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Research Article

The Strong Convergence of Prediction-Correction and Relaxed Hybrid Steepest-Descent Method for Variational Inequalities

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We establish the strong convergence of prediction-correction and relaxed hybrid steepest-descent method (PRH method) for variational inequalities under some suitable conditions that simplify the proof. And it is to be noted that the proof is different from the previous results and also is not similar to the previous results. More importantly, we design a set of practical numerical experiments. The results demonstrate that the PRH method under some descent directions is more slightly efficient than that of the modified and relaxed hybrid steepest-descent method, and the PRH Method under some new conditions is more efficient than that under some old conditions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, let K be a nonempty closed convex subset of H, and let $F : H \to H$ be an operator. Then the variational inequality problem VI(F,K) [1] is to find $x^* \in K$ such that

$$x^* \in K, \quad \langle x - x^*, F(x^*)\rangle \geq 0, \quad \forall x \in K. \tag{1}$$

The literature contains many methods for solving variational inequality problems; see [2–25] and references therein. According to the relationship between the variational inequality problems and a fixed point problem, we can obtain

$$\begin{aligned} &x^* \text{ is the solution of } \mathrm{VI}(F,K) \\ &\Longleftrightarrow x^* = P_K[x^* - \beta F(x^*)], \quad \beta > 0, \end{aligned} \tag{2}$$

where the projection operator P_K is the projection from H onto K, that is,

$$P_K(x) = \operatorname*{argmin}_{y\in K} \|x - y\|, \quad \forall x \in H. \tag{3}$$

In this paper, $F : H \to H$ is an operator with F : κ-Lipschtz and η-strongly monotone; that is, F satisfies the following conditions:

$$\begin{aligned} &\|F(x) - F(y)\| \leq \kappa\|x - y\|, \\ &\langle F(x) - F(y), x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in K. \end{aligned} \tag{4}$$

If β is small enough, then P_K is a contraction. Naturally, the convergence of Picard iterates generated by the right-hand side of (2) is obtained by Banach's fixed point theorem. Such a method is called the projection method or more results about the projection method see [6, 8, 20] and so forth.

In fact, the projection P_K in the contraction methods may not be easy to compute, and a great effort is to compute the projection P_K in each iteration. Yamada and Deutsch have provided a hybrid steepest-descent method for solving the VI(F,K) [2, 3] in order to reduce the difficulty and complexity of computing the projection P_K. Subsequently, the convergence of hybrid steepest-descent methods was given out by Xu and Kim [4] and Zeng et al. [5]. Naturally, by analyzing several three-step iterative methods in each iteration by the fixed pointed equation, we can obtain the Noor iterations. Recently, Ding et al. [7] proposed a three-step relaxed hybrid steepest-descent method for variational inequalities, and the simple proof of three-step relaxed hybrid steepest-descent methods under different conditions was introduced by Yao et al. [24]. The literature [14, 16] described a modified and relaxed hybrid steepest-descent (MRHSD) method and the different convergence of the MRHSD method under the different conditions. A set of practical numerical experiments in the literature [16] demonstrated that the MRHSD method has different efficiency under different conditions. Subsequently, the prediction-correction and relaxed hybrid steepest-descent method (PRH method) [15] makes more use of the history information and less decreases the loss of information than the methods [7, 14]. The PRH method introduced more descent directions than the MRHSD method [14, 16], and computing these descent directions only needs the history information.

In this paper, we will prove the strong convergence of PRH method under different and suitable restrictions imposed on parameters (Condition 12), which differs from that of [15]. Moreover, the proof of strong convergence is different from the previous proof in [15], which is not similar to that in [7] in Step 2. And more importantly, numerical experiments verify that the PRH method under Condition 12 is more efficient than that under Condition 10, and the PRH method under some descent directions is more slightly efficient than that of the MRHSD method [14, 16]. Furthermore, it is easy to obtain these descent directions.

The remainder of the paper is organized as follows. In Section 2, we review several lemmas and preliminaries. We prove the convergence theorem under Condition 12 in Section 3. In Section 4, we give out a series of numerical experiments, which demonstrated that the PRH method under Condition 12 is more efficient than under Condition 10. Section 5 concludes the paper.

2. Preliminaries

In order to proof the later convergence theorem, we introduce several lemmas and the main results in the following.

Lemma 1. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H. \tag{5}$$

The lemma is a basic result of a Hilbert space with the inner product.

Lemma 2 (demiclosedness principle). *Assume that T is a nonexpansive self-mapping on a nonempty closed convex subset K of a Hilbert space H. If T has a fixed point, then $(I-T)$ is demiclosed. That is, whenever x_n is a sequence in K weakly converging to some $x \in K$ and the sequence $(I-T)x_n$ strongly converges to some $y \in H$, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate result of a projection mapping onto a closed convex subset of a Hilbert space.

Lemma 3. *Let K be a nonempty closed convex subset of H. For all $x, y \in H$ and $z \in K$, then*

(1) $\langle P_K(x) - x, z - P_K(y)\rangle \ge 0$,

(2) $\|P_K(x) - P_K(y)\|^2 \le 1\|x-y\|^2 - \|P_K(x) - x + y - P_K(y)\|^2$.

Lemma 4 (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequence in a Banach space X and let $\{\zeta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\zeta_n \le \limsup_{n\to\infty}\zeta_n < 1$. Suppose $x_{n+1} = (1-\zeta_n)y_n + \zeta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\limsup_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 5 ([5, 7]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\tau_n + \gamma_n, \quad \forall n \ge 0, \tag{6}$$

where α_n, τ_n, and γ_n satisfy the following conditions:

(1) $\alpha_n \subset [0,1]$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, *or* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$,

(2) $\lim_{n\to\infty}\sup\tau_n \le 0$,

(3) $\gamma_n \subset [0,\infty)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty}s_n = 0$.

Since F is η-strongly monotone, $\mathrm{VI}(F,K)$ has a unique solution $x^* \in K$ [5]. Assume that $T : H \to H$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) = K$. Obviously $\mathrm{Fix}(P_K) = K$.

For any given numbers $\lambda \in (0,1)$ and $\mu \in (0, 2\eta/\kappa^2)$, we define the mapping $T_\mu^\lambda : H \to H$ by

$$T_\mu^\lambda x : Tx - \lambda\mu F(Tx), \quad \forall x \in H. \tag{7}$$

Lemma 6 (see [5]). *If $0 < \mu < 2\eta/\kappa^2$ and $0 < \lambda < 1$, then T_μ^λ is a contraction. In fact,*

$$\left\|T_\mu^\lambda x - T_\mu^\lambda y\right\| \le (1-\lambda\delta)\|x-y\|, \tag{8}$$

where $\delta = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)}$, for all $x, y \in H$.

Lemma 7 (see [7]). *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers with $\limsup_{n\to\infty}\alpha_n < \infty$ and let $\{\beta_n\}$ be sequence of real numbers with $\limsup_{n\to\infty}\beta_n \le 0$. Then*

$$\limsup_{n\to\infty}\alpha_n\beta_n \le 0. \tag{9}$$

3. Convergence Theorem

Before analyzing the convergence theorem, we first review the PRH method and related results [15].

Algorithm 8 (see [15]). Take three fixed numbers $t, \rho, \gamma \in (0, 2\eta/\kappa^2)$, starting with arbitrarily chosen initial points $x_0 \in H$, compute the sequences $\{x_n\}, \{\overline{x}_n\}, \{\tilde{x}_n\}, \{\hat{x}_n\}$ such that;

Prediction

Step 1: $\overline{x}_n = \gamma_n x_n + (1-\gamma_n)[Tx_n - \lambda''_{n+1}\gamma F(Tx_n)]$,

Step 2: $\tilde{x}_n = \beta_n x_n + (1-\beta_n)[T\overline{x}_n - \lambda'_{n+1}\rho F(T\overline{x}_n)]$,

Step 3: $\hat{x}_n = \theta_n\overline{x}_n + (1-\theta_n)\tilde{x}_n, 0 \le \theta_n \le 1$, Correction

Step 4: $x_{n+1} = \alpha_n \overline{x}_n + (1-\alpha_n)[T\widehat{x}_n - \lambda_{n+1} tF(T\widehat{x}_n)]$,

where $T : H \rightarrow H$ is a nonexpansive mapping.

Let $\{\alpha_n\} \subset [0,1)$, $\{\beta_n\} \subset [0,1]$ and $\{\gamma_n\} \subset [0,1]$, $\{\lambda_n\}$, $\{\lambda'_n\}$, $\{\lambda''_n\} \subset (0,1)$ satisfy the following conditions.

Remark 9. In fact, the PRH method is the MRHSD method when $\theta_n \equiv 0$, for all n.

Condition 10. One has

$$\begin{aligned}
&(1) \quad \sum_{1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \qquad \sum_{1}^{\infty} |\beta_n - \beta_{n-1}| < \infty, \\
&\qquad \sum_{1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty, \\
&(2) \quad \lim_{n\to\infty} \alpha_n = 0, \qquad \lim_{n\to\infty} \beta_n = 1, \qquad \lim_{n\to\infty} \gamma_n = 1, \\
&(3) \quad \lim_{n\to\infty} \lambda_n = 0, \qquad \lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1, \qquad \sum_{1}^{\infty} \lambda_n = \infty, \\
&(4) \quad \lambda_n \geq \max\left\{\lambda'_n, \lambda''_n\right\}, \quad \forall n \geq 1.
\end{aligned} \tag{10}$$

Theorem 11 (see [15]). *In Condition 10, the sequence $\{x_n\}$ converges strongly to $x^* \in K$, and x^* is the unique solution of the $VI(F,K)$.*

We obtain the strong convergence theorem of PRH method for variational inequalities under different assumptions.

Condition 12. One has

$$\begin{aligned}
&(1) \quad 0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1, \\
&\qquad \lim_{n\to\infty} \beta_n = 1, \qquad \lim_{n\to\infty} \gamma_n = 1, \\
&(2) \quad \lim_{n\to\infty} \lambda_n = 0, \qquad \sum_{1}^{\infty} \lambda_n = \infty, \\
&(3) \quad \lambda_n \geq \max\left\{\lambda'_n, \lambda''_n\right\}, \quad \forall n \geq 1.
\end{aligned} \tag{11}$$

Theorem 13. *The sequence $\{x_n\}$ converges strongly to $x^* \in K$, and x^* is the unique solution of the $VI(F,K)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, $\{\lambda_n\}$, $\{\lambda'_n\}$, $\{\lambda''_n\}$ satisfy Condition 12.*

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$, $\{\overline{x}_n\}$, $\{\widetilde{x}_n\}$, and $\{\widehat{x}_n\}$ are bounded. Since F is η-strongly monotone, VI(F,K) (1) has a unique solution $x^* \in K$, and $T_t^{\lambda_{n+1}} x^* = x^* - \lambda_{n+1} tF(x^*)$, $T_\rho^{\lambda'_{n+1}} x^* = x^* - \lambda_{n+1} \rho F(x^*)$, $T_\gamma^{\lambda''_{n+1}} x^* = x^* - \lambda_{n+1} \gamma F(x^*)$.

A series of computations yields

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\alpha_n \overline{x}_n + (1-\alpha_n) T_t^{\lambda_{n+1}} \widehat{x} - x^*\right\| \\
&\leq \alpha_n \|\overline{x}_n - x^*\| + (1-\alpha_n) \left\|T_t^{\lambda_{n+1}} \widehat{x} - x^*\right\| \\
&\leq \alpha_n \|\overline{x}_n - x^*\| + (1-\alpha_n) \\
&\quad \times \left[\left\|T_t^{\lambda_{n+1}} \widehat{x} - T_t^{\lambda_{n+1}} x^*\right\| + \left\|T_t^{\lambda_{n+1}} x^* - x^*\right\|\right] \\
&\leq \alpha_n \|\overline{x}_n - x^*\| + (1-\alpha_n) \\
&\quad \times \left[(1-\lambda_{n+1}\tau) \|\widehat{x}_n - x^*\| + \lambda_{n+1} t \|F(x^*)\|\right],
\end{aligned} \tag{12}$$

where $\tau = 1 - \sqrt{1 - t(2\eta - t\kappa^2)} \in (0,1)$,

$$\begin{aligned}
\|\widehat{x}_n - x^*\| &= \left\|\beta_n x_n + (1-\beta_n) T_\rho^{\lambda'_{n+1}} \overline{x}_n - x^*\right\| \\
&\leq \beta_n \|x_n - x^*\| + (1-\beta_n) \left\|T_\rho^{\lambda'_{n+1}} \overline{x}_n - x^*\right\| \\
&\leq \beta_n \|x_n - x^*\| + (1-\beta_n) \\
&\quad \times \left[\left\|T_\rho^{\lambda'_{n+1}} \overline{x}_n - T_\rho^{\lambda'_{n+1}} x^*\right\| + \left\|T_\rho^{\lambda'_{n+1}} x^* - x^*\right\|\right] \\
&\leq \beta_n \|x_n - x^*\| + (1-\beta_n) \\
&\quad \times \left[\left(1-\lambda'_{n+1}\tau'\right) \|\overline{x}_n - x^*\| + \lambda'_{n+1} \rho \|F(x^*)\|\right] \\
&\leq \beta_n \|x_n - x^*\| + (1-\beta_n) \|\overline{x}_n - x^*\| \\
&\quad + (1-\beta_n) \lambda'_{n+1} \rho \|F(x^*)\|,
\end{aligned} \tag{13}$$

where $\tau' = 1 - \sqrt{1 - \rho(2\eta - t\kappa^2)} \in (0,1)$.

Moreover, we also obtain

$$\begin{aligned}
\|\overline{x}_n - x^*\| &= \left\|\gamma_n x_n + (1-\gamma_n) T_\gamma^{\lambda''_{n+1}} x_n - x^*\right\| \\
&\leq \gamma_n \|x_n - x^*\| + (1-\gamma_n) \left\|T_\gamma^{\lambda''_{n+1}} x_n - x^*\right\| \\
&\leq \gamma_n \|x_n - x^*\| + (1-\gamma_n) \\
&\quad \times \left[\left\|T_\gamma^{\lambda''_{n+1}} x_n - T_\gamma^{\lambda''_{n+1}} x^*\right\| + \left\|T_\gamma^{\lambda''_{n+1}} x^* - x^*\right\|\right] \\
&\leq \gamma_n \|x_n - x^*\| + (1-\gamma_n) \\
&\quad \times \left[\left(1-\lambda''_{n+1}\tau''\right) \|x_n - x^*\| + \lambda''_{n+1} \gamma \|F(x^*)\|\right] \\
&\leq \|x_n - x^*\| + (1-\gamma_n) \lambda''_{n+1} \gamma \|F(x^*)\|,
\end{aligned} \tag{14}$$

where $\tau'' = 1 - \sqrt{1-\gamma(2\eta - t\kappa^2)} \in (0,1)$, subtituting; (14) into (13) and (14) into (12), we immediately obtain

$$\begin{aligned}\|\tilde{x}_n - x^*\| &= \beta_n \|x_n - x^*\| + (1-\beta_n) \\ &\quad \times \left[\left(1-\lambda'_{n+1}\tau'\right)\|\bar{x}_n - x^*\| + \lambda'_{n+1}\rho \|F(x^*)\|\right] \\ &\le \beta_n \|x_n - x^*\| + (1-\beta_n) \\ &\quad \times \left[\left(1-\lambda'_{n+1}\tau'\right)\|x_n - x^*\| \right. \\ &\quad \left. + (1-\gamma_n)\lambda''_{n+1}\gamma \|F(x^*)\| + \lambda'_{n+1}\rho\|F(x^*)\|\right] \\ &\le \|x_n - x^*\| + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\|F(x^*)\|.\end{aligned} \tag{15}$$

Furthermore,

$$\begin{aligned}\|\hat{x}_n - x^*\| &= \|\theta_n \bar{x}_n + (1-\theta_n)\tilde{x}_n - x^*\| \\ &\le \theta_n \|\bar{x}_n - x^*\| + (1-\theta_n)\|\tilde{x}_n - x^*\| \\ &\le \theta_n \left[\|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\|\right] \\ &\quad + (1-\theta_n)[\|x_n - x^*\| + (1-\beta_n)\lambda_{n+1} \\ &\qquad \times (\gamma+\rho)\|F(x^*)\|] \\ &\le \|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\| \\ &\quad + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\|F(x^*)\|, \\ \|x_{n+1} - x^*\| & \\ &\le \alpha_n \|\bar{x}_n - x^*\| + (1-\alpha_n) \\ &\quad \times [(1-\lambda_{n+1}\tau)\|\hat{x}_n - x^*\| + \lambda_{n+1} t\|F(x^*)\|] \\ &\le \alpha_n \left[\|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\|\right] \\ &\quad + (1-\alpha_n)\{(1-\lambda_{n+1}\tau) \\ &\qquad \times [\|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\| \\ &\qquad + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\|F(x^*)\|] \\ &\qquad + \lambda_{n+1} t\|F(x^*)\|\} \\ &\le \alpha_n \|x_n - x^*\| + \alpha_n(1-\gamma_n)\lambda_{n+1}\gamma\|F(x^*)\| + (1-\alpha_n) \\ &\quad \times [(1-\lambda_{n+1}\tau)\|x_n - x^*\| + \lambda_{n+1}(2\gamma+\rho+t)\|F(x^*)\|].\end{aligned} \tag{16}$$

It is easy to obtain the following by induction:

$$\|x_n - x^*\| \le M_0, \quad \forall n \ge 0, \tag{17}$$

where $M_0 = \max\{3\|x_0 - x^*\|, 3(\rho+\gamma+t)\|F(x^*)\|/\tau\}$,

$$\begin{aligned}\|\tilde{x}_n - x^*\| &\le \|x_n - x^*\| + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\|F(x^*)\| \\ &\le \left(1+\frac{\tau}{3}\right)M_0, \\ \|\bar{x}_n - x^*\| &\le \|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\| \\ &\le \left(1+\frac{\tau}{3}\right)M_0, \\ \|\hat{x}_n - x^*\| &\le \theta_n \|\bar{x}_n - x^*\| + (1-\theta_n)\|\tilde{x}_n - x^*\| \\ &\le 2\left(1+\frac{\tau}{3}\right)M_0.\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}&\{Tx_n\}, \{T\bar{x}_n\}, \{T\tilde{x}_n\}, \{T\hat{x}_n\}, \\ &\{F(Tx_n)\}, \{F(T\bar{x}_n)\}, \{F(T\tilde{x}_n)\}, \{F(T\hat{x}_n)\}\end{aligned} \tag{19}$$

are also bounded.

Step 2. Consider $\|x_{n+1} - x_n\| \to 0$.

Indeed, by a series of computations, we have

$$\begin{aligned}&\|\bar{x}_n - \bar{x}_{n-1}\| \\ &\quad = \left\|\gamma_n x_n - \gamma_{n-1}x_{n-1} + (1-\gamma_n)T_\gamma^{\lambda''_{n+1}} x_n \right. \\ &\qquad \left. - (1-\gamma_{n-1})T_\gamma^{\lambda''_n} x_{n-1}\right\| \\ &\quad \le \|\gamma_n x_n - \gamma_{n-1}x_{n-1}\| \\ &\qquad + \left\|(1-\gamma_n)T_\gamma^{\lambda''_{n+1}} x_n - (1-\gamma_{n-1})T_\gamma^{\lambda''_n} x_{n-1}\right\| \\ &\quad \le \|x_n - x_{n-1}\| + \left|(1-\gamma_n)\lambda''_{n+1} - (1-\gamma_{n-1})\lambda''_n\right| \\ &\qquad \times \gamma\|F(Tx_{n-1})\| \\ &\qquad + |\gamma_n - \gamma_{n-1}|(\|x_{n-1}\| + \|Tx_{n-1}\|).\end{aligned} \tag{20}$$

According to (20) and the prediction step of Algorithm 8, we also obtain

$$\begin{aligned}&\|\tilde{x}_n - \tilde{x}_{n-1}\| \\ &\quad = \left\|\beta_n x_n - \beta_{n-1}x_{n-1} + (1-\beta_n)T_\rho^{\lambda'_{n+1}} \bar{x}_n \right. \\ &\qquad \left. - (1-\beta_{n-1})T_\rho^{\lambda'_n} \bar{x}_{n-1}\right\| \\ &\quad \le \|\beta_n x_n - \beta_{n-1}x_{n-1}\| \\ &\qquad + \left\|(1-\beta_n)T_\rho^{\lambda'_{n+1}} \bar{x}_n - (1-\beta_{n-1})T_\rho^{\lambda'_n} \bar{x}_{n-1}\right\|\end{aligned}$$

$$\begin{aligned}&\leq \|x_n - x_{n-1}\| + \left|(1-\beta_n)\lambda'_{n+1} - (1-\beta_{n-1})\lambda'_n\right| \\ &\quad\times \rho \|F(T\bar{x}_{n-1})\| + (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right) \\ &\quad\times |\gamma_n - \gamma_{n-1}| (\|x_{n-1}\| + \|Tx_{n-1}\|) \\ &\quad+ (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right)\left|(1-\gamma_n)\lambda''_{n+1} - \gamma_{n-1}\lambda''_n\right| \\ &\quad\times \gamma \|F(Tx_{n-1})\| + |\beta_n - \beta_{n-1}| \\ &\quad\times (\|x_{n-1}\| + \|T\bar{x}_{n-1}\| + \|T\bar{x}_{n-1}\|). \end{aligned} \tag{21}$$

Also by the prediction step of Algorithm 8 and (20), (21), we have

$$\begin{aligned}\|\hat{x}_n - \hat{x}_{n-1}\| &\leq \theta_n \|\bar{x}_n - \bar{x}_{n-1}\| + (1-\theta_n)\|\tilde{x}_n - \tilde{x}_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| \\ &\quad+ \left|(1-\gamma_n)\lambda''_{n+1} - (1-\gamma_{n-1})\lambda''_n\right| \gamma \|F(Tx_{n-1})\| \\ &\quad+ |\gamma_n - \gamma_{n-1}| (\|x_{n-1}\| + \|Tx_{n-1}\|) \\ &\quad+ \left|(1-\beta_n)\lambda'_{n+1} - (1-\beta_{n-1})\lambda'_n\right| \rho \|F(T\bar{x}_{n-1})\| \\ &\quad+ (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right)|\gamma_n - \gamma_{n-1}| \\ &\quad\times (\|x_{n-1}\| + \|Tx_{n-1}\|) + (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right) \\ &\quad\times \left|(1-\gamma_n)\lambda''_{n+1} - \gamma_{n-1}\lambda''_n\right| \gamma \|F(Tx_{n-1})\| \\ &\quad+ |\beta_n - \beta_{n-1}| (\|x_{n-1}\| + \|T\bar{x}_{n-1}\| + \|T\bar{x}_{n-1}\|). \end{aligned} \tag{22}$$

Let

$$\hat{y}_n = T_t^{\lambda_{n+1}} \hat{x}_n = T\hat{x}_n - \lambda_{n+1} t F(T\hat{x}_n), \tag{23}$$

so we get

$$x_{n+1} = \alpha_n \bar{x}_n + (1-\alpha_n)\hat{y}_n. \tag{24}$$

Furthermore,

$$\begin{aligned}&\|\hat{y}_n - \hat{y}_{n-1}\| \\ &\quad= \|T\hat{x}_n - T\hat{x}_{n-1} + \lambda_n t F(T\hat{x}_{n-1}) - \lambda_{n+1} t F(T\hat{x}_n)\| \\ &\quad\leq \|T\hat{x}_n - T\hat{x}_{n-1}\| + \lambda_n t \|F(T\hat{x}_{n-1})\| \\ &\qquad+ \lambda_{n+1} t \|F(T\hat{x}_n)\| \\ &\quad\leq \|\hat{x}_n - \hat{x}_{n-1}\| + \lambda_n t \|F(T\hat{x}_{n-1})\| \\ &\qquad+ \lambda_{n+1} t \|F(T\hat{x}_n)\|. \end{aligned} \tag{25}$$

Apply $\lim_{n\to\infty}\beta_n = 1$, $\lim_{n\to\infty}\lambda_n = 0$, and $\lim_{n\to\infty}\gamma_n = 1$ and (22), (25) to get

$$\begin{aligned}&\|\hat{y}_n - \hat{y}_{n-1}\| - \|x_n - x_{n-1}\| \\ &\quad\leq \left|(1-\gamma_n)\lambda''_{n+1} - (1-\gamma_{n-1})\lambda''_n\right| \gamma \|F(Tx_{n-1})\| \\ &\qquad+ |\gamma_n - \gamma_{n-1}| (\|x_{n-1}\| + \|Tx_{n-1}\|) \\ &\qquad+ \left|(1-\beta_n)\lambda'_{n+1} - (1-\beta_{n-1})\lambda'_n\right| \rho \|F(T\bar{x}_{n-1})\| \\ &\qquad+ (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right)|\gamma_n - \gamma_{n-1}| (\|x_{n-1}\| + \|Tx_{n-1}\|) \\ &\qquad+ (1-\beta_n)\left(1-\lambda'_{n+1}\tau'\right) \\ &\qquad\times \left|(1-\gamma_n)\lambda''_{n+1} - \gamma_{n-1}\lambda''_n\right| \gamma \|F(T(x_{n-1}))\| \\ &\qquad+ |\beta_n - \beta_{n-1}| (\|x_{n-1}\| + \|T\bar{x}_{n-1}\| + \|T\bar{x}_{n-1}\|) \\ &\qquad+ \lambda_n t \|F(T\hat{x}_{n-1})\| + \lambda_{n+1} t \|F(T\hat{x}_n)\| \longrightarrow 0. \end{aligned} \tag{26}$$

According to Lemma 4, we obtain

$$\lim_{n\to\infty} \|\hat{y}_{n-1} - x_{n-1}\| = 0. \tag{27}$$

Furthermore, by $\lim_{n\to\infty}\gamma_n = 1$, we also get

$$\begin{aligned}&\|\bar{x}_n - x_n\| \\ &\quad= \left\|-(1-\gamma_n)x_n + (1-\gamma_n)\left[Tx_n - \lambda'_{n+1}\gamma F(Tx_n)\right]\right\| \\ &\quad\leq (1-\gamma_n)\|x_n\| + (1-\gamma_n)\|Tx_n\| + \lambda'_{n+1}\gamma \|F(Tx_n)\| \longrightarrow 0. \end{aligned} \tag{28}$$

By (27), (28) and the correction step of Algorithm 8, we immediately conclude that

$$\begin{aligned}&\|x_n - x_{n-1}\| \\ &\quad= \|\alpha_{n-1}\bar{x}_{n-1} + (1-\alpha_{n-1})\hat{y}_{n-1} - x_{n-1}\| \\ &\quad\leq \alpha_{n-1}\|\bar{x}_{n-1} - x_{n-1}\| + (1-\alpha_{n-1})\|\hat{y}_{n-1} - x_{n-1}\| \longrightarrow 0, \end{aligned} \tag{29}$$

so we get

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{30}$$

Step 3. Consider $\|x_{n+1} - Tx_n\| \to 0$.

Indeed, by the prediction step of Algorithm 8, we have

$$\begin{aligned}&\|\tilde{x}_n - x_n\| \\ &\quad= \left\|-(1-\beta_n)x_n + (1-\beta_n)\left[T\bar{x}_n - \lambda'_{n+1}\rho F(T\bar{x}_n)\right]\right\| \\ &\quad\leq (1-\beta_n)\|x_n\| + (1-\beta_n)\left[\|T\bar{x}_n\| + \left\|\lambda'_{n+1}\rho F(T\bar{x}_n)\right\|\right]. \end{aligned} \tag{31}$$

According to the assumption $\lim_{n\to\infty}\beta_n = 1$ and $\lim_{n\to\infty}\lambda_n = 0$, then

$$\|\tilde{x}_n - x_n\| \longrightarrow 0. \tag{32}$$

By (32), we immediately obtain

$$\|\widehat{x}_n - x_n\| \leq \theta_n \|\overline{x}_n - x_n\| + (1-\theta_n)\|\widetilde{x}_n - x_n\| \longrightarrow 0. \quad (33)$$

By a series of computations, we can get

$$\begin{aligned}
&\|x_{n+1} - Tx_n\| \\
&\quad = \left\|\alpha_n(\overline{x}_n - Tx_n) + (1-\alpha_n)\left(T_t^{\lambda_{n+1}}\widehat{x} - Tx_n\right)\right\| \\
&\quad \leq \alpha_n\|\overline{x}_n - Tx_n\| + (1-\alpha_n)\|T\widehat{x}_n - Tx_n\| \\
&\qquad + (1-\alpha_n)\lambda_{n+1}t\|F(T\widehat{x}_n)\| \\
&\quad \leq \alpha_n\|\overline{x}_n - Tx_n\| + \|\widehat{x}_n - x_n\| + \lambda_{n+1}t\|F(T\widehat{x})\| \\
&\quad \leq \alpha_n\|x_{n+1} - Tx_n\| + \alpha_n\|\overline{x}_n - x_{n+1}\| \\
&\qquad + \|\widehat{x}_n - x_n\| + \lambda_{n+1}t\|F(T\widehat{x})\|.
\end{aligned} \quad (34)$$

Hence, by (28), (33), and (34), we also obtain

$$\begin{aligned}
\|x_{n+1} - Tx_n\| &\leq \frac{\alpha_n}{1-\alpha_n}\|\overline{x}_n - x_{n+1}\| \\
&\quad + \frac{\|\widehat{x}_n - x_n\|}{1-\alpha_n} + \frac{\lambda_{n+1}t\|F(T\widehat{x})\|}{1-\alpha_n} \longrightarrow 0.
\end{aligned} \quad (35)$$

Using Steps 2 and 3, it is easy to obtain the following corollary.

Corollary 14. Consider $\|x_n - Tx_n\| \to 0$.

Applying Steps 2 and 3, one gets

$$\|x_{n+1} - Tx_n\| \longrightarrow 0, \qquad \|x_{n+1} - x_n\| \longrightarrow 0, \quad (36)$$

so it is easy to see that

$$\|x_n - Tx_n\| \leq \|x_{n+1} - Tx_n\| + \|x_{n+1} - x_n\| \longrightarrow 0. \quad (37)$$

Step 4. Consider $\lim_{n\to\infty}\sup\langle -F(x^*), T\widehat{x}_n - x^*\rangle \leq 0$.

For some $\widehat{x} \in H$, here exits $\{Tx_{n_i}\} \to \widehat{x}$ weakly and such that

$$\begin{aligned}
&\lim_{n\to\infty}\sup\langle -F(x^*), Tx_n - x^*\rangle \\
&\quad = \lim_{n\to\infty}\sup\left\langle -F(x^*), Tx_{n_i} - x^*\right\rangle.
\end{aligned} \quad (38)$$

According to $\{Tx_{n_i}\} \to \widehat{x}$, we have

$$\widehat{x} \in \operatorname{Fix}(T) = K. \quad (39)$$

By x^* being the unique solution of $\mathrm{VI}(F,K)$, we can obtain

$$\begin{aligned}
&\lim_{n\to\infty}\sup\langle -F(x^*), Tx_n - x^*\rangle \\
&\quad = \lim_{n\to\infty}\sup\langle -F(x^*), \widehat{x} - x^*\rangle \\
&\quad \leq 0.
\end{aligned} \quad (40)$$

Since $\|T\widehat{x}_n - Tx_n\| \leq \|\widehat{x}_n - x_n\| \to 0$, we immediately conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\sup\langle -F(x^*), T\widehat{x}_n - x^*\rangle \\
&\quad \leq \lim_{n\to\infty}\sup\langle -F(x^*), T\widehat{x}_n - Tx_n\rangle \\
&\qquad + \lim_{n\to\infty}\sup\langle -F(x^*), Tx_n - x^*\rangle \\
&\quad \leq \lim_{n\to\infty}\sup\langle -F(x^*), Tx_n - x^*\rangle \\
&\quad \leq 0.
\end{aligned} \quad (41)$$

Step 5. By Step 1 and Lemma 1, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \left\|\alpha_n(\overline{x}_n - x^*) + (1-\alpha_n)\left(T_t^{\lambda_{n+1}}\widehat{x}_n - x^*\right)\right\|^2 \\
&\quad \leq \|\alpha_n(\overline{x}_n - x^*)\|^2 + (1-\alpha_n) \\
&\qquad \times \left\|\left(T_t^{\lambda_{n+1}}\widehat{x}_n - T_t^{\lambda_{n+1}}x^* + T_t^{\lambda_{n+1}}x^* - x^*\right)\right\|^2 \\
&\quad \leq \|\alpha_n(\overline{x}_n - x^*)\|^2 + (1-\alpha_n) \\
&\qquad \times \left[\left\|T_t^{\lambda_{n+1}}\widehat{x}_n - T_t^{\lambda_{n+1}}x^*\right\|^2\right. \\
&\qquad\quad \left. + 2\left\langle T_t^{\lambda_{n+1}}x^* - x^*, T_t^{\lambda_{n+1}}\widehat{x}_n - x^*\right\rangle\right] \\
&\quad \leq \alpha_n\left[\|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\|\right]^2 \\
&\qquad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2 \\
&\qquad \times \left[\|x_n - x^*\| + (1-\gamma_n)\lambda''_{n+1}\gamma\|F(x^*)\|\right. \\
&\qquad\quad \left. + (1-\beta_n)\lambda_{n+1}(\gamma+\rho)\|F(x^*)\|\right]^2 \\
&\qquad + 2t\lambda_{n+1}\langle -F(x^*), T\widehat{x}_n - x^* - t\lambda_{n+1}F(T\widehat{x}_n)\rangle \\
&\quad \leq \alpha_n\|x_n - x^*\|^2 + (1-\gamma_n)\lambda_{n+1}\gamma M \\
&\qquad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2\|x_n - x^*\|^2 \\
&\qquad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2(1-\beta_n)\lambda_{n+1}M \\
&\qquad + (1-\alpha_n)(1-\lambda_{n+1}\tau)^2(1-\gamma_n)\lambda_{n+1}\gamma M \\
&\qquad + 2t\lambda_{n+1}\langle -F(x^*), T\widehat{x}_n - x^* - t\lambda_{n+1}F(T\widehat{x}_n)\rangle \\
&\quad \leq [1 - (1-\alpha_n)\lambda_{n+1}\tau]\|x_n - x^*\|^2 \\
&\qquad + (1-\alpha_n)\lambda_{n+1}\tau w'_{n+1},
\end{aligned} \quad (42)$$

where

$$w'_{n+1} = \frac{2t\left\langle -F\left(x^*\right), T\hat{x}_n - x^* - t\lambda_{n+1}F\left(T\hat{x}_n\right)\right\rangle}{\tau\left(1-\alpha_n\right)} + \frac{\varphi_n}{\tau\left(1-\alpha_n\right)} + \frac{\xi_n}{\tau\left(1-\alpha_n\right)},$$
$$\varphi_n = \left(1-\gamma_n\right)\gamma M,$$
$$\xi_n = \left(1-\alpha_n\right)\left(1-\lambda_{n+1}\tau\right)^2\left(1-\beta_n\right)M + \left(1-\alpha_n\right)\left(1-\lambda_{n+1}\tau\right)^2\left(1-\gamma_n\right)\lambda_{n+1}\gamma M, \quad (43)$$

and $M_0 \ll M < \infty$.

Denote

$$s'_{n+1} = \left\|x_{n+1} - x^*\right\|, \qquad u_n = \left(1-\alpha_n\right)\lambda_{n+1}\tau. \quad (44)$$

We can rewrite (42) as

$$s'_{n+1} \le \left(1-u_n\right)s'_n + u_n w'_n + 0. \quad (45)$$

In fact, u_n, w'_n satisfies Lemma 5; according to

$$\lim_{n\to\infty}\beta_n = 1, \qquad \lim_{n\to\infty}\gamma_n = 1, \qquad \lim_{n\to\infty}\lambda_n = 0, \quad (46)$$

we obtain

$$\frac{\varphi_n}{\tau\left(1-\alpha_n\right)} \longrightarrow 0,$$
$$\frac{\xi_n}{\tau\left(1-\alpha_n\right)} \longrightarrow 0. \quad (47)$$

Moreover, by Step 4, we also obtain

$$\lim_{n\to\infty}\frac{2t\left\langle -F\left(x^*\right), T\hat{x}_n - x^* - t\lambda_{n+1}F\left(T\hat{x}_n\right)\right\rangle}{\tau\left(1-\alpha_n\right)}$$
$$\le \frac{2t}{\tau}\lim_{n\to\infty}\sup\left\{\left\langle -F\left(x^*\right), T\hat{x}_n - x^*\right\rangle + \lambda_{n+1}\left\langle -F\left(x^*\right), -tF\left(T\hat{x}_n\right)\right\rangle\right\}$$
$$\le \frac{2t}{\tau}\lim_{n\to\infty}\sup\left\{\left\langle -F\left(x^*\right), T\hat{x}_n - x^*\right\rangle\right\} + \lim_{n\to\infty}\sup\left\{\lambda_{n+1}\left\langle -F\left(x^*\right), -tF\left(T\hat{x}_n\right)\right\rangle\right\}$$
$$\le 0 + 0 = 0. \quad (48)$$

Furthermore, by (43), (47), and (48), it is easy to obtain

$$\lim_{n\to\infty}\sup w'_n \le 0. \quad (49)$$

Consequently apply Lemma 5 to obtain

$$\left\|x_n - x^*\right\| \longrightarrow 0. \quad (50)$$

□

4. Numerical Experiments

The problem considered in this section is

$$\min\left\{\frac{1}{2}\|X - C\|_F^2 \mid X \in K\right\}, \quad (51)$$

where $\|\cdot\|_F$ is the matrix Fröbenis norm; that is,

$$\|C\|_F = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|C_{ij}\right|^2\right)^{1/2}. \quad (52)$$

Note that the matrix Fröbenis norm is induced by the inner product

$$\langle A, B\rangle = \text{Trace}\left(A^T B\right). \quad (53)$$

The problems arise from finance and statistics, and we form the test problems similarly as in [9, 21].

Let $K = S_+^n \cap \mathfrak{B}$, where

$$S_+^n = \left\{H \in \mathbb{R}^{n\times n} \mid H^T = H, H \ge 0\right\},$$
$$\mathfrak{B} = \left\{H \in \mathbb{R}^{n\times n} \mid H^T = H, H_L \le H \le H_U\right\}. \quad (54)$$

Let H_L, H_U be given $n \times n$ symmetric matrices, and C asymmetric which differs from previous approaches [9, 21], and it is to be noted that the extended contraction method (EC method) [9] has much difficulty in computing the examples when C is asymmetric, where $H_L \le H_U$ in element wise:

$$H_L \le H_U : \left(H_L\right)_{ij} \le \left(H_U\right)_{ij}, \quad \forall i, j \in 1, \ldots, n. \quad (55)$$

Then (51) is equivalent to the following variational inequality:

$$\left\langle X' - X, \nabla\left(\frac{1}{2}\|X - C\|^2\right)\right\rangle \ge 0, \quad \forall X' \in K. \quad (56)$$

So we get

$$\left\langle X' - X, X - C\right\rangle \ge 0, \quad \forall X' \in K. \quad (57)$$

According to Condition 10, we take the following parameter sequences, and let Condition 10 denote the parameter sequences:

$$\alpha_n = \frac{1}{\ln n},$$
$$\lambda_n = \lambda'_n = \lambda''_n = \frac{1}{\ln(n+1)},$$
$$\beta_n = \gamma_n = 1 - \frac{1}{\ln n},$$
$$\gamma = \rho = t = c_0 > 0. \quad (58)$$

According to Condition 12, we take the following parameter sequences, and let Condition 12 denote the parameter sequences:

$$\alpha_n = 0.8 - \frac{1}{(10 * \ln n)}, \quad n = 2k,$$
$$\alpha_n = 0.3 - \frac{1}{(10 * \ln n)}, \quad n = 2k - 1,$$

Table 1: Numerical results for the PRH method and the EC method.

Asymmetric matrix	$c_0 = 0.1$, $\theta_n = 0.8$, tolerance $= 10^{-4}$						
	Condition 10		Condition 12		EC method		
n	It	cpu	It	cpu	It	cpu	tolerance
100	201	8.34	130	5.35	100	14.46	$8.289e+000$
200	333	75.44	208	47.14	100	94.30	$1.010e+002$
300	443	318.02	272	174.70	100	302.29	$4.899e+002$
400	543	789.16	330	446.00	100	686.83	$9.628e+002$
500	647	1747.70	388	972.18	100	1287.36	$1.756e+003$
1000	1082	19884.30	634	11502.13	100	9220.50	$9.826e+003$
2000	>2000	>150000	1052	128504.67	100	>74640.41	$>5.597e+003$

```
Matlab code:
C = zeros(n,n); HU = ones(n,n) * 0.1; HL = −HU;
for i = 1 : n
   for j = 1 : n
      t = mod(t * 42108 + 13846, 46273);
      C(i, j) = t * 2/46273 − 1;
   end;
end;
for i = 1 : n
   C(i,i) = abs(C(i,i)) * 2; HU(i,i) = 1; HL(i,i) = 1;
end;
```

Algorithm 1

Table 2: Numerical results for tolerance 10^{-4}.

Asymmetric matrix	$c_0 = 0.1$, $\theta_n = 0.8$			
	Condition 10		Condition 12	
n	It	cpu	It	cpu
100	204	8.78	130	5.45
200	330	76.08	208	47.72
300	445	323.20	272	175.89
400	548	867.56	330	450.59
500	663	1916.90	388	994.18

Table 3: Numerical results for tolerance 10^{-3}.

Asymmetric matrix	$c_0 = 0.1$, $\theta_n = 0.8$			
	Condition 10		Condition 12	
n	It	cpu	It	cpu
1000	193	3893.63	126	2280.74
2000	318	42981.02	200	28737.65

$$\lambda_n = \lambda_n' = \lambda_n'' = \frac{1}{\ln(n+1)},$$

$$\beta_n = 1 - \frac{1}{\ln n}, \quad n = 2k,$$

$$\beta_n = 1 - \frac{1}{\ln n}, \quad n = 2k-1,$$

$$\gamma_n = 1 - \frac{1}{\ln n}, \quad n = 2k,$$

$$\gamma_n = 1 - \frac{1}{\ln(2n)}, \quad n = 2k-1,$$

$$\gamma = \rho = t = c_0 > 0. \tag{59}$$

Obviously, we have much difficulty in computing the projection of $P_K[X]$, for all $x \in S^n$. In order to reduce the difficulty and complexity of computing the projection P_K, we define TX by

$$TX = H(G(X)), \tag{60}$$

where

$$G(X) = \min(H_U, \max(X, H_L)),$$
$$H(X) = P_{S_+^n}(X), \tag{61}$$

which can be computed without difficulty and the fixed point set of $\mathrm{Fix}(T) = K$. According to Theorems 11 and 13, the sequences generated by Algorithm 8 under Conditions 10 and 12 are convergent.

The computation begins with ones (n,n) in MATLAB and stops as soon as $\|x_{k+1} - x_k\| \le 10^{-3}$ or 10^{-2}. All codes were implemented in MATLAB 7.1 and ran at a Pentium R 1.70G processor, 2G Acer note computer.

We test the problems with $n = 100, 200, 300, 400, 500, 1000$, and 2000. The test results with the PRH method under

Table 4: Numerical results for tolerance 10^{-4}.

Asymmetric matrix	$\gamma = 0.1, \rho = 0.3, t = 0.1$									
	$\theta_n = 0$		$\theta_n = 0.2$		$\theta_n = 0.4$		$\theta_n = 0.6$		$\theta_n = 0.8$	
n	It	cpu	It	cpu	It	cpu	It	cpu	It	cpu
100	132	5.52	134	5.60	128	5.50	134	5.67	132	5.54
200	210	48.04	206	47.22	208	48.04	204	47.15	214	48.58
300	274	177.49	268	176.08	276	178.80	274	177.68	276	178.84
400	336	468.28	328	445.93	336	468.20	334	454.24	330	453.79
500	392	977.79	394	1012.57	378	948.44	386	953.91	390	971.10

```
Matlab code:
C = zeros(n, n); HU = ones(n, n) * 0.1; HL = −HU;
for i = 1 : n
    for j = 1 : n
        C = −1 + 2 * rand(n);
    end;
end;
for i = 1 : n
    C(i, i) = abs(C(i, i)) * 2; HU(i, i) = 1; HL(i, i) = 1;
end;
```

Algorithm 2

different conditions are reported in Tables 1, 2, 3, and 4. And the CPU time is in seconds. It is to be noted that the results of extended contraction method are only given out when the iteration step (It) is less than or equal to 100.

Test Examples 1. In this example we generate the data in a similar manner as in [9]. The entries of diagonal elements of C are randomly generated in the interval $(0, 2)$; the entries of off-diagonal elements of C are randomly generated in the interval $(-1, 1)$ (Algorithm 1):

$$\begin{gathered}(H_U)_{jj} = (H_L)_{jj} = 1,\\ (H_U)_{ij} = -(H_L)_{ij} = 0.1, \quad \forall i \neq j, i, j = 1, 2, \ldots, n.\end{gathered} \tag{62}$$

When $n \geq 1000$ and tolerance 10^{-4}, the computation time of the proposed method is too long, so the results of the PRH method give out approximate solution with $n \geq 1000$ and tolerance 10^{-3} in the following. And the extended contraction method (EC method) has much difficulty in computing the examples when C is asymmetric. Furthermore, by introducing auxiliary variable, the certain projection method or relaxed-PPA method [10] can be implemented by these tests.

Test Examples 2. We form the data of the second problems similarly as in the first test examples. The entries of diagonal elements of C are randomly generated in the interval $(0, 2)$; the entries of off-diagonal elements of C are generated from a uniform distribution in the same interval (Algorithm 2):

$$\begin{gathered}(H_U)_{jj} = (H_L)_{jj} = 1,\\ (H_U)_{ij} = -(H_L)_{ij} = 0.1, \quad \forall i \neq j, i, j = 1, 2, \ldots, n.\end{gathered} \tag{63}$$

From Tables 1 to 3, we found that the iteration numbers and CPU time of PRH under Condition 12 are more efficient than that under Condition 10. In Table 4 of our method, the tests' results give out that the PRH method under some descent directions is more slightly efficient than those of the MRHSD method [14, 16], and it is easy to obtain these descent directions. Furthermore, it is important to find $\gamma, \rho,$ and t by Tables 2 and 4.

5. Conclusions

We have proved the strong convergence of PRH method under Condition 12, which differs from Condition 10. The result can be considered as an improvement and refinement of the previous results [14]. And more importantly, numerical experiments demonstrated that the PRH method under Condition 12 is more efficient than that under Condition 10, and the PRH method under some descent directions is more slightly efficient than that of the MRHSD method. How to select parameters of the PRH method for solving variational inequalities is worthy of further investigations in the future.

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