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Research Article

Peaked and Smooth Solitons for $K^*(4,1)$ Equation

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This paper is contributed to explore all possible single peak solutions for the $K^*(4,1)$ equation $u_t = u_x u^2 + 2\alpha(u u_{xxx} + 2u_x u_{xx})$. Our procedure shows that the $K^*(4,1)$ equation either has peakon, cuspon, and smooth soliton solutions when sitting on a nonzero constant pedestal $\lim_{\xi \to \pm\infty} u = A \neq 0$ or possesses compacton solutions only when $\lim_{\xi \to \pm\infty} u = A = 0$. We present a new smooth soliton solution in an explicit form. Mathematical analysis and numeric graphs are provided for those soliton solutions of the $K^*(4,1)$ equation.

1. Introduction

It is well known that the study of nonlinear wave equations and their solutions is of great importance in many areas of physics.

In 1993, Cooper et al. [1] considered the following generalized KdV equation (GKdV):

$$K^*(l,p): u_t = u_x u^{l-2} + \alpha \left[2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3\right], \quad (1)$$

where $l, p \in Z^+$. These equations are derived from Lagrangian

$$L(l,p) = \int \left[\frac{1}{2}\psi_x \psi_t - \frac{(\psi_x)^l}{l(l-1)} + \alpha(\psi_x)^p(\psi_{xx})^2\right] dx, \quad (2)$$

where $u(x,t)$ is defined by $u(x,t) = \psi_x(x,t)$. These equations have the same terms as the following equations, considered by Rosenau and Hyman [2]:

$$K(m,n): u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1, \quad (3)$$

but relative weights of the terms are quite different leading to the possibility that the integrability properties might be different. Cooper et al. [1] investigated Hamiltonian structure and integrability properties for this class of KdV equations. By using bifurcation theory of dynamical systems, when $l, p \geq 2$, Tang and Li [3] investigated the bifurcation behavior for traveling wave solutions of (1). In [4], by using sine-cosine method and extended tanh method, some new solitary patterns solutions and compactons solutions are formally derived. In [5], by using analytic methods from the dynamical systems theory, some new exact explicit parametric representations of breaking loop-solutions under some fixed parameter conditions are formally derived.

In the development of soliton theory, there exist many different approaches to searching for exact solutions of nonlinear partial differential equations, such as mapping method [6], fan-expansion method [7], and (G'/G)-expansion method [8]. In particular, it is very interesting to investigate the traveling wave solutions on a constant pedestal. Qiao and Zhang [9] discussed the traveling wave solutions for the Camassa-Holm equation on the nonzero constant pedestal $\lim_{\xi \to \pm\infty} u = A \neq 0$ and found new soliton solutions, which are smooth and cusped. Later, Zhang and Qiao [10] investigated the Degasperis-Procesi equation under the boundary condition $\lim_{\xi \to \pm\infty} u = A$ and obtained all possible single peak soliton solutions of the Degasperis-Procesi equation. Recently, Chen and Li [11] studied osmosis $K(2,2)$ equation with a nonzero constant pedestal and obtained smooth, peaked, and cusped soliton solutions of the osmosis $K(2,2)$ equation. More recently, Zhang et al. [12] studied the $K(2,2)$ equation under an inhomogeneous boundary condition and obtained compacton solutions, loop soliton solutions, cusped soliton solutions, and smooth soliton solutions.

In the literature [1, 3–5], the authors did not investigate the existence of peakon soliton for the $K^*(l, p)$ equation. We are thus interested in an important question that should be investigated: does the $K^*(l, p)$ equation have peakon soliton? We hope to answer this problem in this paper. Assume that $l = 4$, $p = 1$. Then, (1) becomes (simply called $K^*(4, 1)$ equation)

$$u_t = u_x u^2 + 2\alpha\left(u u_{xxx} + 2u_x u_{xx}\right). \tag{4}$$

We will study single peak solitary solutions of $K^*(4,1)$ equation under inhomogeneous boundary condition

$$\lim_{\xi \to \pm\infty} u = A. \tag{5}$$

Peakon, compacton, cuspon, and smooth soliton solutions are obtained. Our method is based on phase portrait analysis technique under an inhomogeneous boundary condition.

2. Asymptotic Behavior of Solutions

In this section, we first introduce some notations. Let $C^k(\Omega)$ denote the set of all k times continuously differential functions on the open set Ω. $L^p_{\mathrm{loc}}(R)$ refers to the set of all functions whose restriction on any compact subset is L^p integrable. $H^1_{\mathrm{loc}}(R)$ stands for $H^1_{\mathrm{loc}}(R) = \{u \in L^2_{\mathrm{loc}}(R) \mid u' \in L^2_{\mathrm{loc}}(R)\}$. Assume that $sn(x,k)$, $cn(x,k)$, $dn(x,k)$ are the Jacobian elliptic functions with the modulus $k, K(k)$ is the first kind of complete elliptic integral, $E(\phi, k)$ is the normal elliptic integral of the 2nd kind, and $\prod(\phi, \alpha^2, k)$ is the normal elliptic integral of the 3rd kind [13].

Let us consider the traveling wave solution of $K^*(4,1)$ equation (4) through a generic setting $u(x,t) = u(\xi)$, $\xi = x - ct$, where c is wave speed. Substituting it into (4) yields

$$-cu' = \frac{1}{3}\left(u^3\right)' + \alpha\left[2\left(uu''\right)' + \left(\left(u'\right)^2\right)'\right], \tag{6}$$

where "$'$" is the derivative with respect to ξ. Taking the integration twice on both sides leads to

$$\left(u'\right)^2 = \frac{u^4 + 6cu^2 + 12g_1 u + 12g_2}{-12\alpha u}, \tag{7}$$

where $g_1, g_2 \in R$ are two integration constants. Let us solve (7) with the boundary condition (5). From (7) we obtain that

$$\begin{aligned}\left(u'\right)^2 &= \frac{1}{12\alpha}\left[-u^3 - 6cu - \frac{3A^2\left(A^2+2c\right)}{u} + 4A\left(A^2+3c\right)\right] \\ &= \frac{(u-A)^2\left(u^2+2Au+3A^2+6c\right)}{-12\alpha u}.\end{aligned} \tag{8}$$

If $A^2 + 3c < 0$, then (8) reduces to

$$\left(u'\right)^2 = \frac{(u-A)^2\left(u-B_1\right)\left(u-B_2\right)}{-12\alpha u}, \tag{9}$$

where

$$B_1 = -A + \sqrt{-2\left(A^2+3c\right)}, \qquad B_2 = -A - \sqrt{-2\left(A^2+3c\right)}. \tag{10}$$

Obviously, $B_1 \geq B_2$. We assume that $\alpha > 0$ throughout the paper since there are similar results for the case $\alpha < 0$.

Definition 1. A function $u(\xi)$ is said to be a single peak soliton solution for $K^*(4,1)$ equation (4) if $u(\xi)$ satisfies the following conditions:

(A1) $u(\xi)$ is continuous on R and has a unique peak point ξ_0, where $u(\xi)$ attains its global maximum or minimum value;

(A2) $u(\xi) \in C^3(R - \{\xi_0\})$ satisfies (8) on $R - \{\xi_0\}$;

(A3) $\lim_{\xi \to \pm\infty} u(\xi) = A$.

Definition 2. A wave function u is called a peakon if u is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u'(\xi) = -\lim_{\xi \downarrow \xi_0} u'(\xi) = a$, $a \neq 0$, $a \neq \pm\infty$.

Definition 3. A wave function u is called a cuspon if u is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} u'(\xi) = -\lim_{\xi \downarrow \xi_0} u'(\xi) = +\infty$ (or $-\infty$).

Without loss of generality, one assumes that $\xi_0 = 0$.

Lemma 4. *Equation* (4) *has trivial solution $u \equiv A$, if one of the following three conditions holds:*

(i) $A \geq 0$, $c > 0$;

(ii) $A > \sqrt{-c}$, $c < 0$;

(iii) $-\sqrt{-c} \leq A < 0$, $c < 0$.

Proof. (i) If $A \geq 0$, $c > 0$, then $u \geq 0$, $2(A^2 + 3c) > 0$, $\lim_{\xi \to \pm\infty} u(\xi) = A$, and

$$\left(u'\right)^2 = \frac{(u-A)^2\left[(u+A)^2 + 2\left(A^2+3c\right)\right]}{-12\alpha u} \leq 0. \tag{11}$$

The fact that $(u')^2 \geq 0$ implies that $u' = 0$ and $u \equiv A$.

If $A = 0$, $c > 0$, then $2(A^2 + 3c) > 0$, $\lim_{\xi \to \pm\infty} u(\xi) = 0$, and $(u')^2 = u(u^2 + 6c)/-12\alpha \leq 0$. The fact that $(u')^2 \geq 0$ implies that $u' = 0$ and $u \equiv A$.

(ii) If $A \geq \sqrt{-3c}$, $c < 0$, then $u > 0$, $2(A^2 + 3c) \geq 0$, $\lim_{\xi \to \pm\infty} u(\xi) = A > 0$, and

$$\left(u'\right)^2 = \frac{(u-A)^2\left[(u+A)^2 + 2\left(A^2+3c\right)\right]}{-12\alpha u} \leq 0. \tag{12}$$

The fact that $(u')^2 \geq 0$ implies that $u' = 0$ and $u \equiv A$.

If $\sqrt{-c} < A < \sqrt{-3c}$, $c < 0$, then $2(A^2 + 3c) < 0$, $A > B_1 = -A + \sqrt{-2(A^2 + 3c)} > B_2$, $\lim_{\xi \to \pm\infty} u(\xi) = A > 0$, and

$$\left(u'\right)^2 = \frac{(u-A)^2\left(u-B_1\right)\left(u-B_2\right)}{-12\alpha u} \leq 0. \tag{13}$$

The fact that $(u')^2 \geq 0$ implies that $u' = 0$ and $u \equiv A$.

(iii) If $-\sqrt{-c} \leq A < 0$, $c < 0$, then $2(A^2 + 3c) < 0$, $B_2 \leq A < B_1$, $\lim_{\xi \to \pm\infty} u(\xi) = A < 0$, and

$$\left(u'\right)^2 = \frac{(u - A)^2 (B_1 - u)(u - B_2)}{12\alpha u} \leq 0. \tag{14}$$

The fact that $(u')^2 \geq 0$ implies $u' = 0$ and $u \equiv A$. □

Theorem 5. *Suppose that $u(\xi)$ is a single peak solitary wave solution for (4) at the peak point $\xi_0 = 0$. If $A < 0$, $c > 0$ or $0 \leq A \leq \sqrt{-c}$, $c < 0$ or $-\sqrt{-3c} \leq A < -\sqrt{-c}$, $c < 0$, then $u(0) = 0$ or $u(0) = B_1$ or $u(0) = B_2$.*

Proof. If $u(0) \neq 0$, then $u(\xi) \neq 0$ for any $\xi \in R$ since $u(\xi) \in C^3(R - \{0\})$. Differentiating both sides of (8) yields $u(\xi) \in C^\infty(R)$.

(i) If $A \geq 0$, $c > 0$ or $A > \sqrt{-c}$, $c < 0$ or $-\sqrt{-c} \leq A < 0$, $c < 0$, from Lemma 4, we know that (4) has trivial solution $u \equiv A$.

(ii) For $A = 0$, $c < 0$, we have $(u')^2 = u(\sqrt{-6c} - u)(u + \sqrt{-6c})/12\alpha$, if $u(0) \neq 0$, then, according to the definition of peak point, we have $u'(0) = 0$; thus, $u(0) = B_1 = \sqrt{-6c}$ or $u(0) = B_2 = -\sqrt{-6c}$.

(iii) For $A < 0$, $c > 0$ or $0 < A \leq \sqrt{-c}$, $c < 0$ or $-\sqrt{-3c} \leq A < -\sqrt{-c}$, $c < 0$, if $u(0) \neq 0$, then $u(\xi) \in C^\infty(R)$. According to the definition of peak point, we have $u'(0) = 0$. Thus, $u(0) = B_1$ or $u(0) = B_2$ since $u(0) = A$ contradicts the fact that 0 is the unique peak point. □

Theorem 6. *Suppose that $u(\xi)$ is a single peak solitary wave solution for (4) at the peak point $\xi_0 = 0$. When $A < 0$, $c > 0$ or $0 \leq A \leq \sqrt{-c}$, $c < 0$ or $-\sqrt{-3c} \leq A < -\sqrt{-c}$, $c < 0$, then we have the following classification and asymptotic behavior of solutions.*

(i) *If $u(0) = B_2$ and $A < B_2 < 0$ or $u(0) = B_1$ and $0 < A < B_1$, then $u(\xi)$ is a smooth solitary wave solution.*

(ii) *If $A = 0, c < 0$, $u(0) = B_1$, then $u(\xi)$ is a compacton solution [2].*

(iii) *If $u(0) = 0$ and $A = -\sqrt{-2c}$, then $u(\xi)$ gives the peakon solution*

$$u(\xi) = \sqrt{-2c} \times \left[2 - 3\tanh^2\left(\tanh^{-1}\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} + \frac{1}{4}\sqrt{\frac{\sqrt{-2c}}{\alpha}}|\xi|\right)\right]. \tag{15}$$

(iv) *If $u(0) = 0$ and $A \neq -\sqrt{-2c}$, $c < 0$ and $A < 0$, $c > 0$ or $0 < A \leq \sqrt{-c}$ or $-\sqrt{-3c} \leq A < -\sqrt{-c}$, then $u(\xi)$ is a cuspon soliton solution and*

$$\begin{aligned} u(\xi) &= -\left(\frac{3}{2h_1(0)}\right)^{2/3}|\xi|^{2/3} + O(|\xi|), \quad \xi \longrightarrow 0, \\ u'(\xi) &= -\frac{2}{3}\left(\frac{3}{2h_1(0)}\right)^{2/3}|\xi|^{-1/3} + O(1), \quad \xi \longrightarrow 0, \end{aligned} \tag{16}$$

or

$$\begin{aligned} u(\xi) &= \left(\frac{3}{2h_2(0)}\right)^{2/3}|\xi|^{2/3} + O(|\xi|), \quad \xi \longrightarrow 0, \\ u'(\xi) &= \frac{2}{3}\left(\frac{3}{2h_2(0)}\right)^{2/3}|\xi|^{-1/3} + O(1), \quad \xi \longrightarrow 0. \end{aligned} \tag{17}$$

Proof. (i) From the process of proving Theorem 5, we know that if $u(0) = B_1 > 0$ or $u(0) = B_2 < 0$, then $u(0) \neq 0$; thus, $u(\xi)$ is a smooth solitary wave solution.

(ii) If $A = 0$, then $B_2 = -\sqrt{-6c}$, $u(0) = B_1 = \sqrt{-6c}$, (9) becomes

$$u' = \sqrt{\frac{u\left(\sqrt{-6c} - u\right)\left(u - \sqrt{-6c}\right)}{12\alpha}}\,\text{sign}(\xi). \tag{18}$$

Integrating both sides of (18) on the interval $(0, \sqrt{-6c}]$ leads to a compacton solution with compact support

$$u(\xi) = \begin{cases} \sqrt{-6c}\left(1 - sn^2\left(\dfrac{1}{4}\sqrt{\dfrac{2\sqrt{-6c}}{3\alpha}}|\xi|, \dfrac{\sqrt{2}}{2}\right)\right), & \\ \qquad \text{if } |\xi| \leq 4\sqrt{\dfrac{3\alpha}{2\sqrt{-6c}}}K\left(\dfrac{\sqrt{2}}{2}\right), & \\ 0, \qquad \text{if } |\xi| > 4\sqrt{\dfrac{3\alpha}{2\sqrt{-6c}}}K\left(\dfrac{\sqrt{2}}{2}\right). & \end{cases} \tag{19}$$

The profile of compacton is shown in Figure 2(b).

Remark 1. To the best of our knowledge, the solution (19) of (4) has not been reported in the literature.

(iii) If $u(0) = 0$ and $A = -\sqrt{-2c}$, then (9) becomes

$$u' = \frac{\left(u + \sqrt{-2c}\right)\sqrt{2\sqrt{-6c} - u}}{2\sqrt{3\alpha}}\,\text{sign}(\xi). \tag{20}$$

Integrating both sides of (20) on the interval $(-\sqrt{-2c}, 0]$ leads to a peakon solution

$$u(\xi) = \sqrt{-2c}\left[2 - 3\tanh^2\left(\tanh^{-1}\sqrt{\frac{2}{3}} + \frac{1}{4}\sqrt{\frac{\sqrt{-2c}}{\alpha}}|\xi|\right)\right], \tag{21}$$

with the following properties:

$$\begin{gathered} u(0) = 0, \qquad u(\pm\infty) = A, \\ u'(0+) = c\sqrt{\frac{\sqrt{-2c}}{-12c\alpha}}, \qquad u'(0-) = -c\sqrt{\frac{\sqrt{-2c}}{-12c\alpha}}. \end{gathered} \tag{22}$$

The profile of peakon solution is shown in Figure 2(e).

Remark 2. To the best of our knowledge, the solution (21) of (4) has not been reported in the literature.

(iv) If $u(0) = 0$, $A \neq 0$, and $A \neq -\sqrt{-2c}$, then $u^2 + 2Au + 3(A^2 + 2c)$ does not contain the factor u. From (9) we obtain

$$\begin{aligned} u' &= (u - A)\frac{\sqrt{u^2 + 2Au + 3(A^2 + 2c)}}{2\sqrt{3\alpha}\sqrt{-u}}\,\mathrm{sign}(\xi), \\ &A < 0, \quad c > 0, \quad \text{or} \quad -\sqrt{-3c} \le A < -\sqrt{-2c}, \\ u' &= -(u - A)\frac{\sqrt{-(u^2 + 2Au + 3(A^2 + 2c))}}{2\sqrt{3\alpha}\sqrt{u}}\,\mathrm{sign}(\xi), \\ &0 < A \le \sqrt{-c}. \end{aligned} \tag{23}$$

Let $h_1(u) = 2\sqrt{3\alpha}/(u - A)\sqrt{u^2 + 2Au + 3(A^2 + 2c)}$, then $h_1(0) = -(2/A)\sqrt{\alpha/(A^2 + 2c)}$ and

$$\int \sqrt{-u}h_1(u)\,du = \int \mathrm{sign}(\xi)\,d\xi. \tag{24}$$

Inserting $h_1(u) = h_1(0) + O(u)$ into (24) and using the initial condition $u(0) = 0$, we obtain

$$-\frac{2}{3}(-u)^{3/2}h_1(0)(1 + O(1)) = |\xi|. \tag{25}$$

Thus,

$$u = -|\xi|^{2/3}\left(\frac{3}{2h_1(0)}\right)^{2/3}(1 + O(1))^{-2/3}, \quad \xi \longrightarrow 0, \tag{26}$$

which implies that $u = O(|\xi|^{3/2})$. Therefore, we have

$$u(\xi) = -\left(\frac{3}{2h_1(0)}\right)^{2/3}|\xi|^{2/3} + O(|\xi|), \quad \xi \longrightarrow 0, \tag{27}$$

$$u'(\xi) = -\frac{2}{3}\left(\frac{3}{2h_1(0)}\right)^{2/3}|\xi|^{-1/3} + O(1), \quad \xi \longrightarrow 0. \tag{28}$$

Thus, $u(\xi) \notin H^1_{\mathrm{loc}}(R)$.

Let $h_2(u) = 2\sqrt{3\alpha}/(A - u)\sqrt{-(u^2 + 2Au + 3(A^2 + 2c))}$; then $h_2(0) = (2/A)\sqrt{\alpha/(-A^2 - 2c)}$ and

$$\int \sqrt{u}h_2(u)\,du = \int \mathrm{sign}(\xi)\,d\xi. \tag{29}$$

Inserting $h_2(u) = h_2(0) + O(u)$ into (29) and using the initial condition $u(0) = 0$, we obtain

$$\frac{2}{3}(u)^{3/2}h_2(0)(1 + O(1)) = |\xi|. \tag{30}$$

Thus,

$$u = |\xi|^{2/3}\left(\frac{3}{2h_2(0)}\right)^{2/3}(1 + O(1))^{-2/3}, \quad \xi \longrightarrow 0, \tag{31}$$

which implies that $u = O(|\xi|^{3/2})$. Therefore, we have

$$\begin{aligned} u(\xi) &= \left(\frac{3}{2h_2(0)}\right)^{2/3}|\xi|^{2/3} + O(|\xi|), \quad \xi \longrightarrow 0, \\ u'(\xi) &= \frac{2}{3}\left(\frac{3}{2h_2(0)}\right)^{2/3}|\xi|^{-1/3} + O(1), \quad \xi \longrightarrow 0. \end{aligned} \tag{32}$$

Thus, $u(\xi) \notin H^1_{\mathrm{loc}}(R)$. □

3. Smooth, Peaked, and Cusped Single Peak Solitary Wave Solutions

Theorem 6 gives a classification for all single peak solitary wave solutions for (4). In this section, we will present all possible soliton solutions for (4). We shall discuss the four cases: Case 1: $A < 0$, $c > 0$; Case 2: $A = 0$, $c < 0$; Case 3: $0 < A \le \sqrt{-c}$, $c < 0$ or $-\sqrt{-3c} \le A < -\sqrt{-c}$, $c < 0$.

Case 1 ($A < 0$, $c > 0$, $u(0) = 0$). In this case, according to Theorem 5 and standard phase portrait analytical technique (see Figure 1(a)), we have $A < u \le 0$ and

$$\begin{aligned} u' = &-\mathrm{sign}(A)\frac{u - A}{u} \\ &\times \sqrt{\frac{-u(u^2 + 2Au + 3A^2 + 6c)}{12\alpha}}\,\mathrm{sign}(\xi). \end{aligned} \tag{33}$$

Integrating both sides of (33) on the interval $(A, 0]$ leads to

$$\int_u^0 \frac{u\,du}{(u - A)\sqrt{-u[(u + A)^2 + 2(A^2 + 3c)]}} = -\frac{|\xi|}{2\sqrt{3\alpha}}. \tag{34}$$

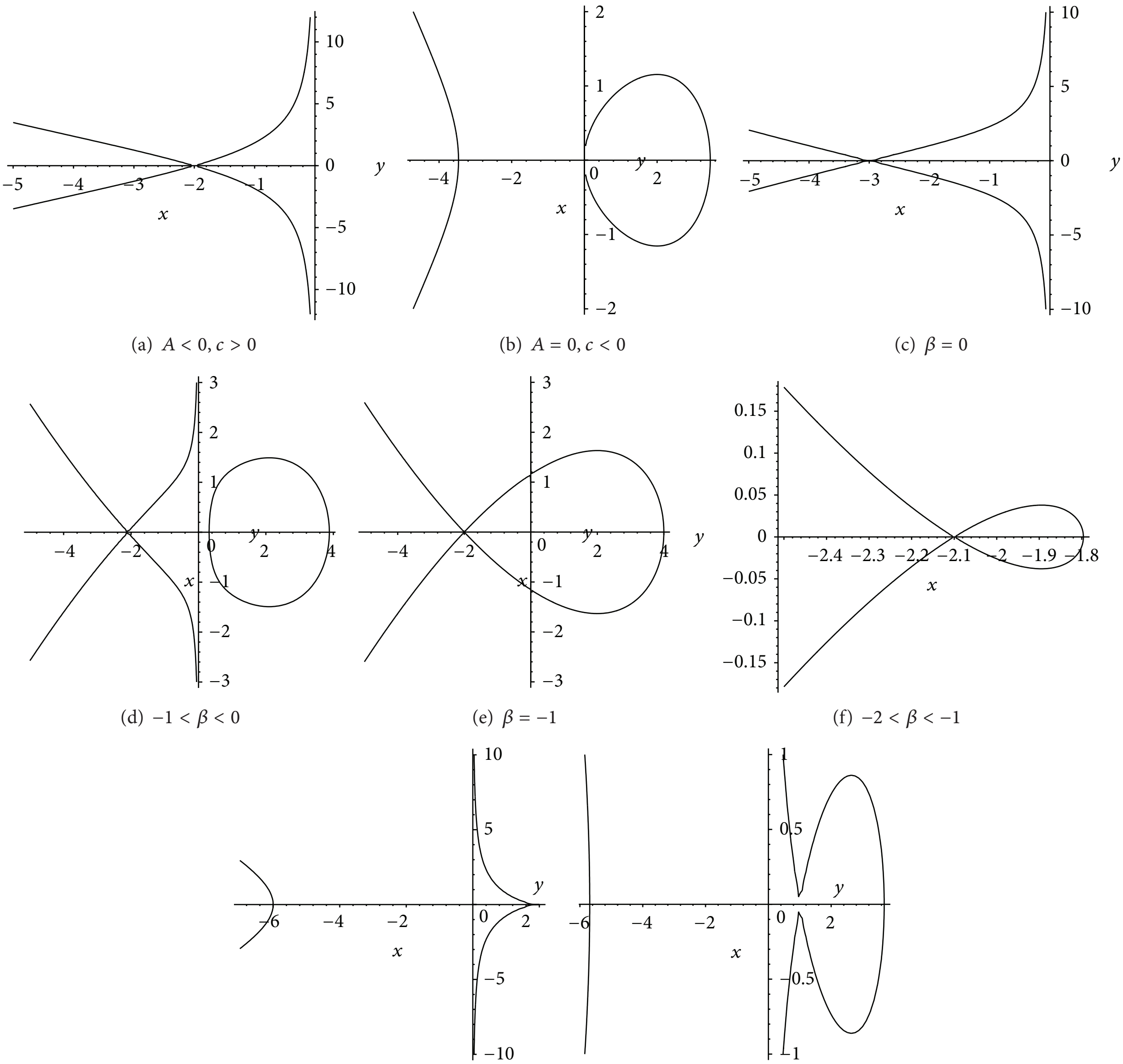
Thus we obtain the implicit solution $u(\xi)$ defined by

$$\begin{aligned} F(u) &\equiv cn^{-1}(\Phi(u), k) - \frac{1}{1 - \alpha_1} \\ &\times \left[\prod\left(\phi, \frac{\alpha_1^2}{\alpha_1^2 - 1}, k\right) - \alpha_1 f_1\right] \\ &= \frac{(A - \delta)|\xi|}{2\sqrt{3\alpha\delta}}, \end{aligned} \tag{35}$$

where $\Phi(u) = (\delta + u)/(\delta - u)$, $\phi = \arccos\Phi(u)$, $f_1 = \sqrt{(1-\alpha_1^2)/(k^2(1-\alpha_1^2)+\alpha_1^2)}\arctan(\sqrt{(k^2(1-\alpha_1^2)+\alpha_1^2)/(1-\alpha_1^2)}\,sd(u_1, k))$, $\alpha_1 = \Phi(A)$, $k^2 = (\sqrt{\delta} + A)/2\sqrt{\delta}$, $\delta = \sqrt{3(A^2 + 2c)}$.

In view of $\phi(u) = ((u - A)/u)\sqrt{-u(u^2 + 2Au + 3A^2 + 6c)/12\alpha} < 0$, we know that $F(u)$ is strictly decreasing on $(A, 0]$ with $F_1(u) = F_{(A,0]}(u)$, which gives a unique cuspon soliton solution $u_1(\xi)$ satisfying $u_1(0) = 0$, $\lim_{\xi \to \pm\infty} u_1(\xi) = A$, $u_1'(0+) = -\infty$, $u_1'(0-) = +\infty$. The profile of cuspon soliton solution is shown in Figure 2(a).

Case 2 ($A = 0$, $c < 0$, $u(0) = B_1 = \sqrt{-6c}$). In this case, by the standard phase portrait analysis (see Figure 1(b)), we



(a) $A < 0, c > 0$ (b) $A = 0, c < 0$ (c) $\beta = 0$

(d) $-1 < \beta < 0$ (e) $\beta = -1$ (f) $-2 < \beta < -1$

(g) $\beta = 2$ (h) $\beta > 2$

FIGURE 1: The phase portraits of (8) on the (u, u') plane.

have $0 < u \leq \sqrt{-6c}$ and then obtain a compacton solution (see (19)). The profile of compacton is shown in Figure 2(b).

Case 3 ($0 < A \leq \sqrt{-c}, c < 0$ or $-\sqrt{-3c} \leq A < -\sqrt{-c}, c < 0$). By virtue of Theorem 5, any single peak soliton solution for (4) must satisfy the following initial and boundary values problem:

$$\begin{gathered}\left(u'\right)^2 = \frac{(u-A)^2\left(B_1-u\right)\left(u-B_2\right)}{12\alpha u}, \\ u(0) \in \left\{0, B_1, B_2\right\}, \\ \lim_{\xi\to\pm\infty} u(\xi) = A.\end{gathered} \tag{36}$$

Equation (36) implies

$$\frac{\left(B_1-u\right)\left(u-B_2\right)}{u} \geq 0, \tag{37}$$

$$\frac{\left(B_1-A\right)\left(A-B_2\right)}{A} \geq 0. \tag{38}$$

(i) When $A > 0$, from (37) we obtain

$$\left(-2+\frac{1}{A}\sqrt{-2\left(A^2+3c\right)}\right)\left(2+\frac{1}{A}\sqrt{-2\left(A^2+3c\right)}\right) \geq 0. \tag{39}$$

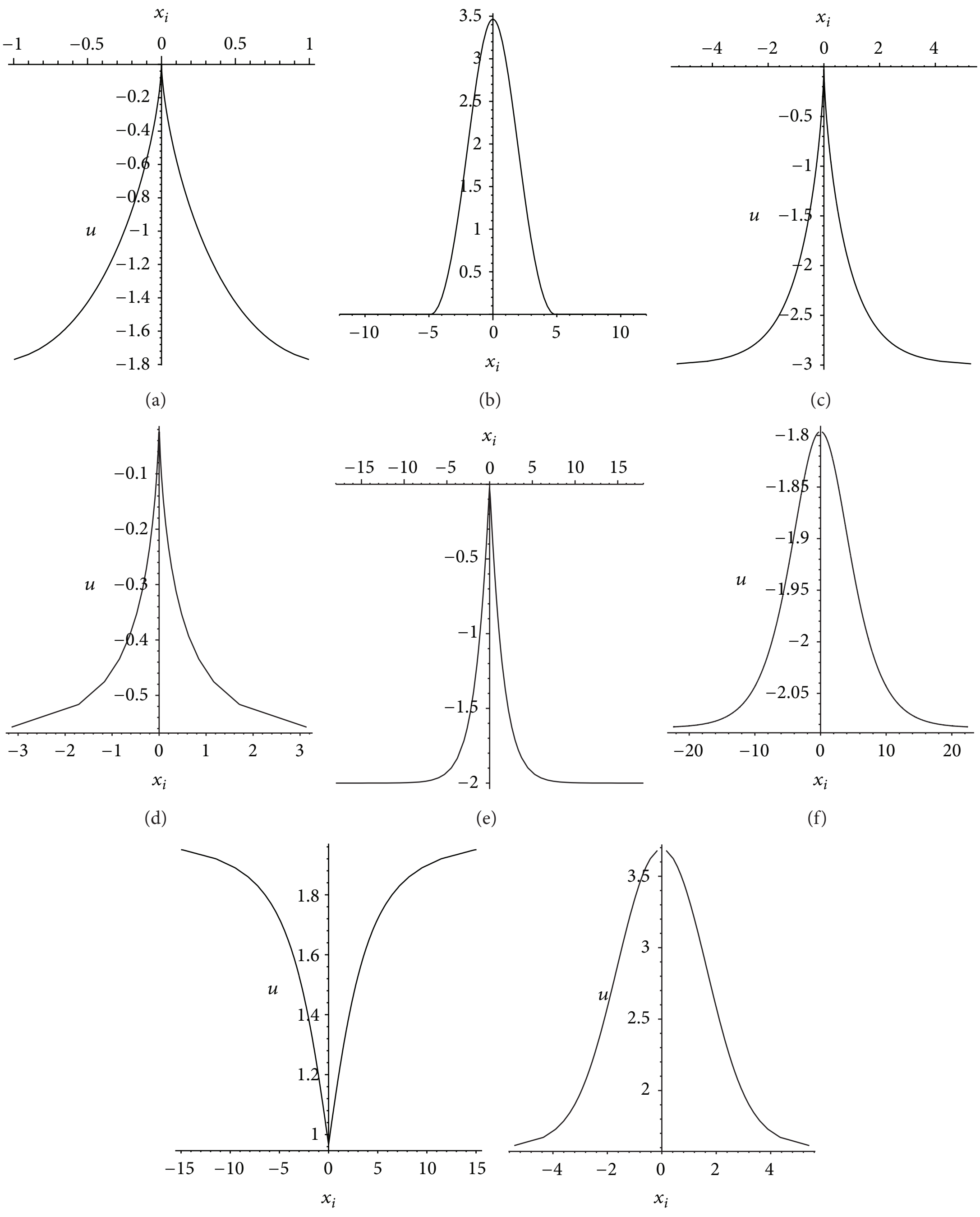


FIGURE 2: The profiles of waves for $\alpha = 1$. (a) $A = -2$, $c = 4$, (b) $A = 0$, $c = -2$, (c) $\beta = 0$, $A = -3$, $c = -3$, (d) $-1 < \beta < 0$, $A = -2.1$, $c = -2$, (e) $\beta = -1$, $A = -2$, $c = -2$, (f) $-2 < \beta < -1$, $A = -2.1$, $c = -4$, (g) $\beta = 2$, $A = 2$, $c = -4$, (h) $\beta > 2$, $A = 1$, $c = -4$.

(ii) When $A < 0$, from (37) we obtain

$$\left(-2 + \frac{1}{A}\sqrt{-2\left(A^2 + 3c\right)}\right)\left(2 + \frac{1}{A}\sqrt{-2\left(A^2 + 3c\right)}\right) \le 0. \tag{40}$$

Since $A^2 + 3c \le 0$, $A \ne 0$, introducing the constant $\beta = (1/A)\sqrt{-2(A^2 + 3c)}$ yields

$$\left(-2 + \beta\right)\left(2 + \beta\right) \ge 0, \quad A > 0,$$
$$\left(-2 + \beta\right)\left(2 + \beta\right) \le 0, \quad A < 0, \tag{41}$$

which implies that for $A > 0$

$$\beta \ge 2 \tag{42}$$

and for $A < 0$

$$-2 \le \beta \le 0. \tag{43}$$

From the standard phase portrait analysis (see Figures 1(c)–1(h)), we know that if $u(\xi)$ is a single peak soliton solution of (4), then

$$u' = -\operatorname{sign}(A)\frac{u-A}{2u}\sqrt{\frac{u(B_1-u)(u-B_2)}{3\alpha}}\operatorname{sign}(\xi). \quad (44)$$

In the following part , let us separate seven cases to discuss.

(1) $\beta = -1$. In this case, by the standard phase portrait analysis (see Figure 1(e)), we have

$$-\sqrt{-2c} = A < B_2 = 0 < B_1 = 2\sqrt{-2c}. \quad (45)$$

(i) $u(0) = B_2 = 0$. If $u(0) = B_2$, then $A < u \le B_2$. In this case, by the standard phase portrait analysis (see Figure 1(e)), we obtain a peakon solution (see (21)). The profile of peakon solution is shown in Figure 2(e).

(ii) $u(0) = B_1$. In this case there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = -\sqrt{-2c}$.

(2) $-2 < \beta < -1$. In this case, by the standard phase portrait analysis (see Figure 1(f)), we have

$$A < B_2 < 0 < B_1. \quad (46)$$

(i) $u(0) = B_2$. If $u(0) = B_2$, then $A < u \le B_2$. Integrating both sides of (47) on the interval $(A, B_2]$ leads to

$$Q(u) \equiv \prod\left(\phi, \frac{A}{A-B_2}, \sqrt{\frac{B_1}{B_1-B_2}}\right) = \frac{A-B_2}{4B_2}\sqrt{\frac{B_1-B_2}{3\alpha}}|\xi|, \quad (47)$$

where $\phi = \arcsin\sqrt{(B_2-u)/-u}$.

From $\phi(u) < 0$, we know that $Q(u)$ is strictly decreasing on the interval $(A, B_2]$, $Q_1(u) = Q_{(A,B_2]}(u)$ has the inverse denoted by

$$u_2(\xi) = B_2\sec^2\left(\Pi^{-1}\left(\frac{A-B_2}{4B_2}\sqrt{\frac{B_1-B_2}{3\alpha}}|\xi|, \frac{A}{A-B_2}, \sqrt{\frac{B_1}{B_1-B_2}}\right)\right). \quad (48)$$

Corresponding to the homoclinic orbit to the saddle point $(A, 0)$ shown in Figure 1(f), $u_2(\xi)$ gives a smooth soliton solution satisfying

$$u_2(0) = B_2, \qquad \lim_{\xi\to\pm\infty} u_2(\xi) = A, \qquad u_2'(0) = 0. \quad (49)$$

The profile of smooth soliton solution is shown in Figure 2(f).

Remark 3. To the best of our knowledge, the solution (48) of (4) has not been reported in the literature.

(ii) $u(0) = B_1$ or $u(0) = 0$. In this case there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = A$.

(3) $-1 < \beta < 0$. In this case, by the standard phase portrait analysis (see Figure 1(d)), we have

$$A < 0 < B_2 < B_1. \quad (50)$$

(i) $u(0) = 0$. If $u(0) = 0$, then $A < u \le 0$. Integrating both sides of (44) on the interval $(A, 0]$ leads to

$$W(u) \equiv \prod\left(\arcsin\omega_1, \frac{A-B_2}{A}, k\right) - sn^{-1}(\omega_1, k) = \frac{B_2-A}{4B_2}\sqrt{\frac{B_1}{3\alpha}}|\xi|, \quad (51)$$

where $\omega_1 = \sqrt{-u/(B_2-u)}$, $k = \sqrt{(B_1-B_2)/B_1}$.

From $((u-A)/2u)\sqrt{u(B_1-u)(u-B_2)/3\alpha} < 0$, we know that $W(u)$ is strictly decreasing on $(A, 0]$,

$$W_1(u) = W_{(A,0]}(u) \quad (52)$$

has the inverse denoted by $u_3(\xi) = W_1^{-1}(((B_2 - A)/4B_2)\sqrt{B_1/3\alpha}|\xi|)$, and $u_3(\xi)$ gives a cuspon soliton solution satisfying

$$u_3(0) = 0, \qquad \lim_{\xi\to\pm\infty} u_3(\xi) = A, \qquad u_3'(0+) = -\infty, \qquad u_3'(0-) = +\infty. \quad (53)$$

The profile of cuspon soliton solution is shown in Figure 2(d).

(ii) $u(0) = B_2$ or $u(0) = B_1$. In this case there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = A$.

(4) $\beta = -2$. In this case, we have $A = B_1 = -\sqrt{-c}$, by virtue of Lemma 4(iii), (4) has trivial solution $u \equiv A$.

(5) $\beta = 0$. In this case, by the standard phase portrait analysis (see Figure 1(c)), we have

$$-\sqrt{-3c} = A < 0 < B_2 = B_1 = \sqrt{-3c}. \quad (54)$$

(i) $u(0) = 0$. If $u(0) = 0$, then $A < u \le 0$. Integrating both sides of (44) on the interval $(A, 0]$ leads to

$$Z(u) \equiv \left(\tanh^{-1}\frac{\sqrt{-u}}{(-3c)^{1/4}} - \arctan\frac{\sqrt{-u}}{(-3c)^{1/4}}\right) = \frac{(-3c)^{1/4}}{2\sqrt{3\alpha}}|\xi|. \quad (55)$$

From $\phi(u) < 0$, we know that $Z(u)$ is strictly decreasing on $(A, 0]$,

$$Z_1(u) = Z_{(A,0]}(u) \quad (56)$$

has the inverse denoted by $u_4(\xi) = Z_1^{-1}(((-3c)^{1/4}/2\sqrt{3\alpha})|\xi|)$, and $u_4(\xi)$ gives a cuspon soliton solution satisfying

$$u_4(0) = 0, \qquad \lim_{\xi\to\pm\infty} u_4(\xi) = A,$$
$$u_4'(0+) = -\infty, \qquad u_4'(0-) = +\infty. \tag{57}$$

The profile of cuspon soliton solution is shown in Figure 2(c).

(ii) $u(0) = B_2 = B_1$. In this case, there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = A$.

(6) $\beta = 2$. In this case, by the standard phase portrait analysis (see Figure 1(g)), we have

$$-3\sqrt{-c} = B_2 < A = B_1 = \sqrt{-c}. \tag{58}$$

(i) $u(0) = 0$. If $u(0) = 0$, then $0 \le u < A$. Integrating both sides of (44) on the interval $[0, A)$ leads to

$$\begin{aligned}\Psi(u) \equiv &-dn\left(sn^{-1}\left(\sqrt{\frac{\sqrt{-c}-u}{\sqrt{-c}}}, \frac{1}{2}\right), \frac{1}{2}\right)\\ &\times cs\left(sn^{-1}\left(\sqrt{\frac{\sqrt{-c}-u}{\sqrt{-c}}}, \frac{1}{2}\right), \frac{1}{2}\right)\\ &-E\left(\arcsin\left(\sqrt{\frac{\sqrt{-c}-u}{\sqrt{-c}}}\right), \frac{1}{2}\right)\\ =\ &\frac{1}{2}\left(\frac{-c}{9\alpha^2}\right)^{1/4}|\xi|.\end{aligned} \tag{59}$$

In view of $\phi(u) < 0$, we know that $\Psi(u)$ is strictly decreasing on $[0, A)$ and

$$\Psi_1(u) = \Psi_{[0,A)}(u) \tag{60}$$

has the inverse which is denoted by $u_5(\xi) = \Psi_1^{-1}(((-3c)^{1/4}/2\sqrt{3\alpha})|\xi|)$, where $u_5(\xi)$ gives a cuspon soliton solution satisfying

$$u_5(0) = 0, \qquad \lim_{\xi\to\pm\infty} u_5(\xi) = A,$$
$$u_5'(0+) = +\infty, \qquad u_5'(0-) = -\infty. \tag{61}$$

The profile of cuspon soliton solution is shown in Figure 2(g).

(ii) $u(0) = B_2$. In this case there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = A$.

(7) $\beta > 2$. In this case, by the standard phase portrait analysis (see Figure 1(h)), we have

$$B_2 < 0 < A < B_1. \tag{62}$$

(i) $u(0) = 0$. If $u(0) = 0$, then $0 \le u < A$. This case is completely similar to the case of $\beta = 2$, $u(0) = 0$.

(ii) $u(0) = B_1$. If $u(0) = B_1$, then $A < u \le B_1$. Integrating both sides of (44) on the interval $(A, B_1]$ leads to

$$\begin{aligned}\Gamma(u) \equiv &-sn^{-1}(\omega_2, k) + \frac{A}{A-B_1}\\ &\times \prod\left(\arcsin\omega_2, \frac{B_1}{B_1-A}, k\right)\\ =\ &\frac{1}{4}\sqrt{\frac{B_1-B_2}{3\alpha}}|\xi|,\end{aligned} \tag{63}$$

where $\omega_2 = \sqrt{(B_1-u)/B_1}$, $k = \sqrt{B_1/(B_1-B_2)}$.

In view of $\phi(u) > 0$, we know that $\Gamma(u)$ is strictly increasing on $[0, A)$ and

$$\Gamma_1(u) = \Gamma_{[0,A)}(u) \tag{64}$$

has the inverse which is denoted by $u_6(\xi) = \Gamma_1^{-1}(((-3c)^{1/4}/2\sqrt{3\alpha})|\xi|)$, where $u_6(\xi)$ gives a smooth soliton solution satisfying

$$u_6(0) = 0, \qquad \lim_{\xi\to\pm\infty} u_6(\xi) = A, \qquad u_6'(0) = 0. \tag{65}$$

The profile of smooth soliton solution is shown in Figure 2(h).

(iii) $u(0) = B_2$. In this case there is no single peak solitary wave solution for the boundary condition $u(\pm\infty) = A$.

Let us summarize our results in the following theorem.

Theorem 7. *Suppose that $u(\xi)$ is a single peak soliton for* (4) *at the peak point $\xi_0 = 0$, which satisfies the boundary condition* (5). *Then we have the following conclusions.*

(i) $A < 0, c > 0$. If $u(0) = 0$, (4) *has cuspon soliton solution which can be expressed as*

$$u(\xi) = F^{-1}_{(A,0]}\left(\frac{(A-\delta)|\xi|}{2\sqrt{3\alpha\delta}}\right), \tag{66}$$

where $\delta = \sqrt{3(A^2 + 2c)}$. It satisfies

$$u(0) = 0, \qquad \lim_{\xi\to\pm\infty} u(\xi) = A, \tag{67}$$

$$u'(0+) = -\infty, \qquad u'(0-) = +\infty. \tag{68}$$

(ii) $A = 0$, $c < 0$. If $u(0) = B_1 = \sqrt{-6c}$, the only possible quasi single peak soliton is compacton:

$$u(\xi) = \begin{cases} \sqrt{-6c}\left(1 - sn^2\left(\frac{1}{4}\sqrt{\frac{2\sqrt{-6c}}{3\alpha}}|\xi|, \frac{\sqrt{2}}{2}\right)\right), \\ \qquad \text{if } |\xi| \le 4\sqrt{\frac{3\alpha}{2\sqrt{-6c}}}K\left(\frac{\sqrt{2}}{2}\right), \\ 0, \qquad \text{if } |\xi| > 4\sqrt{\frac{3\alpha}{2\sqrt{-6c}}}K\left(\frac{\sqrt{2}}{2}\right), \end{cases} \tag{69}$$

with the following properties:

$$u(0) = \sqrt{-6c}, \quad u'(0) = 0. \tag{70}$$

(iii) $0 < A \le \sqrt{-c}$, $c < 0$, or $-\sqrt{-3c} \le A < -\sqrt{-c}$, $c < 0$.

(1) $\beta = -1$. *If $\beta = -1$, then $u(0) = 0 = B_2$, and* (4) *has peakon soliton solution which can be expressed as*

$$u(\xi) = \sqrt{-2c}\left[2 - 3\tanh^2\left(\tanh^{-1}\sqrt{\frac{2}{3}} + \frac{1}{4}\sqrt{\frac{\sqrt{-2c}}{\alpha}}|\xi|\right)\right], \tag{71}$$

with the following properties:

$$\begin{gathered} u(0) = 0, \qquad u(\pm\infty) = A, \\ u'(0+) = c\sqrt{\frac{\sqrt{-2c}}{-12c\alpha}}, \qquad u'(0-) = -c\sqrt{\frac{\sqrt{-2c}}{-12c\alpha}}. \end{gathered} \tag{72}$$

(2) $-2 < \beta < -1$. *If $u(0) = B_2$,* (4) *has smooth soliton solution which can be expressed as*

$$u(\xi) = B_2 sec^2\left(\Pi^{-1}\left(\frac{A - B_2}{4B_2}\sqrt{\frac{B_1 - B_2}{3\alpha}}|\xi|, \frac{A}{A - B_2}, \sqrt{\frac{B_1}{B_1 - B_2}}\right)\right), \tag{73}$$

with the following properties:

$$u(0) = B_2, \quad \lim_{\xi \to \pm\infty} u(\xi) = A, \quad u'(0) = 0. \tag{74}$$

(3) $-1 < \beta < 0$. *If $u(0) = 0$,* (4) *has cuspon soliton solution which can be expressed as*

$$u(\xi) = W_1^{-1}\left(\frac{B_2 - A}{4B_2}\sqrt{\frac{B_1}{3\alpha}}|\xi|\right), \tag{75}$$

with the following properties:

$$\begin{gathered} u(0) = 0, \qquad \lim_{\xi \to \pm\infty} u(\xi) = A, \\ u'(0+) = -\infty, \qquad u'(0-) = +\infty. \end{gathered} \tag{76}$$

(4) $\beta = 0$. *If $u(0) = 0$,* (4) *has cuspon soliton solution which can be expressed as*

$$u(\xi) = Z_1^{-1}\left(\frac{(-3c)^{1/4}}{2\sqrt{3\alpha}}|\xi|\right), \tag{77}$$

with the following properties:

$$\begin{gathered} u(0) = 0, \qquad \lim_{\xi \to \pm\infty} u(\xi) = A, \\ u'(0+) = -\infty, \qquad u'(0-) = +\infty. \end{gathered} \tag{78}$$

(5) $\beta = 2$. *If $u(0) = 0$,* (4) *has cuspon soliton solution which can be expressed as*

$$u(\xi) = \Psi_1^{-1}\left(\frac{(-3c)^{1/4}}{2\sqrt{3\alpha}}|\xi|\right), \tag{79}$$

with the following properties:

$$\begin{gathered} u(0) = 0, \qquad \lim_{\xi \to \pm\infty} u(\xi) = A, \\ u'(0+) = +\infty, \qquad u'(0-) = -\infty. \end{gathered} \tag{80}$$

(6) $\beta > 2$.

(i) $u(0) = 0$. *If $u(0) = 0$, then $0 \le u < A$. This case is completely similar to the case of $\beta = 2, u(0) = 0$.*

(ii) $u(0) = B_1$. *If $u(0) = B_1$,* (4) *has cuspon soliton solution which can be expressed as*

$$u(\xi) = \Gamma_1^{-1}\left(\frac{(-3c)^{1/4}}{2\sqrt{3\alpha}}|\xi|\right), \tag{81}$$

with the following properties:

$$u(0) = 0, \quad \lim_{\xi \to \pm\infty} u(\xi) = A, \quad u'(0) = 0. \tag{82}$$

4. Conclusion

In this paper, we study the single peak solitary wave solutions of $K^*(4,1)$ equation under inhomogeneous boundary condition. The conditions of the existence of peakon, compacton, cuspon, and smooth soliton solutions are given by using phase portrait analytical technique. We have obtained all peakon, compacton, cuspon, and smooth soliton solutions of $K^*(4,1)$ equation and analyzed their analytic and dynamical behavior. We have gotten a new type of smooth soliton, which is expressed in terms of trigonometric functions (see (48)), for $K^*(4,1)$ equation. New peaked solitons and new type of smooth soliton solutions are expected to apply in nonlinear shallow-water wave theory and Newton motion theory because they have a very close relation. Actually, the ODE (8) has a physical meaning and coincides with the Newton equation of a particle in the potential

$$V(u) = \frac{u^3 + (6c - A^2)u}{-12\alpha} + \frac{A^2(3A^2 + 6c)}{12\alpha u}. \tag{83}$$

We solve the Newton equation $(u')^2 = V(u) - V(A)$, for all possible single peak solitary wave solutions, where $V(A) = -A(3A^2 + 6c)/6\alpha$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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