $r > 0$, we may define a single-valued operator $J_r = (I + rS)^{-1} : E \to \text{dom}(S)$, which is called the resolvent of S for $r > 0$. Assume $S^{-1}0 = \{x \in E : 0 \in Sx\}$. It is known that $S^{-1}0 = F(J_r)$ for all $r > 0$.

The following lemma has been proved in [21].

Lemma 18. *Let E be a real Banach space and let S be an m-accretive operator on E. For $r > 0$, let J_r be the resolvent operator associated with S and r. Then*

$$\left\|J_r x - J_s x\right\| \leq \frac{|r-s|}{r}\left\|x - J_r x\right\|, \tag{84}$$

for all $r, s > 0$ and $x \in E$.

We also know the following lemma from [29].

Lemma 19. *Let C be a nonempty, closed, and convex subset of a real Banach space E and let S be an accretive operator on E such that $S^{-1}0 \neq \emptyset$ and $\overline{\text{dom}(S)} \subset C \subset \cap_{r>0} R(I + rS)$. Suppose that $\{r_n\}$ is a sequence of $(0,\infty)$ such that $\inf\{r_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$. Then*

(i) $\sum_{n=1}^{\infty}\sup\{\| J_{r_{n+1}}z - J_{r_n}z \| : z \in B\} < \infty$ for any bounded subset B of C;

(ii) $\lim_{n\to\infty} J_{r_n}z = J_r z$ for all $z \in C$ and $F(J_r) = \cap_{n=1}^{\infty} F(J_{r_n})$, where $r_n \to r$ as $n \to \infty$.

As an application of our main result, we include a concrete example in support of Theorem 15. Using Theorem 15, we obtain the following strong convergence theorem for m-accretive operators.

Theorem 20. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the 2-uniform smooth constant ρ and C a nonempty, closed, and convex subset of E. Suppose that the normalized duality mapping $J : E \to E^*$ is weakly sequentially continuous at zero. Let $A : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$ and let $B : C \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \eta < \sqrt{2}k\rho$, $0 < \mu < \eta/k^2\rho^2$, and $0 \leq \gamma l < \tau_0$, where $\tau_0 = (1 - \sqrt{1 - 2t_0\mu(\eta - t_0\mu k^2\rho^2)})/t_0$ satisfies (34). Let S be an m-accretive operator from E to E^* such that $S^{-1}(0) \neq \emptyset$. Let $r_n > 0$ such that $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$ and let $J_{r_n} = (I + r_n S)^{-1}$ be the resolvent of S. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by

$$\begin{aligned} y_n &= Q_C\left[\alpha_n\gamma B x_n + (I - \alpha_n\mu A) J_{r_n} x_n\right], \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n J_{r_n} y_n, \quad n \in \mathbb{N}, \end{aligned} \tag{85}$$

where Q_C is the sunny nonexpansive retraction from E onto C. Then, the sequence $\{x_n\}$ defined in (85) converges strongly to $x^* \in S^{-1}(0)$.

Proof. Letting $T_n = J_{r_n}, \forall n \in \mathbb{N}$, in Theorem 15, from (62), we obtain (85). It is easy to see that T_n satisfies all the conditions in Theorem 15 for all $n \in \mathbb{N}$. Therefore, in view of Theorem 15 we have the conclusions of Theorem 20. This completes the proof. □

Remark 21. Theorem 20 improves and extends Theorems 4.2, 4.3, 4.4, and 4.5 in [19].

Acknowledgments

The authors would like to thank the editor and the referees for sincere evaluation and constructive comments which improved the paper considerably. The work of Eskandar Naraghirad was conducted with a postdoctoral fellowship at the National Sun Yat-sen University of Kaohsiung, Taiwan.

References

[1] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.

[2] H. K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[3] L. C. Ceng, H. K. Xu, and J. C. Yao, "The viscosity approximation method for asymptotically nonexpansive mappings in Banach

spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 4, pp. 1402–1412, 2008.

[4] G. Marino and H. K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[5] Y. Liu, "A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4852–4861, 2009.

[6] X. Qin, M. Shang, and S. M. Kang, "Strong convergence theorems of modified Mann iterative process for strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1257–1264, 2009.

[7] I. Yamada, "The hybrid steepest descent method for the variational inequality problems over the intersection of fixed point sets of nonexpansive mappings," in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Studies in Computational Mathematics*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[8] M. Tian, "A general iterative algorithm for nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 3, pp. 689–694, 2010.

[9] L. C. Ceng and S. Huang, "Modified extragradient methods for strict pseudo-contractions and monotone mappings," *Taiwanese Journal of Mathematics*, vol. 13, no. 4, pp. 1197–1211, 2009.

[10] L. C. Ceng, S. Huang, and Y. C. Liou, "Hybrid proximal point algorithms for solving constrained minimization problems in banach spaces," *Taiwanese Journal of Mathematics*, vol. 13, no. 2, pp. 805–820, 2009.

[11] L. C. Ceng, S. Huang, and A. Petruşel, "Weak convergence theorem by a modified extragradient method for nonexpansive mappings and monotone mappings," *Taiwanese Journal of Mathematics*, vol. 13, no. 1, pp. 225–238, 2009.

[12] L. C. Ceng and N. C. Wong, "Viscosity approximation methods for equilibrium problems and fixed point problems of nonlinear semigroups," *Taiwanese Journal of Mathematics*, vol. 13, no. 5, pp. 1497–1513, 2009.

[13] L. C. Ceng and J. C. Yao, "Relaxed viscosity approximation methods for fixed point problems and variational inequality problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3299–3309, 2008.

[14] L. C. Zeng, Q. H. Ansari, D. S. Shyu, and J. C. Yao, "Strong and weak convergence theorems for common solutions of generalized equilibrium problems and zeros of maximal monotone operators," *Fixed Point Theory and Applications*, vol. 2010, Article ID 590278, 33 pages, 2010.

[15] L. C. Ceng, Q. H. Ansari, and J. C. Yao, "Some iterative methods for finding fixed points and for solving constrained convex minimization problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 16, pp. 5286–5302, 2011.

[16] Y. Yao, J. C. Yao, and H. Zhou, "Approximation methods for common fixed points of infinite countable family of nonexpansive mappings," *Computers and Mathematics with Applications*, vol. 53, no. 9, pp. 1380–1389, 2007.

[17] L. C. Ceng and J. C. Yao, "Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 729–741, 2008.

[18] L. C. Ceng, S. M. Guu, and J. C. Yao, "A general composite iterative algorithm for nonexpansive mappings in Hilbert spaces," *Computers and Mathematics with Applications*, vol. 61, no. 9, pp. 2447–2455, 2011.

[19] C. Klin-Eam and S. Suantai, "Strong convergence of composite iterative schemes for a countable family of nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 2, pp. 431–439, 2010.

[20] Y. Wang and W. Xu, "Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities," *Fixed Point Theory and Applications*, vol. 2013, article 121, 2013.

[21] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokahama Publishers, Yokahama, Japan, 2000.

[22] J. P. Gossez and E. L. Dozo, "Some geometric properties related to the fixed point theory for nonexpansive mappings," *Pacific Journal of Mathematics*, vol. 40, no. 3, pp. 565–573, 1972.

[23] K. Geobel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, New York, NY, USA, 1990.

[24] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[25] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[26] H. Zhou, "Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 3160–3173, 2008.

[27] H. K. Xu and T. H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 185–201, 2003.

[28] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[29] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.