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Research Article

A Generalization of Lacunary Equistatistical Convergence of Positive Linear Operators

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In this paper we consider some analogs of the Korovkin approximation theorem via lacunary equistatistical convergence. In particular we study lacunary equi-statistical convergence of approximating operators on H_{w_2} spaces, the spaces of all real valued continuous functions f dened on $K = [0, \infty)^m$ and satisfying some special conditions.

1. Introduction

Approximation theory has important applications in the theory of polynomial approximation, in various areas of functional analysis, numerical solutions of integral and differential equations [1–6]. In recent years, with the help of the concept of statistical convergence, various statistical approximation results have been proved [7]. In the usual sense, every convergent sequence is statistically convergent, but its converse is not always true. And, statistical convergent sequences do not need to be bounded.

Recently, Aktuğlu and Gezer [8] generalized the idea of statistical convergence to lacunary equi-statistical convergences. In this paper, we first study some Korovkin type approximation theorems via lacunary equi- statistical convergence in H_{w_2} spaces. Then using the modulus of continuity, we study rates of lacunary equi-statistically convergence in H_{w_2}.

We recall here the concepts of equi-statistical convergence and lacunary equi-statistical convergence.

Let f and f_r belong to $C(X)$, which is the space of all continuous real valued functions on a compact subset X of the real numbers. $\{f_r\}$ is said to be equi-statistically convergent to f on X and denoted by $f_r \twoheadrightarrow f$ (equistat) if for every $\varepsilon > 0$, the sequence of real valued functions

$$p_{r,\varepsilon}(x) := \frac{1}{r}\left|\{m \le r : f_m(x) - f(x) \ge \varepsilon\}\right| \quad (1)$$

converges uniformly to the zero function on X, which means that

$$\lim_{r\to\infty}\left\|p_{r,\varepsilon}(\cdot)\right\|_{C(X)} = 0. \quad (2)$$

A lacunary sequence $\theta = \{k_r\}$ is an integer sequence such that

$$k_0 = 0, \qquad h_r = k_r - k_{r-1} \longrightarrow \infty \quad \text{as } r \longrightarrow \infty. \quad (3)$$

In this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let θ be a lacunary sequence then $\{f_r\}_{r\in\mathbb{N}}$ is said to be lacunary equi-statistically convergent to f on X and denoted by $f_r \twoheadrightarrow f$ (θ-equistat) if for every $\varepsilon > 0$, the sequence of real valued functions $\{s_{r,\varepsilon}\}_{n\in\mathbb{N}}$ defined by

$$s_{r,\varepsilon}(x) := \frac{1}{h_r}\left|\{m \in I_r : f_m(x) - f(x) \ge \varepsilon\}\right| \quad (4)$$

uniformly converges to zero function on X, which means that

$$\lim_{n\to\infty}\left\|s_{r,\varepsilon}(\cdot)\right\|_{C(X)} = 0. \quad (5)$$

A Korovkin type approximation theorem by means of lacunary equi-statistical convergence was given in [8]. We can state this theorem now. An operator L defined on a linear space of functions Y is called linear if $L(\alpha f + \beta g, x) = \alpha L(f, x) + \beta L(g, x)$, for all $f, g \in Y$, $\alpha, \beta \in \mathbb{R}$ and is called positive, if $L(f, x) \ge 0$, for all $f \in Y$, $f \ge 0$. Let X be a compact subset of $\mathbb{R}$, and let $C(X)$ be the space of all continuous real valued functions on X.

Lemma 1 (see [8]). *Let θ be a lacunary sequence, and let $L_r : C(X) \to C(X)$ be a sequence of positive linear operators satisfying*

$$L_r(t^\nu, x) \twoheadrightarrow x^\nu, \ (\theta\text{-equistat}), \quad \nu = 0, 1, 2, \tag{6}$$

then for all $f \in C(X)$,

$$L_r(f, x) \twoheadrightarrow f, \ (\theta\text{-equistat}). \tag{7}$$

We turn to introducing some notation and the basic definitions used in this paper. Throughout this paper $I = [0, \infty)$. Let

$$C(I) := \{f : f \text{ is a real-valued continuous function on } I\}, \tag{8}$$

and

$$C_B(I) := \{f \in C(I) : f \text{ is bounded function on } I\}. \tag{9}$$

Consider the space H_w of all real-valued functions f defined on I and satisfying

$$|f(x) - f(y)| \le w\left(f; \left|\frac{x}{1+x} + \frac{y}{1+y}\right|\right), \tag{10}$$

where w is the modulus of continuity defined by

$$w(f;\delta) := \sup_{\substack{x,y\in I \\ |x-y|\le\delta}} |f(x) - f(y)|, \quad \text{for any } \delta > 0 \tag{11}$$

(see [9]). Let $K := I^2 = [0,\infty) \times [0,\infty)$, then the norm on $C_B(K)$ is given by

$$\|f\| := \sup_{(x,y)\in K} |f(x,y)|, \quad f \in C_B(K), \tag{12}$$

and also denote the valued of Lf at a point $(x, y) \in K$ is denoted by $L(f; x, y)$ [10, 11].

$w_2(f; \delta_1, \delta_2)$ is the type of modulus of continuity for the functions of two variables satisfying the following properties: for any real numbers δ_1, δ_2, δ_1', δ_2', δ_1'', and $\delta_2'' > 0$,

(i) $w_2(f; \delta_1, \cdot)$ and $w_2(f; \cdot, \delta_2)$ are nonnegative increasing functions on $[0, \infty)$,

(ii) $w_2(f; \delta_1' + \delta_1'', \delta_2) \le w_2(f; \delta_1', \delta_2) + w_2(f; \delta_1'', \delta_2)$,

(iii) $w_2(f; \delta_1, \delta_2' + \delta_2'') \le w_2(f; \delta_1, \delta_2') + w_2(f; \delta_1, \delta_2'')$,

(iv) $\lim_{\delta_1,\delta_2\to 0} w_2(f; \delta_1, \delta_2) = 0$.

The space H_{w_2} is of all real-valued functions f defined on K and satisfying

$$\begin{aligned} &|f(u,v) - f(x,y)| \\ &\quad \le w_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right). \end{aligned} \tag{13}$$

It is clear that any function in H_{w_2} is continuous and bounded on K.

2. Lacunary Equistatistical Approximation

In this section, using the concept of Lacunary equistatistical convergence, we give a Korovkin type result for a sequence of positive linear operators defined on $C(I^m)$, the space of all continuous real valued functions on the subset I^m of R^m, and the real m-dimensional space. We first consider the case of $m = 2$. Following [7] we can state the following theorem.

Theorem 2. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let $\{L_r\}$ be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. L_r is satisfying $L_r(f_\nu; x, y) \twoheadrightarrow f_\nu(x, y)$ (θ-equistat), $\nu = 0, 1, 2$, where $f_k(u, v) \in H_{w_2}$, $k = 0, 1, 2, 3$,*

$$\begin{aligned} f_0(u,v) &= 1, \\ f_1(u,v) &= \frac{u}{u+1}, \\ f_2(u,v) &= \frac{v}{v+1}, \\ f_3(u,v) &= \left(\frac{u}{u+1}\right)^2 + \left(\frac{v}{v+1}\right)^2, \end{aligned} \tag{14}$$

then for all $f \in H_{w_2}$,

$$L_r(f; x, y) \twoheadrightarrow f(x, y), \ (\theta\text{-equistat}). \tag{15}$$

Proof. Let $(x, y) \in K$ be a fixed point, $f \in H_{w_2}$, and assume that (14) holds. For every $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(u, v) - f(x, y)| < \varepsilon$ holds for all $(u, v) \in K$ satisfying

$$\left|\frac{u}{u+1} - \frac{x}{x+1}\right| < \delta_1, \qquad \left|\frac{v}{v+1} - \frac{y}{y+1}\right| < \delta_2. \tag{16}$$

Let

$$\begin{aligned} K_{\delta_1,\delta_2} := \Big\{(u,v) \in K : &\left|\frac{u}{1+u} - \frac{x}{1+x}\right| < \delta_1, \\ &\left|\frac{v}{v+1} - \frac{y}{y+1}\right| < \delta_2\Big\}. \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} |f(u,v) - f(x,y)| &= |f(u,v) - f(x,y)|_{\chi_{K_{\delta_1,\delta_2}}(u,v)} \\ &\quad + |f(u,v) - f(x,y)|_{\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v)} \\ &< \varepsilon + 2M_{\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v)}, \end{aligned} \tag{18}$$

where χ_P denotes the characteristic function of the set P. Observe that

$$\chi_{K\setminus K_{\delta_1,\delta_2}}(u,v) \le \frac{1}{\delta_1^2}\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \frac{1}{\delta_2^2}\left(\frac{v}{v+1} - \frac{y}{y+1}\right)^2. \tag{19}$$

Using (18), (19), and $M := \|f\|$ we have

$$\left|f(u,v) - f(x,y)\right| \leq \varepsilon + \frac{2M}{\delta^2}\left\{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{v}{v+1} - \frac{y}{y+1}\right)^2\right\}, \tag{20}$$

where $\delta := \min\{\delta_1, \delta_2\}$.

By the linearity and positivity of the operators $\{L_r\}$ and by (18), we have

$$\begin{aligned}
L_r&\left(\left(f_1 - \frac{u}{1+u}f_0\right)^2 + \left(f_2 - \frac{v}{1+v}f_0\right)^2; x, y\right)\\
&\leq L_r(f_3; x, y)\\
&\quad - 2\left[\frac{x}{1+x}L_r(f_1; x, y) + \frac{y}{1+y}L_r(f_2; x, y)\right]L_r(f_3; x, y)\\
&\quad - 2\left[\frac{x}{1+x}L_r(f_1; x, y) + \frac{y}{1+y}L_r(f_2; x, y)\right]\\
&\quad + \left[\left(\frac{x}{1+x}\right)^2 + \left(\frac{y}{1+y}\right)^2\right]L_r(f_0; x, y).
\end{aligned} \tag{21}$$

Hence, we get

$$\begin{aligned}
&\left|L_r(f; x, y) - f(x,y)\right|\\
&\quad\leq L_r\left(\left|f(u,v) - f(x,y)\right|; x, y\right)\\
&\qquad + \left|f(x,y)\right|\left|L_r(f_0; x, y) - f_0(x,y)\right|\\
&\quad\leq L_r\left(\varepsilon + \frac{2M}{\delta^2}\left\{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2 + \left(\frac{v}{v+1} - \frac{y}{1+y}\right)^2\right\}; x, y\right)\\
&\qquad + M\left|L_r(f_0; x, y) - f_0(x,y)\right|\\
&\quad\leq L_r\left(\varepsilon + \frac{2M}{\delta^2}\left\{\left(f_1 - \frac{x}{1+x}\cdot f_0\right)^2 + \left(f_2 - \frac{y}{1+y}\cdot f_0\right)^2\right\}; x, y\right)\\
&\qquad + M\left|L_r(f_0; x, y) - f_0(x,y)\right|\\
&\quad= L_r(\varepsilon; x, y) + L_r\left(\frac{2M}{\delta^2}\left(f_1 - \frac{x}{1+x}f_0\right)^2 + \left(f_2 - \frac{y}{1+y}f_0\right)^2; x, y\right)\\
&\qquad + M\left|L_r(f_0; x, y) - f_0(x,y)\right|\\
&\quad\leq \varepsilon + \frac{2M}{\delta^2}\left|L_r(f_3; x, y) - f_3(x,y)\right|\\
&\qquad + \frac{4M}{\delta^2}\left|L_r(f_2; x, y) - f_2(x,y)\right|\\
&\qquad + \frac{4M}{\delta^2}\left|L_r(f_1; x, y) - f_1(x,y)\right|\\
&\qquad + \left(\varepsilon + M + \frac{4M}{\delta^2}\right)\left|L_r(f_0; x, y) - f_0(x,y)\right|\\
&\quad= \frac{2M}{\delta^2}\left|L_r(f_3; x, y) - f_3(x,y)\right|\\
&\qquad + \frac{4M}{\delta^2}\left|L_r(f_2; x, y) - f_2(x,y)\right|\\
&\qquad + \frac{4M}{\delta^2}\left|L_r(f_1; x, y) - f_1(x,y)\right|\\
&\qquad + \varepsilon + N\left|L_r(f_0; x, y) - f_0(x,y)\right|,
\end{aligned} \tag{22}$$

where $N := \varepsilon + M + 4M/\delta^2$. For a given $\mu > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \mu$. Define the following sets:

$$\begin{aligned}
D_\mu(x,y) &:= \left\{m \in \mathbb{N} : \left|L_m(f; x, y) - f(x,y)\right| \geq \mu\right\},\\
D_\mu^\nu(x,y) &:= \left\{m \in \mathbb{N} : \left|L_m(f; x, y) - f(x,y)\right| \geq \frac{\mu - \varepsilon}{4N}\right\},
\end{aligned} \tag{23}$$

where $\nu = 0, 1, 2, 3$. Then from (22) we clearly have

$$D_\mu(x,y) \subseteq \bigcup_{\nu=0}^{3} D_\mu^\nu(x,y). \tag{24}$$

Therefore define the following real valued functions:

$$\begin{aligned}
s_{r,\mu}(x,y) &:= \frac{1}{h_r}\left|\left\{m \in I_r : \left|L_m(f; x, y) - f(x,y)\right| \geq \mu\right\}\right|,\\
s_{r,\mu}^\nu(x,y) &:= \frac{1}{h_r}\left|\left\{m \in I_r : \left|L_m(f_\nu; x, y) - f(x,y)\right| \geq \frac{\mu - \varepsilon}{4N}\right\}\right|,
\end{aligned} \tag{25}$$

where $\nu = 0, 1, 2, 3$. Then by the monotonicity and (24) we get

$$s_{r,\mu}(x,y) \leq \sum_{\upsilon=0}^{3} s_{r,\mu}^\nu(x,y) \tag{26}$$

for all $x \in X$, and this implies the inequality

$$\left\|s_{r,\mu}(\cdot)\right\|_K \leq \sum_{\nu=0}^{3}\left\|s_{r,\mu}^\nu(\cdot)\right\|_K. \tag{27}$$

Taking limit in (27) as $r \to \infty$ and using (14) we have

$$\lim_{r\to\infty}\left\|s_{r,\mu}(\cdot)\right\|_K = 0. \tag{28}$$

Then for all $f \in H_{w_2}$, we conclude that

$$L_r(f;x,y) \twoheadrightarrow f(x,y), \ (\theta\text{-equistat}). \tag{29}$$

□

Now replace I^2 by $I^m = [0,\infty) \times \cdots \times [0,\infty)$ and by an induction, we consider the modulus of continuity type function w_m as in the function w_2. Then let H_{w_m} be the space of all real-valued functions f satisfying

$$\begin{aligned} &|f(u_1,u_2,\dots,u_m) - f(x_1,x_2,\dots,x_m)| \\ &\quad \le w_2\left(f;\left|\frac{u_1}{u_1+1}-\frac{x_1}{x_1+1}\right|,\dots,\left|\frac{u_m}{u_m+1}-\frac{x_m}{x_m+1}\right|\right). \end{aligned} \tag{30}$$

Therefore, using a similar technique in the proof of Lemma 1 one can obtain the following result immediately.

Theorem 3. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let $\{L_r\}$ be a sequence of positive linear operators from H_{w_m} into $C_B(I^m)$. L_r is satisfying*

$$\begin{aligned} &L_r(f_v;x,y) \twoheadrightarrow f_v(x,y), \ (\theta\text{-equistat}), \\ &\qquad v = 0,1,2,\dots,m+1, \end{aligned} \tag{31}$$

where $f_k(u_1,u_2,\dots,u_m) \in H_{w_m}$, $k = 0,1,2,\dots m+1$,

$$\begin{aligned} f_0(u_1,u_2,\dots,u_m) &= 1, \\ f_1(u_1,u_2,\dots,u_m) &= \frac{u_1}{u_1+1}, \\ &\vdots \\ f_m(u_1,u_2,\dots,u_m) &= \frac{u_m}{u_m+1}, \\ f_{m+1}(u_1,u_2,\dots,u_m) &= \left(\frac{u_1}{u_1+1}\right)^2 \\ &\quad + \left(\frac{u_2}{u_2+1}\right)^2 + \cdots + \left(\frac{u_m}{u_m+1}\right)^2. \end{aligned} \tag{32}$$

Then for all $f \in H_{w_m}$,

$$L_r(f;u_1,u_2,\dots,u_m) \twoheadrightarrow f(u_1,u_2,\dots,u_m), \ (\theta\text{-equistat}). \tag{33}$$

Assume that $I = [0,\infty)$, $K := I \times I$. One considers the following positive linear operators defined on $H_{w_2}(K)$:

$$\begin{aligned} B_n(f;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\ &\quad \times \sum_{k=0}^{n}\sum_{l=0}^{n} f\left(\frac{k}{n-k+1},\frac{l}{n-l+1}\right)\binom{n}{k}\binom{n}{l}x^k y^l, \end{aligned} \tag{34}$$

where $f \in H_{w_2}$, $(x,y) \in K$ and $n \in \mathbb{N}$.

Lemma 4. *Let $\theta = \{k_r\}$ be a lacunary sequence, and let*

$$\begin{aligned} B_n(f;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\ &\quad \times \sum_{k=0}^{n}\sum_{l=0}^{n} f\left(\frac{k}{n-k+1},\frac{l}{n-l+1}\right)\binom{n}{k}\binom{n}{l}x^k y^l \end{aligned} \tag{35}$$

be a sequence of positive linear operators from H_{w_2} into $C_B(K)$. If B_n is satisfying

$$\begin{aligned} &B_n(f_v;x,y) \twoheadrightarrow f_v(x,y), \ (\theta\text{-equistat}), \quad v = 0,1,2,3, \\ &f_0(u,v) = 1, \\ &f_1(u,v) = \frac{u}{u+1}, \\ &f_2(u,v) = \frac{v}{v+1}, \\ &f_3(u,v) = \left(\frac{u}{u+1}\right)^2 + \left(\frac{v}{v+1}\right)^2, \end{aligned} \tag{36}$$

then for all $f \in H_{w_2}$,

$$B_n(f;x,y) \twoheadrightarrow f(x,y), \ (\theta\text{-equistat}). \tag{37}$$

Proof. Assume that (36) holds, and let $f \in H_{w_2}$. Since

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k, \qquad (1+y)^n = \sum_{l=0}^{n}\binom{n}{l}y^l, \tag{38}$$

it is clear that, for all $n \in \mathbb{N}$,

$$\begin{aligned} B_n(f_0;x,y) &= \frac{1}{(1+x)^n(1+y)^n}\sum_{k=0}^{n}\sum_{l=0}^{n}\binom{n}{k}x^k\binom{n}{l}y^k \\ &= \frac{1}{(1+x)^n(1+y)^n}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)\sum_{l=0}^{n}\binom{n}{l}y^k = 1. \end{aligned} \tag{39}$$

Now, by assumption we have

$$B_n(f_0;x,y) \twoheadrightarrow f_0(x,y), \ (\theta\text{-equistat}). \tag{40}$$

Using the definition of B_n, we get

$$\begin{aligned} B_n(f_1;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \\ &\quad\times \sum_{k=1}^{n} \frac{k/(n-k+1)}{(k/(n-k+1))+1} \binom{n}{k} x^k \sum_{l=0}^{n} \binom{n}{l} y^l \\ &= \frac{1}{(1+x)^n(1+y)^n} \sum_{k=1}^{n} \frac{k}{n+1} \binom{n}{k} x^k \sum_{l=0}^{n} \binom{n}{l} y^l \\ &= \frac{1}{(1+x)^n} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \binom{n}{k+1} x^{k+1} \\ &= \frac{x}{(1+x)^n} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \binom{n}{k+1} x^{k}. \end{aligned} \tag{41}$$

Since

$$\binom{n}{k+1} = \binom{n-1}{k} \frac{n}{k+1} \tag{42}$$

we get

$$\begin{aligned} B_n(f_1;x,y) &= \frac{x}{(1+x)^n} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \frac{n}{k+1} \binom{n-1}{k} x^k \\ &= \frac{x}{(1+x)^n} \sum_{k=0}^{n-1} \frac{n}{n+1} \binom{n-1}{k} x^k \\ &= \frac{x}{(1+x)(1+x)^{n-1}} \cdot \frac{n}{n+1} \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} x^k \\ &= \frac{n}{n+1} \left(\frac{x}{x+1}\right). \end{aligned} \tag{43}$$

So, we have

$$|B_n(f_1;x,y) - f_1(x,y)| = \frac{x}{x+1} \left| \frac{n}{n+1} - 1 \right|. \tag{44}$$

The fact that $\lim_{n\to\infty}(n/(n+1)) = 1$ and using a similar technique as in the proof of Lemma 1, we get

$$\lim_{r\to\infty} \left\| \frac{1}{h_r} \left| \{ m \in I_r : |B_n(f_1;x,y) - f_1(x,y)| \geq \varepsilon \} \right| \right\| = 0. \tag{45}$$

Hence we have

$$B_n(f_2;x,y) \twoheadrightarrow f_2(x,y), \ (\theta\text{-equistat}). \tag{46}$$

Also we have

$$B_n(f_3;x,y) \twoheadrightarrow f_3(x,y), \ (\theta\text{-equistat}). \tag{47}$$

To see this, by the definition of B_n, we first write

$$\begin{aligned} B_n(f_3;x,y) &= \frac{1}{(1+x)^n(1+y)^n} \sum_{k=0}^{n}\sum_{l=0}^{n} \binom{n}{k}\binom{n}{l} x^k y^l \\ &\quad\times \left[\frac{k^2}{(n+1)^2} + \frac{l^2}{(n+1)^2} \right] \\ &= \frac{1}{(1+x)^n(1+y)^n} \sum_{k=1}^{n} \frac{k^2}{(n+1)^2} \binom{n}{k} x^k \sum_{l=0}^{n} \binom{n}{l} y^l \\ &\quad + \frac{1}{(1+x)^n(1+y)^n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{l=1}^{\infty} \frac{l^2}{(n+1)^2} \binom{n}{l} y^l \\ &= \frac{1}{(1+x)^n} \sum_{k=1}^{n} \frac{k}{(n+1)^2} x^k \\ &\quad + \frac{1}{(1+x)^n} \sum_{k=2}^{n} \frac{k(k-1)}{(n+1)^2} \binom{n}{k} x^k \\ &\quad + \frac{1}{(1+y)^n} \sum_{l=2}^{n} \frac{l(l-1)}{(n+1)^2} \binom{n}{l} y^l \\ &\quad + \frac{1}{(1+y)^n} \sum_{l=1}^{n} \frac{l}{(n+1)^2} y^l. \end{aligned} \tag{48}$$

Then,

$$\begin{aligned} B_n(f_3;x,y) &= \frac{n(n-1)}{(n+1)^2} \frac{x^2}{(x+1)^2} \\ &\quad + \frac{n}{(n+1)^2} \frac{x}{(x+1)} + \frac{n}{(n+1)^2} \frac{y}{y+1} \\ &\quad + \frac{y^2}{(y+1)^2} \frac{n(n-1)}{(n+1)^2} \end{aligned} \tag{49}$$

which implies that

$$\begin{aligned} &|B_n(f_3;x,y) - f_3(x,y)| \\ &\quad = \left(\frac{x^2}{(x+1)^2} + \frac{y^2}{(y+1)^2} \right) \left| \frac{n(n-1)}{(n+1)^2} - 1 \right| \\ &\quad + \frac{n}{(n+1)^2} \left| \frac{x}{x+1} + \frac{y}{y+1} \right|. \end{aligned} \tag{50}$$

Since

$$\lim_{n\to\infty} \frac{n(n-1)}{(n+1)^2} = 1, \qquad \lim_{n\to\infty} \frac{n}{(n+1)^2} = 0, \tag{51}$$

we get

$$\lim_{r\to\infty} \left\| \frac{1}{h_r} \left| \{ m \in I_r : |B_n(f_3;x,y) - f_3(x,y)| \geq \varepsilon \} \right| \right\| = 0. \tag{52}$$

Thus $B_n(f_3;x,y) \twoheadrightarrow f_3(x,y)$, ($\theta$-equistat). Therefore we obtain that for all $f \in H_{w_2}$, $B_n(f;x,y) \twoheadrightarrow f(x,y)$, ($\theta$-equistat). □

3. Rates of Lacunary Equistatistical Convergence

In this section we study the order of lacunary equi-statistical convergence of a sequence of positive linear operators acting on $H_{w_2}(K)$, where $K = I^m$. To achieve this we first consider the case of $m = 2$.

Definition 5. A sequence $\{f_r\}$ is called lacunary equi-statistically convergent to a function f with rate $0 < \beta < 1$ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{s_{r,\varepsilon}(x,y)}{r^{-\beta}} = 0, \tag{53}$$

where $s_{r,\varepsilon}(x,y)$ is given in Lemma 1. In this case it is denoted by

$$f_r - f = o\left(r^{-\beta}\right),\ (\theta\text{-equistat}) \quad \text{on } K = I\times I. \tag{54}$$

Lemma 6. *Let $\{f_r\}$ and $\{g_r\}$ be two sequences of functions in $H_{w_2}(K)$, with*

$$\begin{aligned} f_r - f &= o\left(r^{-\beta_1}\right),\ (\theta\text{-equistat}),\\ g_r - g &= o\left(r^{-\beta_2}\right),\ (\theta\text{-equistat}). \end{aligned} \tag{55}$$

Then one has

$$(f_r + g_r) - (f+g) = o\left(r^{-\beta}\right),\ (\theta\text{-equistat}), \tag{56}$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Assume that $f_r - f = o(r^{-\beta_1})$, ($\theta$-equistat) and $g_r - g = o(r^{-\beta_2})$, ($\theta$-equistat) on K. For all $\varepsilon > 0$, consider the following functions:

$$\begin{aligned} &s_{r,\varepsilon}(x,y)\\ &\quad := \frac{1}{h_r}\left|\{n\in I_r : |(f_n+g_n)(x,y) - (f+g)(x,y)| \geq \varepsilon\}\right|,\\ &s^1_{r,\varepsilon}(x,y) := \frac{1}{h_r}\left|\left\{n\in I_r : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\right\}\right|,\\ &s^2_{r,\varepsilon}(x,y) := \frac{1}{h_r}\left|\left\{n\in I_r : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2}\right\}\right|. \end{aligned} \tag{57}$$

Then we have

$$\begin{aligned} \lim_{r\to\infty}\frac{\|s_{r,\varepsilon}(x,y)\|_{H_{w_2}(K)}}{r^{-\beta}} &= \lim_{r\to\infty}\frac{\left\|s^1_{r,\varepsilon}(x,y)\right\|}{r^{-\beta}} + \lim_{r\to\infty}\frac{\left\|s^2_{r,\varepsilon}(x,y)\right\|}{r^{-\beta}},\\ \frac{s_{r,\varepsilon}(x,y)}{r^{-\beta}} &\leq \frac{s^1_{r,\varepsilon}(x,y)}{r^{-\beta}} + \frac{s^2_{r,\varepsilon}(x,y)}{r^{-\beta}}\\ &\leq \frac{s^1_{r,\varepsilon}(x,y)}{r^{-\beta_1}} + \frac{s^2_{r,\varepsilon}(x,y)}{r^{-\beta_2}}, \end{aligned} \tag{58}$$

and hence

$$\frac{\|s_{r,\varepsilon}(x,y)\|_{H_{w_2}(K)}}{r^{-\beta}} \leq \frac{\left\|s^1_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta_1}} + \frac{\left\|s^2_{r,\varepsilon}(x,y)\right\|_{H_{w_2}(K)}}{r^{-\beta_2}}. \tag{59}$$

Taking limit as $r \to \infty$ and using the assumption complete the proof. □

Now we give the rate of lacunary equi-statistical convergence of a positive linear operators $L_r(f; x, y)$ to $f(x, y)$ with the help of modulus of continuity.

Theorem 7. *Let $K = I\times I$, and let $L_r : H_{w_2}(K) \to H_{w_2}(K)$ be a sequence of positive linear operators. Assume that*

(i) *$L_r(f_0; x, y) - f_0 = o(r^{-\beta_1})$, ($\theta$-equistat) on K,*

(ii) *$w(f; \delta_{r,x}, \delta_{r,y}) = o(r^{-\beta_2})$, ($\theta$-equistat) on K with*

$$\begin{aligned} \delta_{r,x} &= \sqrt{L_r\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2, x\right)},\\ \delta_{r,y} &= \sqrt{L_r\left(\left(\frac{v}{1+v} - \frac{y}{1+y}\right)^2, y\right)}. \end{aligned} \tag{60}$$

Then

$$L_r(f; x, y) - f(x,y) = o\left(r^{-\beta}\right),\ (\theta\text{-equistat}) \ \text{ on } K, \tag{61}$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. Let $f \in H_{w_2}(K)$ and $x \in K$. Use

$$\begin{aligned} &|L_r(f; x,y) - f(x,y)|\\ &\quad\leq L_r(|f(u,v) - f(x,y)|; x,y)\\ &\qquad + |f(x,y)|\,|L_r(f_0; x,y) - f(x,y)|\\ &\quad\leq L_r\left(w_2\left(f; \left|\frac{u}{1+u} - \frac{x}{1+x}\right|, \left|\frac{v}{1+v} - \frac{y}{1+y}\right|\right); x, y\right)\\ &\qquad + |f(x,y)|\,|L_r(f_0; x,y) - f_0(x,y)|\\ &\quad\leq (1 + L_r(f_0; x,y))\, w_2\left(f; \delta_{r,x}, \delta_{r,y}\right)\\ &\qquad + M\,|L_r(f_0; x,y) - f_0(x,y)|\\ &\quad= 2w_2\left(f; \delta_{r,x}, \delta_{r,y}\right) + M\,|L_r(f_0; x,y) - f_0(x,y)|\\ &\qquad + w_2\left(f; \delta_{r,x}, \delta_{r,y}\right)|L_r(f_0; x,y) - f_0(x,y)|, \end{aligned} \tag{62}$$

where $M = \|f\|_{H_{w_2}(K)}$. Using inequality (62), conditions (i) and (ii) we get

$$\lim_{r\to\infty}\frac{1}{r^{-\beta}}\left\|\frac{1}{h_r}\left|\{r\in I_r : |L_r(f; x,y) - f(x,y)| \geq \varepsilon\}\right|\right\| = 0, \tag{63}$$

so we have

$$L_r(f;x,y) - f(x,y) = o\left(r^{-\beta}\right), \ (\theta\text{-equistat}) \ \text{on} \ K. \tag{64}$$

□

Finally we give the rate of lacunary equi-statistical convergence for the operators $L_r(f,x)$ by using the Peetre's K-functional in the space $H_{w_2}(K)$. The Peetre K-functional of function $f \in H_{w_2}(K)$ is defined by

$$K\left(f;\delta_{r,x},\delta_{r,y}\right) = \inf_{g\in H_{w_2}(K)} \left\{\|f-g\|_{C_B(K)} + \delta\|g\|_{C_B(K)}\right\}, \tag{65}$$

where

$$\|f\|_{C_B(K)} = \sup_{(x,y)\in K} |f(x,y)|. \tag{66}$$

Theorem 8. *Let $f \in H_{w_2}(K)$ and $\{K(f;\delta_{r,x},\delta_{r,y})\}$ be the sequence of Peetre's K-functional. If*

$$\begin{aligned}\delta_{r,x} &= \left\|L_r\left(\left(f_1 - \frac{x}{1+x}\right);x,y\right)\right\|_{C_B(K)} \\ &\quad + \left\|L_r\left(\left(f_1 - \frac{x}{1+x}\right)^2;x,y\right)\right\|_{C_B(K)} \|g\|_{C_B(K)}, \\ \delta_{r,y} &= \left\|L_r\left(\left(f_2 - \frac{y}{1+y}\right);x,y\right)\right\|_{C_B(K)} \\ &\quad + \left\|L_r\left(\left(f_2 - \frac{y}{1+y}\right)^2;x,y\right)\right\|_{C_B(K)} \|g\|_{C_B(K)}, \\ &\lim_{r\to\infty}\left\|\delta_{r,x}\right\| = 0, \ (\theta\text{-equistat}) \\ &\lim_{r\to\infty}\left\|\delta_{r,y}\right\| = 0, \ (\theta\text{-equistat})\end{aligned} \tag{67}$$

on $x, y \in K$, then

$$\left\|L_r(f;x,y) - f(x,y)\right\|_{C_B(K)} \le K\left(f;\delta_{r,x},\delta_{r,y}\right). \tag{68}$$

Proof. For each $g \in H_{w_2}(K)$, we get

$$\begin{aligned}&\left\|L_r(g;x,y) - g(x,y)\right\|_{C_B(K)} \\ &\le \|g\|_{C_B(K)}\left(\left\|L_r\left(\left(f_1 - \frac{x}{1+x}\right);x,y\right)\right\|_{C_B(K)} \right. \\ &\qquad \left. + \left\|L_r\left(\left(f_1 - \frac{x}{1+x}\right)^2;x,y\right)\right\|\right) \\ &\quad + \|g\|_{C_B(K)}\left(\left\|L_r\left(f_2 - \frac{y}{1+y}\right);x,y\right\|_{C_B(K)}\right) \\ &\quad + \left\|L_r\left(\left(f_2 - \frac{y}{1+y}\right)^2\right);x,y\right\| \\ &= \left(\delta_{r,x},\delta_{r,y}\right)\|g\|_{C_B(K).}\end{aligned} \tag{69}$$

□

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