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Research Article

Hybrid Projection Algorithm for Two Countable Families of Hemirelatively Nonexpansive Mappings and Applications

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Two countable families of hemirelatively nonexpansive mappings are considered based on a hybrid projection algorithm. Strong convergence theorems of iterative sequences are obtained in an uniformly convex and uniformly smooth Banach space. As applications, convex feasibility problems, equilibrium problems, variational inequality problems, and zeros of maximal monotone operators are studied.

1. Introduction

Throughout this paper, we always assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E and $\langle \cdot, \cdot \rangle$ is the pairing between E, and E^*. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively.

Let $f : C \times C \to \mathbb{R}$ be a bifunction and $A : C \to E^*$ a nonlinear mapping. The "so-called" generalized mixed equilibrium problem is to find $x \in C$ such that

$$f(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions to (1) is denoted by $\mathrm{GMEP}(f, A, \varphi)$, that is,

$$\mathrm{GMEP}(f, A, \varphi) = \{x \in C : f(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}. \tag{2}$$

1.1. Analysis of Special Cases. (1) If $\varphi(\cdot) \equiv 0$, the problem (1) reduces to the generalized equilibrium problem, which is to find $x \in C$ such that

$$f(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The set of solutions to (3) is denoted by $\mathrm{GEP}(f, A)$.

(2) If $A \equiv 0$, the problem (1) reduces to the mixed equilibrium problem, which is to find $x \in C$ such that

$$f(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{4}$$

The set of solutions to (4) is denoted by $\mathrm{MEP}(f, \varphi)$.

(3) If $f(\cdot, \cdot) \equiv 0$, the problem (1) reduces to the mixed variational inequality of Browder type, which is to find $x \in C$ such that

$$\langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{5}$$

The set of solutions to (5) is denoted by $\mathrm{MVI}(A, \varphi, C)$.

(4) If $f(\cdot, \cdot) \equiv 0$ in (3), the problem (3) reduces to the classic variational inequality, which is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \tag{6}$$

which is called the Hartmann-Stampacchia variational inequality. The set of solutions to (6) is denoted by $\mathrm{VI}(A, C)$.

(5) If $A \equiv 0$ in (3), the problem (3) reduces to the classic equilibrium problem, which is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{7}$$

The set of solutions to (7) is denoted by $\mathrm{EP}(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x \rangle$ for all

$x, y \in C$. Then $p \in \mathrm{EP}(f)$ if and only if $\langle Tp, y - p\rangle \geq 0$ for all $y \in C$; that is, p is a solution of the variational inequality.

(6) If $f(\cdot,\cdot) \equiv 0$ in (4), the problem (4) reduces to the minimize problem, which is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{8}$$

The set of solutions to (8) is denoted by Argmin(φ).

The problem (1) is very general in the sense that it includes, as special case, optimization problems, variational inequalities, minimax problems, monotone inclusion problems, saddle point problems, vector equilibrium problems, and the Nash equilibrium problem in noncooperative games. Numerous problems in physics, optimization, and economics reduce to finding a solution of some special case or the problem (1). Some solution methods have been proposed to solve the problems (1), (3)–(8) in Hilbert spaces and Banach spaces; see, for example, [1–7] and references therein.

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $S_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E, and define $f : S_E \times S_E \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by

$$f(x, y, t) = \frac{\|x + ty\| - \|x\|}{t} \tag{9}$$

for $x, y \in S_E$ and $t \in \mathbb{R} \setminus \{0\}$. A Banach space E is said to be smooth if the limit $\lim_{t\to 0} f(x, y, t)$ exists for each $x, y \in S_E$. It is also said to be uniformly smooth if the limit $\lim_{t\to 0} f(x, y, t)$ is attained uniformly for $(x, y) \in S_E \times S_E$.

The modulus of convexity of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \right. \\ \left. \|x\| = \|y\| = 1,\ \|x - y\| \geq \varepsilon\right\}. \tag{10}$$

A Banach space E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. Observe that every p-uniformly convex is uniformly convex. One should note that no Banach space is p-uniformly convex for $1 < p < 2$. It is well known that $L_p(l_p)$ or W_m^p is p-uniformly convex if $p \geq 2$ and 2-uniformly convex if $1 < p \leq 2$; see [8] for more details.

For each $p > 1$, the generalized duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \right. \\ \left. \|x^*\| = \|x\|^{p-1}\right\}, \quad \forall x \in E. \tag{11}$$

In particular, if $p = 2$, J_p is called the normalized duality mapping. If E is a Hilbert space, then $J_p = I$, where I is the identity mapping. In this paper, We denote by J the normalized duality mapping. It is known that the duality mapping J has the following properties:

(i) if E is smooth, then J is single valued;

(ii) if E is strictly convex, then J is one to one;

(iii) if E is reflexive, then J is surjective;

(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(v) if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is single valued and also one to one (see [9–12]).

Let E be a smooth Banach space. Consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{12}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{13}$$

We also know that $\phi(x, y) = 0$ if and only if $x = y$ (see [13]). Moreover, if E is a Hilbert space, (12) reduces to $\phi(x, y) = \|x - y\|^2$, for any $x, y \in E$.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A point p in C is said to be a strong asymptotic fixed point of T [14] if C contains a sequence $\{x_n\}$ which converges strong to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Let $T : C \to C$ be a mapping, and recall the following definition:

(a) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \tag{14}$$

(b) T is called relatively nonexpansive if $\widehat{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T); \tag{15}$$

(c) a mapping T is said to be weak relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T); \tag{16}$$

(d) a mapping T is called hemirelatively nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T). \tag{17}$$

Remark 1. From the definitions, it is obvious that a relatively nonexpansive mapping is a weak relatively nonexpansive mapping, and a weak relatively nonexpansive mapping is a hemi-relatively nonexpansive mapping, but the converse is not true.

Next, we give an example which is a closed hemirelatively nonexpansive mapping.

Example 2. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset $C \subset E$. Then Π_C is a relatively nonexpansive mapping, and then it is also a closed hemi-relatively nonexpansive mapping.

In 2005, Matsushita and Takahashi [13] obtained strong convergence theorems for a single relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space E. To be more precise, they proved the following theorem.

Theorem MT (see Matsushita and Takahashi [13, Theorem 3.1]). *Let E be precisely a uniformly convex and uniformly smooth Banach space and C a nonempty closed convex subset of E, and let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{gathered} x_0 = x \in C, \\ y_n = \Pi_C\left(\alpha_n J x_n + (1-\alpha_n) J T x_n\right), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{gathered} \tag{18}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Since then, algorithms constructed for solving the same equilibrium problem, variational inequality problems, and fixed point of relatively nonexpansive mappings (or weak relatively nonexpansive mappings or hemi-relatively nonexpanisve mappings) have been further developed by many authors. For a part of works related to these problems, please see [4, 15–18], and for the hybrid algorithm projection methods for these problems, please see [19–44] and the references therein.

Motivated and inspired by the results in the literature, in this paper we focus our attention on finding a common fixed point of two countable families of hemi-relatively nonexpansive mappings (we shall give the definition of a countable family of hemi-relatively nonexpansive mappings in the next section) by using a simple hybrid algorithm. Furthermore, we will give some applications of our main result in equilibrium problems, variational inequality problems, and convex feasibility problems.

2. Preliminaries

Let C be a closed convex subset of E, and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into itself. We denote by F the set of common fixed points of $\{T_n\}_{n=0}^{\infty}$. That is, $F = \bigcap_{n=0}^{\infty} F(T_n)$, where $F(T_n)$ denote the set of fixed points of T_n, for all $n \in \mathbb{N} \cup \{0\}$.

Recall that $\{T_n\}_{n=0}^{\infty}$ is said to be uniformly closed, if $p \in \bigcap_{n=1}^{\infty} F(T_n)$, whenever $\{x_n\} \subset C$ converges strongly to p and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$ (see [45] for more details).

A point $p \in C$ is said to be an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$. The set of asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$.

A point $p \in C$ is said to be a strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|T_n x_n - x_n\| = 0$. The set of strong asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Using the definition of (strong) asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$, Su et al. [46] introduced the following definitions.

Definition 3 (see Su et al. [46]). Countable family of mappings $\{T_n\}$ is said to be countable family of relatively nonexpansive mappings if $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$ and

$$\phi(p, T_n x) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T_n),\ n \in \mathbb{N} \cup \{0\}. \tag{19}$$

Definition 4 (see Su et al. [46]). Countable family of mappings $\{T_n\}$ is said to be countable family of weak relatively nonexpansive mappings if $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$ and

$$\phi(p, T_n x) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T_n),\ n \in \mathbb{N} \cup \{0\}. \tag{20}$$

Now, we introduce the definition of countable family of hemi-relatively nonexpansive mappings which is more general than countable family of relatively nonexpansive mappings and countable family of weak relatively nonexpansive mappings.

Definition 5. Countable family of mappings $\{T_n\}$ is said to be countable family of hemi-relatively nonexpansive mappings if $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$ and

$$\phi(p, T_n x) \le \phi(p, x), \quad \forall x \in C,\ p \in F(T_n),\ n \in \mathbb{N} \cup \{0\}. \tag{21}$$

Remark 6. From Definitions 3–5, one has the following facts.

(1) The definitions of relatively nonexpansive mapping, weak relatively nonexpansive mapping, and hemi-relatively nonexpansive mapping are special cases of Definitions 3, 4, and 5 as $T_n \equiv T$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) Countable family of hemi-relatively nonexpansive mappings, which do not need the restriction $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty})$ (or $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty})$), is more general than countable family of relatively nonexpansive mappings (or countable family of weak relatively nonexpansive mappings).

Next we give an example which is a countable family of hemi-relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings.

Example 7. Let E be any smooth Banach space and $x_0 = (1 + 1/n)^n x_0 \neq 0$ any element of E. Define a countable family of mappings $T_n : E \to E$ as follows: for all $n \geq 1$,

$$T_n(x) = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) x_0, & \text{if } x = \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, \\ -x, & \text{if } x \neq \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0. \end{cases} \quad (22)$$

Then $\{T_n\}_{n=1}^{\infty}$ is a countable family of hemi-relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings.

Proof. First, it is obvious that T_n has a unique fixed point 0; that is, $F(T_n) = \{0\}$ for all $n \geq 1$. In addition, one easily sees that

$$\|T_n x\| \leq \|x\|, \quad \forall x \in E, \ n \geq 1. \quad (23)$$

This implies that

$$\|T_n x\|^2 - \|x\|^2 \leq 2\langle 0, JT_n x - Jx\rangle = 2\langle p, JT_n x - Jx\rangle, \quad (24)$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from the above inequality that

$$\|p\|^2 - 2\langle p, JT_n x\rangle + \|T_n x\|^2 \leq \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2, \quad (25)$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $x \in E$. That is,

$$\phi(p, T_n x) \leq \phi(p, x), \quad (26)$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $x \in E$. Hence, $\{T_n\}_{n=1}^{\infty}$ is a countable family of hemi-relatively nonexpansive mappings. On the other hand, letting

$$x_n = \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, \quad \forall n \geq 1, \quad (27)$$

from the definition of T_n, one has

$$T_n x_n = \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) x_0, \quad \forall n \geq 1, \quad (28)$$

which implies that $\|x_n - T_n x_n\| \to 0$ and $x_n \to e\check{x}_0 (x_n \rightharpoonup e\check{x}_0)$ as $n \to \infty$. That is, $e\check{x}_0 \in \widehat{F}(\{T_n\}_{n=0}^{\infty})$ but $e\check{x}_0 \notin F(\{T_n\}_{n=0}^{\infty})$, which shows that $\{T_n\}_{n=1}^{\infty}$ is not a countable family of relatively nonexpansive mappings. □

In what follows, we will need the following lemmas.

Lemma 8 (see Alber [47]). *Let C be a convex subset of a smooth real Banach space E. Let $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx\rangle \geq 0, \quad \forall z \in C. \quad (29)$$

Lemma 9 (see Alber [47]). *Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth real Banach space E, and let $x \in E$. Then for each $y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x). \quad (30)$$

Lemma 10 (see Kamimura and Takahashi [48]). *Let E be a uniformly convex and smooth real Banach space, and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

3. Main Results

Now, we give our main results in this paper.

Theorem 11. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $\{S_n\}$, $\{T_n\}$ be two uniformly closed countable families of hemi-relatively nonexpansive mappings from C into itself such that*

$$\mathcal{F} = \left\{\bigcap_{n=1}^{\infty} F(S_n)\right\} \bigcap \left\{\bigcap_{n=1}^{\infty} F(T_n)\right\} \neq \emptyset. \quad (31)$$

For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{gathered} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, S_n y_n) \leq \phi(z, T_n x_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{gathered} \quad (32)$$

where the sequences $y_n = T_n x_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from C onto $\mathcal{F}$.

Proof. We first show that C_{n+1} is closed and convex. It is obvious that C_{n+1} is closed. Since

$$\begin{aligned} &\phi(z, S_n y_n) \leq \phi(z, T_n x_n) \\ &\quad \Longleftrightarrow \|S_n y_n\|^2 - \|T_n x_n\| \\ &\quad - 2\langle z, JS_n y_n - JT_n x_n\rangle \geq 0, \end{aligned} \quad (33)$$

$$\begin{aligned} &\phi(z, T_n x_n) \leq \phi(z, x_n) \\ &\quad \Longleftrightarrow \|T_n x_n\|^2 - \|x_n\| \\ &\quad - 2\langle z, JT_n x_n - Jx_n\rangle \geq 0, \end{aligned} \quad (34)$$

C_{n+1} is convex. Therefore, C_{n+1} is closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Let $u \in \mathcal{F}$; from the definition of S_n and T_n, we have

$$\phi(u, S_n y_n) \leq \phi(u, y_n) = \phi(u, T_n x_n) \leq \phi(u, x_n). \quad (35)$$

Hence, we have $u \in C_{n+1}$. This implies that $\mathcal{F} \subset C_{n+1}$ for arbitrary $n \in \mathbb{N} \cup \{0\}$.

Noticing $x_n = \Pi_{C_n} x_0$, from Lemma 8, we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \quad (36)$$

Since $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$, we arrive at

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0, \quad \forall p \in \mathcal{F}. \quad (37)$$

From Lemma 9, we have

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0) \end{aligned} \quad (38)$$

for each $p \in \mathcal{F} \subset C_n$ and for all $n \in \mathbb{N} \cup \{0\}$. So the sequence $\{\phi(x_n, x_0)\}$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{39}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, we have that $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for any positive integer $m \ge n$. It follows that

$$\phi(x_m, x_n) = \phi\left(x_m, \Pi_{C_n} x_0\right) \le \phi(x_m, x_0) - \phi(x_n, x_0). \tag{40}$$

Letting $m, n \to \infty$ in (40), by the existence of the limit of $\{\phi(x_n, x_0)\}$, we have $\phi(x_m, x_n) \to 0$. It follows from Lemma 10 that $x_n - x_m \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $q \in C$ such that $x_n \to q$ as $n \to \infty$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have from the definition of C_{n+1} that

$$\begin{aligned} \phi(x_{n+1}, S_n y_n) &\le \phi(x_{n+1}, T_n x_n) \\ &\le \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{41}$$

From the inequality above, we have

$$\begin{aligned} \phi(x_{n+1}, T_n x_n) &\le \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}, \\ \phi(x_{n+1}, S_n y_n) &\le \phi(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{42}$$

On the other hand, taking $m = n + 1$ in (40), we have

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{43}$$

From (42) and (43), we have that

$$\begin{aligned} \lim_{n \to \infty} \phi(x_{n+1}, T_n x_n) &= 0, \\ \lim_{n \to \infty} \phi(x_{n+1}, S_n y_n) &= 0. \end{aligned} \tag{44}$$

By using Lemma 10, the inequalities (43) and (44) follow that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0, \tag{45}$$

$$\lim_{n \to \infty} \|x_{n+1} - T_n x_n\| = 0, \tag{46}$$

$$\lim_{n \to \infty} \|x_{n+1} - S_n y_n\| = 0. \tag{47}$$

Respectively, noticing that

$$\begin{aligned} \|x_n - T_n x_n\| &= \|x_n - x_{n+1} + x_{n+1} - T_n x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n x_n\|. \end{aligned} \tag{48}$$

It follows from (45) and (46) that

$$\lim_{n \to \infty} \|x_n - T_n x_n\| = 0. \tag{49}$$

From uniform closedness of $\{T_n\}$, we get $q \in \bigcap_{n=1}^{\infty} F(T_n)$. On the other hand, noticing that $y_n = T_n x_n$, we have

$$\begin{aligned} \lim_{n \to \infty} y_n &= \lim_{n \to \infty} x_n = q, \\ \|y_n - S_n y_n\| &= \|y_n - x_{n+1} + x_{n+1} - S_n y_n\| \\ &\le \|y_n - x_{n+1}\| + \|x_{n+1} - S_n y_n\| \\ &\le \|T_n x_n - x_{n+1}\| + \|x_{n+1} - S_n y_n\|. \end{aligned} \tag{50}$$

It follows from (46) and (47) that

$$\lim_{n \to \infty} \|y_n - S_n y_n\| = 0. \tag{51}$$

From uniform closedness of $\{S_n\}$, we also have $q \in \bigcap_{n=1}^{\infty} F(S_n)$. Therefore, $q \in \mathcal{F}$.

Finally, we show that $q = \Pi_{\mathcal{F}} x_0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - p, Jx_0 - Jx_n \rangle \ge 0, \quad \forall p \in \mathcal{F} \subset C_n. \tag{52}$$

Taking the limit as $n \to \infty$ in (52), we obtain

$$\langle q - p, Jx_0 - Jq \rangle \ge 0, \quad \forall p \in \mathcal{F}, \tag{53}$$

and hence $p = \Pi_{\mathcal{F}} x_0$ from Lemma 8. This completes the proof. □

Remark 12. Theorem 11 improves Theorem 3.15 of Zhang et al. [49] in the following senses:

(1) from the class of a countable family of weak relatively nonexpansive mappings to the one of a countable family of hemi-relatively nonexpansive mappings;

(2) from a single countable family of mappings to two countable families of mappings.

When $T_n = I$ in (32), we can obtain the following corollary immediately.

Corollary 13. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $\{S_n\}$ be a uniformly closed countable family of hemi-relatively nonexpansive mappings from C into itself such that*

$$\mathcal{F} = \left\{ \bigcap_{n=1}^{\infty} F(S_n) \right\} \ne \emptyset. \tag{54}$$

For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{aligned} C_0 &= C, \\ C_{n+1} &= \{z \in C_n : \phi(z, S_n x_n) \le \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0. \end{aligned} \tag{55}$$

Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from C onto $\mathcal{F}$.

Remark 14. We notice that if $\{S_n\}$ is a countable family of weak relatively nonexpansive mappings, Corollary 13 is still held. Therefore, Corollary 13 extends and improves Theorem 3.15 in [49].

4. Applications to Convex Feasibility Problems

In this section, we consider the following convex feasibility problem (CFP):

$$\text{finding an } x \in \bigcap_{n=1}^{\infty} C_n, \tag{56}$$

where $n \in \mathbb{N} \cup \{0\}$, and $\{C_n\}_{n=0}^{\infty}$ is an intersecting closed convex subset sequence of a Banach space E. This problem is a frequently appearing problem in diverse areas of mathematical and physical sciences. There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration [50–53], computer tomography [54], and radiation therapy treatment planning [55]. In computer tomography with limited data, in which an unknown image has to be reconstructed from a priori knowledge and from measured results, each piece of information gives a constraint which in turn gives rise to a convex set C_n to which the unknown image should belong (see [56]).

Using Theorem 11, we discuss the convex feasibility problems as an application.

Theorem 15. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $\{\Omega_n\}_{n=0}^{\infty}$, $\{\Omega_n^*\}_{n=0}^{\infty}$ be two countable families of nonempty closed convex subset of C such that*

$$\Omega = \left\{\bigcap_{n=0}^{\infty} \Omega_n\right\} \bigcap \left\{\bigcap_{n=0}^{\infty} \Omega_n^*\right\} \neq \emptyset. \tag{57}$$

For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{gathered} C_0 = C, \\ C_{n+1} = \left\{z \in C_n : \phi\left(z, \Pi_{\Omega_n} y_n\right) \leq \phi\left(z, \Pi_{\Omega_n^*} x_n\right) \leq \phi\left(z, x_n\right)\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{gathered} \tag{58}$$

where the sequences $y_n = \Pi_{\Omega_n^*} x_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\Omega} x_0$, where Π_{Ω} is the generalized projection from C onto Ω.

Proof. From Lemma 9, we easily have that $\{\Pi_{\Omega_n}\}$ and $\{\Pi_{\Omega_n^*}\}$ are two countable families of hemi-relatively nonexpansive mappings. In view of the continuity of Π_{Ω_n} and $\Pi_{\Omega_n^*}$, we have that $\{\Pi_{\Omega_n}\}$ and $\{\Pi_{\Omega_n^*}\}$ are two uniformly closed countable families of hemi-relatively nonexpansive mappings. Thus, by using Theorem 11, we have that the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\Omega} x_0$. This completes the proof. □

If we only consider a countable family of nonempty closed convex subset of C, the following corollary can be obtained by using Theorem 15.

Corollary 16. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $\{\Omega_n\}_{n=0}^{\infty}$ be a countable family of nonempty closed convex subset of C such that*

$$\Omega = \left\{\bigcap_{n=0}^{\infty} \Omega_n\right\} \neq \emptyset. \tag{59}$$

For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{gathered} C_0 = C, \\ C_{n+1} = \left\{z \in C_n : \phi\left(z, \Pi_{\Omega_n} x_n\right) \leq \phi\left(z, x_n\right)\right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{gathered} \tag{60}$$

Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\Omega} x_0$, where Π_{Ω} is the generalized projection from C onto Ω.

Proof. Putting $\Pi_{\Omega_n^*} \equiv I$ for all $n \in \mathbb{N} \cup \{0\}$ in algorithm (58), the conclusion can be obtained from Theorem 15 immediately. □

5. Applications to Generalized Mixed Equilibrium Problems

In this section, we apply our main results to prove some strong convergence theorems concerning generalized mixed equilibrium problems in a Banach space E.

Let $A : C \to E^*$ be a mapping. First, we recall the following definition:

(I) A is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C; \tag{61}$$

(II) A is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{62}$$

We remark here that an α-inverse strongly monotone A is $(1/\alpha)$-Lipschitz continuous.

For solving the generalized mixed equilibrium problem (1), let us assume that the nonlinear mapping $A : C \to E^*$ is monotone and continuous, the function $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous, and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A_1) $f(x, x) = 0$, for all $x \in C$;

(A_2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;

(A_3) $\limsup_{t \downarrow 0} f(x + t(z - x), y) \leq f(x, y)$, for all $x, y, z \in C$;

(A_4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous for all $x \in C$.

The following result can be found in Blum and Oettli [1].

Lemma 17 (see Blum and Oettli [1]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space*

E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{63}$$

Lemma 18. *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let $A : C \to E^*$ be a monotone and continuous mapping, let the function $\varphi : C \to \mathbb{R}$ be convex and lower semicontinuous, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). Then, $f(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$ satisfies (A_1)–(A_4).*

Proof. For convenience, we set $F(x, y) = f(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$. So, we only need to prove that $F(x, y)$ satisfies (A_1)–(A_4).

(I) We show that $F(x, x) = 0$, for all $x \in C$. Since $f(x, y)$ satisfies (A_1), we have

$$\begin{aligned} F(x, x) = {} & f(x, x) + \langle Ax, x - x\rangle \\ & + \varphi(x) - \varphi(x) = f(x, x) = 0, \quad \forall x \in C. \end{aligned} \tag{64}$$

(II) We show that F is monotone; that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in C$; since A is continuous and monotone, from (A_2), we have

$$\begin{aligned} & F(x, y) + F(y, x) \\ & \quad = f(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \\ & \qquad + f(y, x) + \langle Ay, x - y\rangle + \varphi(x) - \varphi(y) \\ & \quad = f(x, y) + \langle Ax, y - x\rangle + f(y, x) + \langle Ay, x - y\rangle \\ & \quad \leq 0 + \langle Ax - Ay, y - x\rangle = -\langle Ay - Az, y - x\rangle \leq 0. \end{aligned} \tag{65}$$

(III) We show that $\limsup_{t\downarrow 0} F(x + t(z - x), y) \leq F(x, y)$, for all $x, y, z \in C$; Since A is continuous and φ is lower semicontinuous, we have

$$\begin{aligned} & \limsup_{t\downarrow 0} F(x + t(z - x), y) \\ & \quad = \limsup_{t\downarrow 0} f(x + t(z - x), y) \\ & \qquad + \limsup_{t\downarrow 0} \langle A(x + t(z - x)), y - (x + t(z - x))\rangle \\ & \qquad + \limsup_{t\downarrow 0} [\varphi(y) - \varphi(x + t(z - x))] \\ & \quad \leq f(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) = F(x, y). \end{aligned} \tag{66}$$

(IV) We show that the function $y \mapsto F(x, y)$ is convex and lower semicontinuous for each $x \in C$.

For each $x \in C$, for all $t \in (0, 1)$ and for all $y, z \in C$, since f satisfies (A_4) and φ is convex, we have

$$\begin{aligned} & F(x, ty + (1 - t)z) \\ & \quad = f(x, ty + (1 - t)z) \\ & \qquad + \langle Ax, ty + (1 - t)z - x\rangle \\ & \qquad + \varphi(ty + (1 - t)z) - \varphi(x) \\ & \quad = t[f(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)] \\ & \qquad + (1 - t)[f(x, t) + \langle Ax, z - x\rangle + \varphi(z) - \varphi(x)] \\ & \quad = tF(x, y) + (1 - t)F(x, z). \end{aligned} \tag{67}$$

This completes the proof. □

Lemma 19 (see Takahashi and Zembayashi [17]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$\begin{aligned} T_r(x) = \Big\{ & z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \\ & \geq 0, \forall y \in C \Big\} \end{aligned} \tag{68}$$

for all $x \in E$. Then, the following properties hold:

(1) *T_r is single valued;*

(2) *T_r is a firmly nonexpansive-type mapping; that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{69}$$

(3) $\mathrm{EP}(f) = F(T_r) = \widehat{F}(T_r)$;

(4) *$\mathrm{EP}(f)$ is closed and convex;*

(5) *$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, for all $x \in E$, $q \in F(T_r)$.*

Lemma 20 (see Zhang et al. [57]). *Let E be a p-uniformly convex with $p \geq 0$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n = r > 0$. Then the sequence of mappings T_{r_n} is uniformly closed.*

Next, we shall apply Theorem 11 to solve two generalized mixed equilibrium problems. To accomplish this purpose, let $A, B : C \to E^*$ be two monotone and continuous mappings, let the function $\varphi, \psi : C \to \mathbb{R}$ be convex and lower semicontinuous, and let f and g be a bifunction from

$C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). For $r > 0$ and $x \in E$, define two mappings $J_r, K_r : E \to C$ as follows:

$$J_r(x) = \left\{ z \in C : f(z, y) + \langle Az, y - z \rangle + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}, \tag{70}$$

$$K_r(x) = \left\{ z \in C : g(z, y) + \langle Bz, y - z \rangle + \psi(y) - \psi(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}. \tag{71}$$

Theorem 21. *Let E be a p-uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $A, B : C \to E^*$ be two monotone and continuous mappings, let the function $\varphi, \psi : C \to \mathbb{R}$ be convex and lower semicontinuous, and let f and g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) such that $\mathfrak{F} = \mathrm{GMEP}(f, A, \varphi) \bigcap \mathrm{GMEP}(g, B, \psi) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$C_0 = C,$$
$$C_{n+1} = \{ z \in C_n : \phi(z, v_n) \leq \phi(z, u_n) \leq \phi(z, x_n) \}, \tag{72}$$
$$x_{n+1} = \Pi_{C_{n+1}} x_0,$$

where $u_n = J_{r_n} x_n$, $v_n = K_{r_n} u_n$, and $\lim_{n \to \infty} r_n = r$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathfrak{F}} x_0$, where $\Pi_{\mathfrak{F}}$ is the generalized projection from C onto $\mathfrak{F}$.

Proof. From Lemmas 18 and 20, we learn that $\{J_{r_n}\}$ and $\{K_{r_n}\}$ are uniformly closed. And by Lemma 19 (5), one can easily get that $\{J_{r_n}\}$ and $\{K_{r_n}\}$ are uniformly closed countable families of hemi-relatively nonexpansive mappings. Notice that if E is p-uniformly convex, it must be uniformly convex. Therefore, by using Theorem 11, we can obtain the conclusion of Theorem 21. This completes the proof. □

Theorem 22. *Let E be a p-uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a monotone and continuous mappings, let the function $\varphi : C \to \mathbb{R}$ be convex and lower semicontinuous and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) such that $\mathfrak{F} = \mathrm{GMEP}(f, A, \varphi) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$C_0 = C,$$
$$C_{n+1} = \{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \}, \tag{73}$$
$$x_{n+1} = \Pi_{C_{n+1}} x_0,$$

where $u_n = J_{r_n} x_n$ and $\lim_{n \to \infty} r_n = r$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathfrak{F}} x_0$, where $\Pi_{\mathfrak{F}}$ is the generalized projection from C onto $\mathfrak{F}$.

Proof. From Lemmas 18 and 20, we learn that $\{J_{r_n}\}$ is uniformly closed. And by Lemma 19(5), one can easily get that $\{J_{r_n}\}$ is an uniformly closed countable family of hemi-relatively nonexpansive mappings. Notice that if E is p-uniformly convex, it must be uniformly convex. Therefore, by using Corollary 13, we can obtain the conclusion of Theorem 21. This completes the proof. □

If we let $f \equiv 0$, $\varphi \equiv 0$ in (70) and $B \equiv 0$, $\psi \equiv 0$ in (71), the following corollary can be obtained by using Theorem 21.

Corollary 23. *Let E be a p-uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) and $A : C \to E^*$ a monotone and continuous mapping. Suppose that $\mathfrak{F} = \mathrm{VI}(A, C) \bigcap \mathrm{EP}(g) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$C_0 = C,$$
$$C_{n+1} = \{ z \in C_n : \phi(z, v_n) \leq \phi(z, u_n) \leq \phi(z, x_n) \}, \tag{74}$$
$$x_{n+1} = \Pi_{C_{n+1}} x_0,$$

where $u_n = J_{r_n} x_n$, $v_n = K_{r_n} u_n$, and $\lim_{n \to \infty} r_n = r$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathfrak{F}} x_0$, where $\Pi_{\mathfrak{F}}$ is the generalized projection from C onto $\mathfrak{F}$.

Remark 24. By analysis of special cases for generalized mixed equilibrium problem, we can obtain the corresponding results based on Theorems 21 and 22 in sequence. Here, we do not itemize these results.

6. Applications to Maximal Monotone Operators

Let $\mathscr{A}$ be a multivalued operator from E to E^* with domain $D(\mathscr{A}) = \{z \in E : \mathscr{A}z \neq \emptyset\}$ and range $R(\mathscr{A}) = \{z \in E : z \in D(\mathscr{A})\}$. An operator $\mathscr{A}$ is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(\mathscr{A}), \quad y_1 \in \mathscr{A}x_1, \quad y_2 \in \mathscr{A}x_2. \tag{75}$$

A monotone operator $\mathscr{A}$ is said to be maximal if its graph $G(\mathscr{A}) = \{(x, y) : y \in \mathscr{A}x\}$ is not properly contained in the graph of any other monotone operator. It is well known that if $\mathscr{A}$ is a maximal monotone operator, then $\mathscr{A}^{-1}0$ is closed and convex.

The following result is also well known.

Lemma 25 (see Rockafellar [58]). *Let E be a reflexive, strictly convex, and smooth Banach space and $\mathscr{A}$ a monotone operator from E to E^*. Then $\mathscr{A}$ is maximal if and only if $R(J + r\mathscr{A}) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space and $\mathscr{A}$ a maximal monotone operator from E to E^*. Using Lemma 25 and the strict convexity of E, it follows that, for all $r > 0$ and $x \in E$, there exists a unique $x_r \in D(\mathscr{A})$ such that

$$Jx \in Jx_r + r\mathscr{A}x_r. \tag{76}$$

If $J_r x = x_r$, then we can define a single-valued mapping $J_r : E \to D(\mathscr{A})$ by $J_r = (J + r\mathscr{A})^{-1}J$ and such a J_r is called the resolvent of $\mathscr{A}$. We know that $\mathscr{A}^{-1}0 = F(J_r)$ for all $r > 0$ (see [10, 59] for more details).

First, we give an important lemma for this section and remark that the following lemma can be as example of a countable family of hemi-relatively nonexpansive mappings.

Lemma 26. *Let E be a strictly convex and uniformly smooth Banach space, let $\mathscr{A}$ be a maximal monotone operator from E to E^* such that $\mathscr{A}^{-1}0$ is nonempty, and let $\{r_n\}$ be a sequence of positive real numbers which is bounded away from 0 such that $J_{r_n} = (I + r_n\mathscr{A})^{-1}$. Then $\{J_{r_n}\}$ is a uniformly closed countable family of hemi-relatively nonexpansive mappings.*

Proof. One has $\bigcap_{n=0}^{\infty} F(J_{r_n}) = \mathscr{A}^{-1}0 \neq \emptyset$. Firstly, we show J_{r_n} is uniformly closed. Let $\{z_n\}$ be a sequence such that $z_n \to p$ and $\lim_{n\to\infty}\|z_n - J_{r_n}z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}\left(Jz_n - JJ_{r_n}z_n\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{77}$$

It follows from

$$\frac{1}{r_n}\left(Jz_n - JJ_{r_n}z_n\right) \in \mathscr{A}J_{r_n}z_n \tag{78}$$

and the monotonicity of B that

$$\left\langle w - J_{r_n}z_n, w^* - \frac{1}{r_n}\left(Jz_n - JJ_{r_n}z_n\right)\right\rangle \geq 0 \tag{79}$$

for all $w \in D(\mathscr{A})$ and $w^* \in \mathscr{A}w$. Letting $n \to \infty$, one has $\langle w - p, w^*\rangle \geq 0$ for all $w \in D(\mathscr{A})$ and $w^* \in \mathscr{A}w$. Therefore, from the maximality of $\mathscr{A}$, one obtains $p \in \mathscr{A}^{-1}0 = F(J_{r_n})$. Hence, J_{r_n} is uniformly closed.

In addition, for any $w \in E$ and $p \in \bigcap_{n=0}^{\infty} F(J_{r_n})$, from the monotonicity of $\mathscr{A}$, one has

$$\begin{aligned}
\phi\left(p, J_{r_n}w\right) &= \|p\|^2 - 2\left\langle p, JJ_{r_n}w\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&= \|p\|^2 + 2\left\langle p, Jw - JJ_{r_n}w - Jw\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w\rangle - 2\left\langle p, Jw\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&= \|p\|^2 - 2\left\langle J_{r_n}w - p - J_{r_n}w, Jw - JJ_{r_n}w - Jw\right\rangle \\
&\quad - 2\left\langle p, Jw\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&= \|p\|^2 - 2\left\langle J_{r_n}w - p, Jw - JJ_{r_n}w - Jw\right\rangle \\
&\quad + 2\left\langle J_{r_n}w, Jw - JJ_{r_n}w\right\rangle - 2\left\langle p, Jw\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&\leq \|p\|^2 + 2\left\langle J_{r_n}w, Jw - JJ_{r_n}w\right\rangle \\
&\quad - 2\left\langle p, Jw\right\rangle + \left\|J_{r_n}w\right\|^2 \\
&= \|p\|^2 - 2\left\langle p, Jw\right\rangle + \|w\|^2 \\
&\quad - \left\|J_{r_n}w\right\|^2 + 2\left\langle J_{r_n}w, Jw\right\rangle - \|w\|^2 \\
&= \phi\left(p, w\right) - \phi\left(J_{r_n}w, w\right) \leq \phi\left(p, w\right),
\end{aligned} \tag{80}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{J_{r_n}\}$ is a countable family of hemi-relatively nonexpansive mappings. Hence, $\{J_{r_n}\}$ is a uniformly closed countable family of hemi-relatively nonexpansive mappings. □

We consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [4, 13, 49]. Using Theorem 11, we obtain the following result.

Theorem 27. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $\mathscr{A}$, $\mathscr{B}$ be two maximal monotone operators from E to E^* such that $\mathscr{F} = \mathscr{A}^{-1}0 \bigcap \mathscr{B}^{-1}0 \neq 0$, and let $\{r_n\}$ be a sequence of positive real numbers which is bounded away from 0 such that $J_{r_n}^{\mathscr{A}} = (I + r_n\mathscr{A})^{-1}$ and $J_{r_n}^{\mathscr{B}} = (I + r_n\mathscr{B})^{-1}$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{gathered}
C_0 = C, \\
C_{n+1} = \left\{z \in C_n : \phi\left(z, J_{r_n}^{\mathscr{B}}y_n\right) \leq \phi\left(z, J_{r_n}^{\mathscr{A}}x_n\right) \leq \phi\left(z, x_n\right)\right\}, \\
x_{n+1} = \Pi_{C_{n+1}}x_0,
\end{gathered} \tag{3.1}$$

where the sequences $y_n = J_{r_n}^{\mathscr{A}}x_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathscr{F}}x_0$, where $\Pi_{\mathscr{F}}$ is the generalized projection from C onto $\mathscr{F}$.

Proof. From Lemma 26, we know that $\{J_{r_n}^{\mathscr{A}}\}$ and $\{J_{r_n}^{\mathscr{B}}\}$ are two uniformly closed countable families of hemi-relatively nonexpansive mappings. Furthermore, applying Theorem 11, one sees that the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathscr{F}}x_0$. □

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